

Unimodular hunting

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October 24, 2024

Abstract

We develop a method initiated by Bacher and Venkov, and based on a study of the Kneser neighbors of the standard lattice \mathbb{Z}^n , which allows to classify the integral unimodular Euclidean lattices of rank n . As an application, of computational flavour, we determine the isometry classes of unimodular lattices of rank 26 and 27.

1. Introduction

1.1. The classification of unimodular lattices

Consider the standard Euclidean space \mathbb{R}^n , with inner product $x.y = \sum_i x_i y_i$. Recall that a lattice $L \subset \mathbb{R}^n$ is called *integral* if we have $x.y \in \mathbb{Z}$ for all $x, y \in L$, and *unimodular* if its covolume is 1 (see §2 for the basics on integral lattices). We denote by \mathcal{L}_n the set of all integral unimodular lattices in \mathbb{R}^n and by X_n the (finite) set of isometry classes of such lattices. The most trivial element of \mathcal{L}_n , that will nevertheless play a major role here, is the *standard* or *square* lattice:

$$(1.1) \quad I_n := \mathbb{Z}^n.$$

For $n \leq 7$ the isometry class of I_n is the unique element of X_n , a well-known fact with famous contributions from Lagrange, Gauss, Hermite and Minkowski. For $n = 8$ there is also the E_8 lattice, which is the unique other isometry class in X_8 by Mordell, and the first example of an *even* unimodular lattice. Thanks to the works of many authors, including Witt, Kneser, Niemeier, Conway & Sloane and Borcherds, see *e.g.* [KNE57, NIE73, CS99, BOR00], representatives of X_n have then been determined before this work up to $n = 25$.

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n	1–7	8–11	12–13	14	15	16	17	18	19	20	21	22	23	24	25
$ \mathbf{X}_n $	1	2	3	4	5	8	9	13	16	28	40	68	117	297	665

Table 1.1: The size of \mathbf{X}_n for $n \leq 25$.

The subset $\mathbf{X}_n^\emptyset \subset \mathbf{X}_n$ of classes of lattices with no vector of norm¹ ≤ 2 is especially interesting. By the aforementioned classifications, we know $|\mathbf{X}_n^\emptyset| = 0$ for $n < 23$ or $n = 25$, $|\mathbf{X}_{23}^\emptyset| = 1$ (Short Leech lattice), $|\mathbf{X}_{24}^\emptyset| = 2$ (Leech and Odd Leech lattices). Moreover, Borcherds showed $|\mathbf{X}_{26}^\emptyset| = 1$ in [BOR00, Thm. 3.6], and Bacher and Venkov proved $|\mathbf{X}_{27}^\emptyset| = 3$ and $|\mathbf{X}_{28}^\emptyset| = 38$ in [BV01]. Our first main result in this paper is the following:

Theorem A. *We have $|\mathbf{X}_{26}| = 2566$ and $|\mathbf{X}_{27}| = 17059$.*

Let us mention that general lower bounds on $|\mathbf{X}_n|$ may be obtained using the Minkowski-Siegel-Smith mass formula, although they are very bad for “small” n as in our range. Much better lower bounds have been obtained by O. King in [KIN03] in the case $n \leq 32$. He proved in particular $|\mathbf{X}_{26}| \geq 2307$ and $|\mathbf{X}_{27}| \geq 14179$. Our computations show that King’s estimate were not too far from the actual values. As we shall see later, King’s computations also played an important role in our search. See Sect. 12 for a brief analysis of the isometry groups of the lattices of Theorem A.

1.2. The cyclic d -neighbors of \mathbf{I}_n

Our goal in proving Theorem A is actually not only to classify all the aforementioned lattices, but also to provide constructions of all of them as *cyclic neighbors* of the simplest lattice of all, namely of \mathbf{I}_n . The cyclic d -neighbors of a unimodular lattice $L \in \mathcal{L}_n$ are the unimodular lattices $N \in \mathcal{L}_n$ with $L/(N \cap L) \simeq \mathbb{Z}/d$. This is a fairly classical variant of Kneser’s original definition of neighbor lattices [KNE57, BAC97, BV01, SCH09, CL19, VOI23], see §3 for some background on this notion.² In the sequel, we usually omit the adjective *cyclic* and just talk about d -neighbors. We now recall the concrete construction of the d -neighbors of \mathbf{I}_n .

Fix $d \geq 1$ an integer and $x = (x_i)$ in \mathbf{I}_n with $\gcd(d, x_1, \dots, x_n) = 1$. Then the image of x in $\mathbf{I}_n \otimes \mathbb{Z}/d$ generates a *line*, *i.e.* a cyclic subgroup $l := l(x)$ of order d . The orthogonal of this subgroup in \mathbf{I}_n is the lattice

$$(1.2) \quad \mathbf{M}_d(x) := \mathbf{M}_d(l) := \left\{ v \in \mathbf{I}_n \mid \sum_{i=1}^n x_i v_i \equiv 0 \pmod{d} \right\}$$

¹Following a standard abuse of language, the *norm* of an element v is defined as $v.v$.

²For our purposes, it will be important to allow d to be an arbitrary integer. Many references only treat in details the case where d is odd, or assume the lattices to be even for d even.

and it satisfies $\mathbb{I}_n/M_d(x) \simeq \mathbb{Z}/d$. Any subgroup $M \subset \mathbb{I}_n$ with $\mathbb{I}_n/M \simeq \mathbb{Z}/d$ has the form $M = M_d(l)$ for a unique line $l \subset \mathbb{I}_n \otimes \mathbb{Z}/d$. Set $e = 1$ if d is odd and $e = 2$ otherwise. We say that x (or l) is *d-isotropic* if we have

$$(1.3) \quad \sum_{i=1}^n x_i^2 \equiv 0 \pmod{ed}.$$

It is a fact that there is $N \in \mathcal{L}_n$ with $N \cap \mathbb{I}_n = M_d(x)$ if, and only if, x is *d-isotropic*, and if so, there are exactly e such N . The following formula, in which we choose $x' \in \mathbb{I}_n$ arbitrarily with $x' \equiv x \pmod{d}$ and $x' \cdot x' \equiv 0 \pmod{d^2}$, defines the 1 or 2 possible unimodular lattices N with $N \cap \mathbb{I}_n = M_d(x)$:

$$(1.4) \quad N_d(x') = M_d(x) + \mathbb{Z} \frac{x'}{d}.$$

It is easily checked that Formula (1.4) indeed defines an integral unimodular lattice. For d odd, the lattice $N_d(x')$ does not depend on the choice of x' as above, and we simply denote it by $N_d(x)$. For d even, we temporarily denote by $N_d(x)^\pm$ the two possibilities for $N_d(x')$, postponing to §3.11 the discussion of how to distinguish them. The lattices $N_d(x)^+$ and $N_d(x)^-$ are not isometric in general, but they are isometric if we have $x_i \equiv d/2 \pmod{d}$ for some i .

1.3. Theoretical exhaustion

Before giving examples, we mention a key fact, proved by Hsia and Jöchner in [HJ97, Cor. 4.1], asserting that given any (say) odd $L, L' \in \mathcal{L}_n$, there are infinitely many primes p such that L' is isometric to a p -neighbor of L . In the companion paper [CHE22], we proved several quantitative variants of this result (by very different methods). We give here yet another variant of these results. Assume d odd to simplify. Then any odd element in \mathcal{L}_n has the same (explicit) number $c_n(d)$ of d -neighbors, and for $n > 2$ we have $c_n(d) \sim d^{n-2}$ for $d \rightarrow +\infty$. If L is an Euclidean lattice we denote by $O(L)$ its (finite) isometry group and define its *mass* by $\text{mass}(L) = \frac{1}{|O(L)|}$. Also, we denote by m_n^{odd} the mass of the genus of odd unimodular lattices of rank n (see § 6.2).

Theorem B. *Let $L, L' \in \mathcal{L}_n$ be any odd unimodular lattices of rank n . For an integer $d \geq 1$, denote by $n_d(L, L')$ the number of d -neighbors of L which are isometric to L' . Then we have*

$$\frac{n_d(L, L')}{c_n(d)} \rightarrow \frac{\text{mass}(L')}{m_n^{\text{odd}}} \quad \text{for } d \text{ odd and } d \rightarrow +\infty.$$

In particular, for odd $d \rightarrow \infty$, any odd $L' \in \mathcal{L}_n$ appears as a d -neighbor of \mathbb{I}_n with a probability proportional to its mass $\text{mass}(L')$. Theorem B follows

from [CHE22, Thm. A] if one assumes d prime in its statement : see Sect. 13 for the general case. Let us stress that this result, however, does not say anything about the smallest integer d such that a given $L \in \mathcal{L}_n$ is isometric to a d -neighbor of I_n . This quantity is one of the most difficult to predict, and deserves a definition.

Definition 1. For $L \in \mathcal{L}_n$, the *farness* of L , denoted by $\text{far}(L)$, is the smallest integer $d \geq 1$ such that L is isometric to a (cyclic) d -neighbor of I_n .

1.4. First examples

We now give a few interesting examples. The element $1^n = (1, 1, \dots, 1)$ of I_n is 2-isotropic for $n \equiv 0 \pmod{4}$, so we have a unimodular lattice $N_2(1^n)^\pm$ for such an n . We must have $N_2(1^4)^\pm \simeq I_4$ but recognize $N_2(1^8)^\pm \simeq E_8$. Also, $N_2(1^{12})^\pm$ is the unique rank 12 unimodular lattice with no vector of norm 1, and for $n \equiv 0 \pmod{8}$ and $n > 8$ the lattice $N_2(1^n)^\pm$ is even with root system D_n and is sometimes denoted by E_n or D_n^+ in the literature. The *Leech lattice* also has the following beautiful description due to Thompson (see [CS99] p. lvi) as a 94-neighbor of I_{24} :

$$(1.5) \quad \text{Leech} \simeq N_{94}(x)^\pm, \quad x = (1, 3, 5, 7, \dots, 47) \in \mathbb{Z}^{24}.$$

In the same vein, we have $1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1) \equiv 0 \pmod{2n+1}$ for $n \not\equiv 1 \pmod{3}$, hence a unimodular lattice

$$(1.6) \quad N_{2n+1}(1, 2, 3, \dots, n) \in \mathcal{L}_n, \quad n \not\equiv 1 \pmod{3}.$$

It may be shown that this lattice has no vector of norm ≤ 2 for $n \geq 23$ (see Sect. 8 for a study of those lattices). For $n = 23, 24$ and 26 we recover this way respectively the short Leech lattice, the odd Leech lattice and Borchers' lattice in X_n^\emptyset ! It is hard to think of a simpler definition for these lattices than those ones. All of our lattices will be given in this form:

Theorem C. A *list* of (d, x) such that the $N_d(x')$ are representatives for all the unimodular lattices of Theorem A is given in [CHE20b].

This extends previous work by Bacher [BAC97] in the case $n \leq 24$, as well as the study of X_n^\emptyset for $n \leq 28$ in [BV01]. This (partially aesthetic) wish of giving all of our lattices in the form $N_d(x')$ added in practice a number of extra difficulties, and forced us to find neighbor constructions of some lattices more naturally defined in other ways (see e.g. Sect. 9 & 11).

Remark 2. An further question is to determine, for each lattice $L = N_d(x')$ given in our list, the farness of L . We obviously have $\text{far} L \leq d$, and a neighbor form (d, x') will be called *optimal* if we have $\text{far}(N_d(x')) = d$. As an example, it is easy to see that we have $\text{far}(\text{Leech}) \geq 94$ (see Example 3), so that Thompson's construction (1.5) is optimal. Many neighbor forms in our lists are actually optimal, but certainly not all, and we leave as an open question to determine optimal forms for all of them.

1.5. The general method

The basic strategy we follow to prove Theorem A, which of course heavily relies on computer calculations,³ is well-known: if we are able to produce non-isometric lattices L_1, \dots, L_h in \mathcal{L}_n such that $\sum_{i=1}^h \text{mass}(L_i)$ coincides with the total mass of \mathcal{L}_n , whose exact value is known thanks to the mass formulae (see e.g. [CS99, Chap. 16] p. 409), then L_1, \dots, L_h must be representatives of X_n . This strategy requires at least three ingredients:

(M1) Mass computations. We obviously use the Plesken-Souvignier algorithm [PS97], which efficiently computes $|\mathcal{O}(L)|$ from a given Gram matrix of L with small diagonal. This algorithm turned out to work well for unimodular lattices of rank ≤ 27 , which tend to be generated by vectors of norm ≤ 3 , especially when combined with root system arguments introduced in [CHE20a] : see Remark 4.6 for a discussion about this point.

(M2) Finding invariants. Although the Plesken-Souvignier algorithm also allows to check whether two lattices are isometric or not, it is unrealistic to rely on such an isometry test in situations like ours where millions of lattices will have to be compared. Instead, *we define a few ad hoc (and easy enough to compute) invariants, and we bet in our search that they be enough to distinguish the elements in X_n for our specific n .* The most obvious invariants of a lattice L are its *configuration of vectors of norm i or $\leq i$*

$$(1.7) \quad R_i(L) = \{v \in L \mid v.v = i\} \quad \text{and} \quad R_{\leq i}(L) = \{v \in L \mid v.v \leq i\},$$

say viewed as a finite (Euclidean) metric spaces (see Sect. 4). For later use we also set $r_i(L) := |R_i(L)|$. As is well-known, for any integral lattice L there is a unique decomposition

$$(1.8) \quad L = A \perp B \quad \text{with} \quad A \simeq I_m \quad \text{and} \quad r_1(B) = 0$$

(and we have $2m = r_1(L)$). So we may and do restrict to classify the lattices $L \in \mathcal{L}_n$ with $r_1(L) = 0$. The Euclidean set $R_2(L)$ is called the *root system* of L , and is a disjoint union of classical ADE root systems (see §4.3). It is very easy to determine in practice (see Remark 4.4).

A well-known but curious fact about unimodular lattices of rank ≤ 23 , and about Niemeier lattices, is that they are uniquely determined by their $R_{\leq 2}$. This is however only obtained as a by-product of the classification in these cases. It does not hold anymore for X_{24} , which contains for instance two odd lattices with empty R_1 and same root system $8\mathbf{A}_1 4\mathbf{A}_3$. It is natural to study $R_{\leq 3}$ to go further. It turns out that most lattices in our range are indeed spanned (over \mathbb{Z}) by vectors of norm 3. The main difficulty is that contrary to the case $R_{\leq 2}$ we are not aware of any existing study or classification of configurations of vectors of norm ≤ 3 . We refer to §4.7, as well as the

³In all this work, we heavily used the open-source computer algebra system PARI/GP.

companion paper [AC], for a few invariants of $R_{\leq 3}$ that proved useful in our situations. As a by-product of Theorem A, we obtain the following result.

Corollary D. *Let $L, L' \in \mathcal{L}_n$ with $n \leq 27$. Then L and L' are isometric if, and only if, the Euclidean sets $R_{\leq 3}(L)$ and $R_{\leq 3}(L')$ are isometric.*

(M3) Neighbors enumeration. The basic idea is to enumerate, with the computer, and for increasing integers $d = 2, 3, 4, \dots$, all the d -isotropic vectors x in I_n and study their associated lattices $N_d(x')$, by computing their invariant and, if they are new, their mass, and so on until the total mass of \mathcal{L}_n is exhausted. In order to exclude trivial isometries between neighbors induced by $O(I_n)$ (permutations and sign changes of the coordinates of x), we may restrict the enumeration to d -isotropic vectors $x \in \mathbb{Z}^n$ satisfying⁴

$$(1.9) \quad 0 \leq x_1 \leq x_2 \leq \dots \leq x_d \leq d/2.$$

As two d -isotropic vectors generating the same \mathbb{Z}/d -line give rise to the same neighbor, we also usually further assume $x_1 = 1$ and that x_1 has the maximal multiplicity among the x_i 's (this is not a restriction at all for d prime).

1.6. The splitting root system by root system

Most of the computation time in the algorithm (M3) above is spent computing our invariants of the lattices $N_d(x')$ found in the enumeration. These invariants always include the much faster computation of the root system of $N_d(x')$. Also, as will be clear later when discussing the visible root system, the algorithm starts by finding the lattices with the biggest root systems (and, often, quickly finds all of them). It is thus highly desirable to split our search root system by root system and not to compute the full invariant when all the lattices with a given root system have already been found.

This is fortunately permitted by the aforementioned work [KIN03]. Indeed, as explained *loc. cit.*, although the main computation there concerns even unimodular lattices of rank 32, it allows to determine, for any $n \leq 30$ and any root system of rank n , the mass of the subset of \mathcal{L}_n consisting of lattices with root system isomorphic to R . The details of this step are not fully given in [KIN03]: we give another point of view on it in § 6, and explain as well how to deduce similar results for several other genera of interest. The table below compares the number of possible root systems, given by King, to the actual number of lattices in known cases (Table 1.1 and Theorem A), in dimension $n \leq 30$:

⁴This step of dealing with the orbits, immediate here, gives a clear advantage to this method compared to Kneser's original one, consisting of computing successive 2-neighbors until exhaustion. Another important advantage is that I_n has both a large root system and a large automorphism group, which will allow to efficiently bias our search.

n	12	14–17	18	19	20	21	22	23	24	25	26	27	28	29	30
rs	1	1	4	3	12	12	28	49	149	327	1086	2797	4722	11 085	18 220
min	1	1	4	3	12	12	28	49	156	360	1626	11671	312 287	37 604 456	20 131 670 647
‡	1	1	4	3	12	12	28	49	156	368	1901	14493	?	?	?

Table 1.2: Number **rs** of isometry classes of root systems of unimodular lattices in \mathcal{L}_n with no norm 1 vectors [KIN03], lower bound **min** for the number of such classes by the method in [KIN03], compared to the actual number **‡** of isometry classes.

Remark 1.7. (Reduced mass) *Assume we are interested in the groupoid \mathcal{G} of unimodular lattices L having a given rank and root system R . For each such L , then $|\mathbf{W}(R)|$ divides $|\mathbf{O}(L)|$, where $\mathbf{W}(R)$ denotes the Weyl group of R . Thus it is often more natural to multiply the mass of L (and \mathcal{G}) by $|\mathbf{W}(R)|$: we call this the reduced mass of L or \mathcal{G} (see §4.3).*

1.8. The visible part of a d -neighbor of \mathbf{I}_n

Although this search root system by root system is necessary, it is still by far not enough to find all unimodular lattices. Indeed, the number of $\mathbf{O}(\mathbf{I}_n)$ -orbits of d -isotropic lines in \mathbf{I}_n is $\geq \frac{d^{n-2}}{n!2^{n-1}}$, and in practice it is very lengthy to run over all d -isotropic lines already for d about 30 in our range, whereas the farness of many unimodular lattices is much bigger than 30. In any case, it would be ridiculous to enumerate naively all isotropic lines ! Indeed, certain "visible" properties of the neighbors, in the sense that they can be directly read off from the isotropic lines defining them, substantially bias our search and suggest more clever choices of isotropic lines. This is one of the main topics of this paper.

For $i \geq 1$ and N a d -neighbor of \mathbf{I}_n , there is usually a part of mystery in the Euclidean set $\mathbf{R}_i(N)$, but what we do control is its subset $\mathbf{R}_i(M) = \mathbf{R}_i(\mathbf{I}_n) \cap \mathbf{R}_i(N)$. We have $M = M_d(x)$ for some (d -isotropic) $x \in \mathbb{Z}^n$ by Formula (1.2), and in the canonical basis $\varepsilon_1, \dots, \varepsilon_n$ of \mathbb{R}^n , we also have

$$(1.10) \quad \begin{aligned} \mathbf{R}_1(\mathbf{I}_n) &= \{ \pm \varepsilon_i \mid 1 \leq i \leq n \}, \\ \mathbf{R}_2(\mathbf{I}_n) &= \{ \pm \varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n \} \simeq \mathbf{D}_n. \end{aligned}$$

For instance, we see $\mathbf{R}_1(M_d(x)) = \{ \pm \varepsilon_i \mid x_i \equiv 0 \pmod{d} \}$. In particular, we have $r_1(M_d(x)) = 0$ if, and only if, $x_i \not\equiv 0 \pmod{d}$ for all i ,⁵ an assumption that we shall always make since we are only interested in neighbors N with $r_1(N) = 0$. The root system $\mathbf{R}_2(M_d(x))$ is also *visible*, in the sense that it immediately follows from an inspection of the (i, j) with $1 \leq i \leq j \leq n$ and $x_j \equiv \pm x_i \pmod{d}$: see Section 5. It has the form

$$(1.11) \quad \mathbf{R}_2(M) \simeq \mathbf{A}_{n_1-1} \mathbf{A}_{n_2-1} \cdots \mathbf{A}_{n_s-1} \mathbf{D}_m, \quad \text{with } n = n_1 + n_2 + \cdots + n_s + m.$$

⁵If x is d -ordered, i.e. as in (1.9), this is equivalent to $x_1 \geq 1$.

Definition 1.9. For $L \in \mathcal{L}_n$, and N a d -neighbor of L , the visible root system of N is $R_2(M)$ with $M = L \cap N$. It is a sub root system of $R_2(N)$, namely $R_2(N) \cap L$.

Properties of the visible root system are studied in Section 5. Other visible objects will be studied in this paper and play some important roles, such as *visible isometries* in Section 7, and *visible exceptional vectors* in Section 9.

Example 3. (Empty root system) *As a trivial example, consider the problem of searching for d -neighbors $N = N_d(x')$ of I_n with $R_{\leq 2}(N) = \emptyset$. For such an N , the visible root system has of course to be empty, which forces $x_i \neq 0$ and $x_i \not\equiv \pm x_j \pmod{d}$ for all $1 \leq i < j \leq n$. In particular we have $d \geq 2n + 1$. Better, this shows that for $d = 2n + 1$ the unique possibility up to isometry is the lattice $N_{2n+1}(1, 2, \dots, n)$ for $n \not\equiv 1 \pmod{3}$ already introduced in (1.6). A similar reasoning, taking into account that for $N_d(x')$ to be even we must have d even and all coordinates of x odd, immediately leads to Thompson's construction of Leech.*

1.10. The Biased Neighbor Enumeration algorithm

Consider now the problem of searching for unimodular lattices in \mathcal{L}_n with a given (arbitrary) root system R . The basic idea would be to restrict the enumeration (M3) to d -isotropic lines x such that the visible root system of $N_d(x')$ is R . However, this cannot not work in general for at least two different reasons.

- First of all, R may not be of the form of the right hand side of (1.11) at all. This happens either if it contains some component of type \mathbf{E} , of several components of type \mathbf{D} , or if the union of its \mathbf{A} components have a too large total rank. For instance, in dimension n there is no visible root system of type $k \mathbf{A}_1$ with $k > n/2 + 1$, although of course there may be unimodular lattices with such a root system.

- Worse, even if R may occur as a visible root system in dimension n , it may be the case that certain unimodular lattices of rank n and root system R cannot be obtained as a neighbor with visible root system R . One reason for this is that the visible root system of $N = N_d(x')$ is always saturated in I_n , hence closed to be so in N . More precisely, the visible root system of N is a d -kernel of $R_2(N)$ in our terminology : see § 5.3. For this strategy, it becomes important to classify all d -kernels of \mathbf{ADE} root systems, we do so in § 5.15 using properties of affine Weyl groups.

The good news is that the two obstacles above are essentially the only constraints. Indeed, this is a special case of [CHE22, Thm. 7.1], which extends Thm. B, and that we now try to state in its simplest form. For R a root system and $L' \in \mathcal{L}_n$ we denote by $\text{emb}(R, L')$ the set of isometric embeddings⁶

⁶We denote by $Q(R)$ the root lattice generated by R .

$Q(R) \rightarrow L'$ with saturated image and odd orthogonal complement. We also define $m_n^{\text{odd}}(R)$ as the mass of the groupoid of pairs (L, ι) with $L \in \mathcal{L}_n$ and $\iota \in \text{emb}(R, L)$. The following result follows from the special case $A = Q(R)$ of [CHE22, Thm. 7.10], as well as Corollary 7.12 *loc. cit.*

Theorem E. *Let L, L' be odd unimodular lattices of rank n . Assume $R \subset R_2(L)$ is a rank r saturated sub root system whose orthogonal complement in L contains a sublattice isometric to I_3 . For p a prime, let $n_p(L, L', R)$ be the number of p -neighbors of L isometric to L' and with visible root system containing R . Then we have*

$$\frac{n_p(L, L', R)}{p^{n-r-2}} \longrightarrow \frac{|\text{emb}(R, L')| \text{mass}(L')}{m_n^{\text{odd}}(R)} \text{ when } p \rightarrow +\infty.$$

In other words, by prescribing the visible root system to contain R we bias the statistics of Theorem B exactly by the factor $|\text{emb}(R, L')|$. We apply this result to $L = I_n$. Prescribing the visible root system in the enumeration is then immediate : it just amounts to restrict to d -isotropic lines having certain equal coordinates. In practice, this method is extremely efficient, and allows in only a few seconds to find a unimodular lattice with given root system, and even all of those lattices L' having the largest possible $|\text{emb}(R, L')| \text{mass}(L')$. As an example, *it allows to reproduce the full list of all unimodular lattices of rank ≤ 25 in just a few minutes*. It is sometimes delicate to understand which visible root system is the best to use in our search : see § 5.9 & 5.15 for this question, which often boils down to exercises in root systems and coding theory.

We give a concrete example of application of this Biased Neighbor Enumeration algorithm in dimension 26 in the next section, and others in § 10. We also refer to §4 of the companion paper [AC] for a more formal exposition of this algorithm, and to *loc. cit.* §5 for several other examples.

1.11. An example : the root system $10 \mathbf{A}_1$ in X_{26}

Let us consider the problem of finding all unimodular lattices of rank 26 with root system $10 \mathbf{A}_1$ (and with no norm 1 element, we will not repeat this condition). By King, the reduced mass of this groupoid of lattices is 4424507/116121600 (see Remark 1.7). Moreover, we may show that any lattice with root system $10 \mathbf{A}_1$ contains a saturated $8 \mathbf{A}_1$ (see § 5.9 and especially Example 5.14), so it looks safe to search for our lattices with such a visible root system. This means we restricts to enumerating d -isotropic $x \in \mathbb{Z}^{26}$ satisfying (1.9) and with exactly 8 *pairs of equal coordinates* modulo d . In particular, this forces $d \geq 2 \cdot (8 + 10) = 36$.

For $d = 36$ we do instantly find 108 such isotropic lines, and 49 of them happen to lead to a neighbor with root system $10 \mathbf{A}_1$ (and no norm 1 element). This high ratio 49/108 attests that our bias is successful : we chosed very

well the visible root system. Those 49 lattices happen to contain 4 different isometry classes, with respective reduced mass $1/64$, $1/96$, $1/96$ and $1/640$: see Table 1.3 for representatives.

For $d = 37$ and 38 the ratio of d -neighbors with root system $10\mathbf{A}_1$ are respectively $41/73$ and $458/1095$, but no new lattices are found. For $d = 39$, the ratio is $820/1821$ and we find the fifth lattice below, with reduced mass $1/12288$: this later finding fits the fact that the probability to find this lattices was smaller, as so is its mass, by Theorem E.

Unfortunately, for $40 \leq d \leq 49$, a systematic enumeration of more than 2 millions of isotropic lines do not lead to any new lattice. This is not really a surprise. Indeed, the remaining reduced mass is

$$4424507/116121600 - 1/64 - 1/96 - 1/96 - 1/640 - 1/12288 = 17/116121600.$$

This is only about $4 \cdot 10^{-6}$ of the initial mass, hence extremely small, so unless we have at our disposal many cores (and time to waste), it is not reasonable to search for the remaining lattices just by pursuing the enumeration in the ‘‘coupon collector’’ style (see §1.12). It is thus highly desirable to have other methods to find the remaining lattices. We propose two methods here : one that we call the *adding \mathbf{D}_m method*, and that we will explain in details in Sect. 11 (see especially Prop.11.4 and Remark 11.5), and another one based on visible isometries, studied in Sect. 7.

The first idea is to observe that if L is a rank 26 unimodular lattice with root system $10\mathbf{A}_1$ and no norm 1 vector, then the orthogonal of any $2\mathbf{A}_1$ in L has root system $8\mathbf{A}_1$, rank 24, and index 2 in some unimodular lattice. For our L of interest, this lattice has a presumably high farness, and we do see in rank 24 an odd unimodular lattice L_0 with root system $8\mathbf{A}_1$ appearing in the end of our lists. It has the reduced mass $1/20643840$, which is close enough to $1/116121600$ since the quotient is $45/8$, so this is promising. Actually, we can immediately discover this lattice using the visible root system $7\mathbf{A}_1$ in rank 24 : we have *e.g.* $L_0 \simeq N_{35}(x)$ for

$$x = (1, 1, 2, 3, 4, 4, 5, 6, 7, 7, 8, 9, 9, 10, 11, 12, 12, 13, 14, 15, 15, 16, 17, 17) \in \mathbb{Z}^{24}.$$

This suggests to look at the 2-neighbors of $L_0 \oplus \mathbf{I}_2$, which are specific 70-neighbors of \mathbf{I}_{26} . By restricting to those with same visible root system $7\mathbf{A}_1$, there are now only 2^{16} lines to check. We find 16384 isotropic lines, 1816 leading to a neighbor with root system $10\mathbf{A}_1$. Among those, two lattices are found, with reduced masses $1/7372800$ and $1/92897280$ respectively, which fulfills the mass : see Table 1.3 below.

d	$x \in \mathbb{Z}^{26}$	reduced mass	$ \text{emb}(8\mathbf{A}_1, -) $
36	(1, 1, 2, 3, 4, 5, 6, 6, 7, 7, 8, 8, 9, 10, 11, 12, 12, 13, 13, 14, 14, 15, 16, 16, 17, 18)	1/64	45
36	(1, 1, 2, 3, 4, 5, 6, 6, 7, 8, 9, 10, 10, 11, 11, 12, 13, 13, 14, 14, 15, 16, 16, 17, 18, 18)	1/96	45
36	(1, 1, 2, 3, 4, 5, 6, 7, 8, 8, 9, 10, 11, 11, 12, 12, 13, 13, 14, 14, 15, 16, 16, 17, 18, 18)	1/96	45
36	(1, 1, 2, 2, 3, 4, 5, 6, 6, 7, 8, 8, 9, 9, 10, 10, 11, 12, 13, 14, 14, 15, 16, 17, 17, 18)	1/640	45
39	(1, 1, 2, 3, 4, 5, 5, 6, 7, 7, 8, 8, 9, 10, 10, 11, 12, 13, 14, 14, 15, 16, 16, 17, 19, 19)	1/12288	45
70	(1, 1, 33, 3, 4, 4, 5, 29, 7, 7, 27, 9, 9, 25, 11, 12, 12, 13, 21, 15, 15, 19, 18, 18, 35, 35)	1/7372800	25
70	(1, 1, 33, 3, 4, 4, 5, 29, 7, 7, 27, 9, 9, 25, 11, 23, 23, 13, 21, 15, 15, 19, 17, 17, 35, 35)	1/92897280	9

Table 1.3: The 7 lattices with no norm 1 elements and root system $10\mathbf{A}_1$ in X_{26} .

Another way to find the last lattices in Table 1.3 amounts to use the theory of visible isometries. Indeed, we have $116121600 = 2^{13} 3^4 5^2 7$, so we knew for instance that some lattice had an order 7 automorphism (namely, the last in the table above). Applying the method of Section 7 for searching for elements of X_{26} with root system $10\mathbf{A}_1$ and a visible automorphism with characteristic polynomial $\Phi_7^3 \Phi_1^8$ immediately leads to many constructions of this lattice, such as one⁷ for $d = 29 \cdot 27 = 783$. Here 29 is the prime $\equiv 1 \pmod{7}$.

1.12. A coupon collector's problem and choices of invariants

Assume we want to find all unimodular lattices of given rank n by running through all isotropic lines in $I_n \otimes \mathbb{Z}/d$, with $d = 2, 3, \dots$. Theorem B shows that we are in the situation of a coupon collector's problem (or "Panini album"). Indeed, studying a new line (which costs some time to the computer) corresponds to buying a new box, and the mass of a lattice is proportional to the probability of find it in a box. Recall that in a uniform situation with N coupons, then on average we need to buy $N \log N$ boxes to collect all coupons (see e.g. §2.4.1 in [MU17]). In our non-uniform situation this can be much worse, since certain lattices have much smaller masses than others. By Theorem E, the idea of fixing the visible root system substantially helps reducing this non uniformity, and it does work in most cases. Sometimes, however, it is not enough as shown by the Example in Sect. 1.11, and it is preferable to use other methods to find the lattices with smallest mass.

A bit surprisingly, those statistic arguments have also been very helpful in order to discover that some invariants were not fine enough. Indeed, when searching for lattices with a given root system using a certain invariant, if start finding new lattices with very small mass compared to the remaining

⁷The corresponding $x \in \mathbb{Z}^{26}$ is congruent to $(1^{14}, 3^7, 4, 8, 9, 10, 11) \pmod{27}$ and to $(1, 16, 24, 7, 25, 23, 20, 1, 16, 24, 7, 25, 23, 20, 1, 16, 24, 7, 25, 23, 20, 0^5) \pmod{29}$.

mass for this root system, it is a strong indication that we missed the most likely ones : our chosen invariant is not fine enough. See *e.g.* the discussion of the case $R \simeq 2 \mathbf{A}_1 2 \mathbf{A}_2 2 \mathbf{A}_3 2 \mathbf{A}_4$ in dimension 26 in Sect. 10 for an example of such a situation.

1.13. Proofs of Theorems A and C : the full lists

Although most of the lattices in the lists of Theorem C have been found by applying the naive algorithm described in (M3) of §1.5, the main work was to deal with the remaining ones. A large part of them were found by using the biased algorithm of §1.10, using for each remaining root system clever choices of visible root systems as explained in Sect. 5. The most resisting (and interesting!) lattices were then dealt with more specific methods, some of them already encountered in the introduction : enumerating the 2-neighbors of well-chosen lattices, “addition of \mathbf{D}_m ” method (see Sect. 11), separate study of exceptional lattices (see Sect. 9)...⁸

The whole computation required so many case by case considerations that it would be not be reasonable to list them here. For instance, several hundreds of the 2797 possible root systems in dimension 27 had to be treated separately (as in Example 1.11). Instead, our expository choice in this paper was to explain the theoretical aspects underlying each method that we used, and to only provide a few detailed examples as illustrations. We refer to the companion paper [AC] for more examples of our method, including some emphasize (and improvements) on some computational aspects only briefly discussed in this paper. We mention that all of our computations have been made using PARI/GP and a processor Intel(R) Xeon(R) CPU E5-2650 v4 @ 2.20GHz with 65 GB of memory; the total CPU time was about 1 month in dimension 26 and 1 year in dimension 27.

Actually, if our goal is just to prove Theorems A and C, rather than understanding the mathematical ideas involved, it is not really necessary to explain how our lists in [CHE20b] were discovered ! Indeed, it may be checked independently and a posteriori that these lists are complete : it is enough to check that all the given lattices have distinct invariants and that the sum of their masses equals the mass formula. See [CHE20b] for the relevant PARI/GP source code for this check. This is of course much shorter : it only requires 5 hours in dimension 26, and 40 hours in dimension 27. As a consequence, this also provides an independent verification of King’s computations.

In the companion paper [AC], in collaboration with Bill Allombert, we pursue the ideas of this paper and determine in particular X_{28} and X_{29}^\emptyset .

⁸Actually, when we first made these computations in 2020 we had not yet discovered the theory of visible isometries explained in §7, and mostly used instead the ideas of §7.9 to construct lattices with small reduced mass. Many such lattices can also be found faster using the arguments of §7 (although with a worst farness!).

ACKNOWLEDGMENTS. We warmly thank the developers of PARI/GP for their work, as well as the LMO for sharing their machine `pascaline`. We also thank Bill Allombert and Olivier Taïbi for useful discussions.

Contents

1	Introduction	1
2	General conventions and notations	13
3	Cyclic Kneser neighbors of unimodular lattices	15
4	Some invariants of lattices	19
5	The visible root system of a d-neighbor of I_n	25
6	King's mass formulas	31
7	Visible isometries	36
8	An example : the lattice $N_{2n+1}(1, 2, \dots, n)$	41
9	Exceptional lattices and visible characteristic vectors	43
10	A few more examples in dimension 26	49
11	A few constructions of lattices in neighbor form	50
12	The Jordan-Hölder factors of $O(L)$ for $L \in \mathcal{L}_n$ and $n \leq 27$	52
13	Proof of Theorem B.	53
14	References	55

2. General conventions and notations

In this paper, group actions will be on the left. We denote by $|X|$ the cardinality of the set X . For $n \geq 1$ an integer, we denote by S_n the symmetric group on $\{1, \dots, n\}$, by $\text{Alt}_n \subset S_n$ the alternating subgroup, and we also denote by \mathbb{Z}/n the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

(i) If V is an Euclidean space, we usually denote by $x \cdot y$ its inner product. A *lattice* in V is a subgroup generated by a basis of V , or equivalently, a discrete subgroup L with finite covolume, denoted $\text{covol } L$. Its *dual lattice* is the lattice L^\sharp defined as $\{v \in V \mid v \cdot x \in \mathbb{Z}, \forall x \in L\}$. Recall that L is *integral* if we have $L \subset L^\sharp$. An integral lattice is called *even* if we have $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$, *odd* otherwise. The orthogonal group of V is denoted by $O(V)$, and we also denote by $O(L) = \{\gamma \in O(V), \gamma(L) = L\}$ the isometry group of L (a finite group).

(ii) Assume $L \subset V$ is an integral lattice. The finite abelian group $\text{res } L := L^\sharp/L$ (sometimes called the *discriminant group* [Nik79], the *glue group* [CS99]

or the *residue* [CL19]) is equipped with a non-degenerate \mathbb{Q}/\mathbb{Z} -valued symmetric bilinear form, defined by $(x, y) \mapsto x \cdot y \pmod{\mathbb{Z}}$. We have $(\text{covol } L)^2 = |\text{res } L|$. This integer, also denoted $\det L$, is also the *determinant* of the *Gram matrix* $\text{Gram}(e) = (e_i \cdot e_j)_{1 \leq i, j \leq n}$ of any \mathbb{Z} -basis $e = (e_1, \dots, e_n)$ of L .

(iii) A subgroup $I \subset \text{res } L$ is called *isotropic*, if we have $x \cdot y \equiv 0$ for all $x, y \in I$, and a *Lagrangian* if we have furthermore $|I|^2 = |\text{res } L|$. The map $\beta_L : M \mapsto M/L$ defines a bijection between the set of integral lattices containing L and the set of isotropic subgroups of $\text{res } L$. In this bijection, M/L is a Lagrangian if and only if M is unimodular. If I is finite abelian group, we denote by $H(I)$ the *hyperbolic* symmetric bilinear space $I \oplus I^*$, with $I^* = \text{Hom}(I, \mathbb{Q}/\mathbb{Z})$, defined by $(x + \phi) \cdot (x' + \phi') = \phi(x') + \phi'(x)$.

(iv) Assume furthermore that L is an even lattice. Then the finite symmetric bilinear space $\text{res } L$ has a canonical quadratic form $q : \text{res } L \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $q(x + y) - q(x) - q(y) \equiv x \cdot y$, defined by $q(x) = \frac{x \cdot x}{2} \pmod{\mathbb{Z}}$. In the bijection β_L above, the even lattices M correspond to the *quadratic* isotropic subspace $I \subset \text{res } L$, *i.e.* with $q(I) = 0$. We also denote by $\text{qm} : \text{res } L \rightarrow \mathbb{Q}_{\geq 0}$ the *Venkov map*, defined by $\text{qm}(x) = \text{Min}_{y \in x+L} \frac{y \cdot y}{2}$. It satisfies $\text{qm}(x) \equiv q(x) \pmod{\mathbb{Z}}$, $\text{qm}(x) = \text{qm}(-x)$ and $\text{qm}(x) > 0$ for $x \neq 0$.

(v) (*Standard lattices*) Here \mathbb{R}^n denotes the standard Euclidean space, for $n \geq 0$, with canonical basis $\varepsilon_1, \dots, \varepsilon_n$. We set $D_n = \{x \in \mathbb{Z}^n \mid \sum_i x_i \equiv 0 \pmod{2}\}$ and $A_n = \{x \in \mathbb{Z}^{n+1} \mid \sum_i x_i = 0\}$. The E_8 *lattice* is $D_8 + \mathbb{Z}e$ with $e = \frac{1}{2} \sum_{i=1}^8 \varepsilon_i$, the E_7 (resp. E_6) *lattice* is the orthogonal of $\varepsilon_7 + \varepsilon_8$ (resp. of $\varepsilon_7 - \varepsilon_6$ and $\varepsilon_7 + \varepsilon_8$) in E_8 . All these lattices are even. Their Venkov map are well-known (theory of *minuscule weights*), with nonzero values given by Table 2.1:

L	A_n	$D_n, n > 0$ even	D_n, n odd	E_6	E_7	E_8
$\text{res } L$	$\mathbb{Z}/(n+1)$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	0
qm	$\frac{i(n+1-i)}{2(n+1)}$ with $1 \leq i \leq n$	$\frac{n}{8}, \frac{1}{2}, \frac{n}{8}$		$\frac{2}{3}, \frac{2}{3}$	$\frac{3}{4}$	

Table 2.1: The nonzero values of qm on $\text{res } L$ (with multiplicities).

More precisely, for $L = A_n$ there is a group isomorphism $\phi : \mathbb{Z}/(n+1) \xrightarrow{\sim} \text{res } A_n$ with $\text{qm}(\phi(i \pmod{n+1})) = \frac{i(n+1-i)}{2(n+1)}$ for $1 \leq i \leq n$.

(vi) A subgroup A of a lattice L is called *saturated* if the abelian group L/A is torsion free, or equivalently, if A is a direct summand of L as \mathbb{Z} -module. The *saturation* of A in L , defined as $\text{Sat}_L(A) = L \cap (A \otimes \mathbb{Q})$, is the smallest saturated subgroup S of L containing A .

3. Cyclic Kneser neighbors of unimodular lattices

3.1. Definitions and notations

Let V be an Euclidean space, L a unimodular integral lattice in V , and $d \geq 1$ an integer. A d -neighbor of L is a unimodular integral lattice $N \subset V$ such that we have a group isomorphism

$$L/(L \cap N) \simeq \mathbb{Z}/d.$$

Such lattices N are sometimes called *cyclic* d -neighbors of L , but from now on we will omit the adjective cyclic for short. A few remarks are in order:

(Na) If N is a d -neighbor of L , then L is a d -neighbor of N . Indeed, if we set $M = L \cap N$, then we have $M^\sharp = N^\sharp + L^\sharp = N + L$, and thus $\text{res } M = N/M \oplus L/M$ with N/M and L/M Lagrangians. As a consequence, the pairing of $\text{res } M$ identifies N/M with $\text{Hom}(L/M, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/d$.

(Nb) As L is unimodular, the subgroups $M \subset L$ with $L/M \simeq \mathbb{Z}/d$ are the

$$M_d(L; x) := \{m \in L \mid m \cdot x \equiv 0 \pmod{d}\}$$

where $x \in L$ is a d -primitive vector. By this mean an element of L whose image in L/dL generates a subgroup of order d . For d -primitive $x, x' \in L$, we have $M_d(L; x) = M_d(L; x')$ if, and only if, x and x' generate the same subgroup in L/dL . We thus denote as well by $M_d(L; \ell)$ the lattice $M_d(L; x)$, if $\ell \simeq \mathbb{Z}/d$ is the subgroup in L/dL generated by x .

Proposition-Definition 3.2. *Let L be a unimodular integral lattice in V , $d \geq 1$ an integer, $x \in L$ a d -primitive element. Set $M = M_d(L; x)$, and set also $e = 1$ for d odd and $e = 2$ for d even.*

(i) *If $x \cdot x \equiv 0 \pmod{ed}$, there are exactly e cyclic d -neighbors N of L with $N \cap L = M$, and none otherwise. These d -neighbors are the*

$$(3.1) \quad M + \mathbb{Z} \frac{\tilde{x}}{d}$$

where \tilde{x} is any element of L with $\tilde{x} \equiv x \pmod{dL}$ and $\tilde{x} \cdot \tilde{x} \equiv 0 \pmod{d^2}$.

(ii) *For d odd, the lattice (3.1) does not depend on the choice of \tilde{x} and we denote it $N_d(L; x)$. For d even, it only depends on the element $\epsilon \in \mathbb{Z}/2$ defined by $\tilde{x} \cdot x \equiv \frac{x \cdot x}{2} + \epsilon \frac{d^2}{2} \pmod{d^2}$, and we denote it $N_d(L; x; \epsilon)$.*

Proof. As x is d -primitive and L is unimodular, we may and do choose $y \in L$ with $y \cdot x \equiv 1 \pmod{d}$. We clearly have $L \subset M^\sharp$ and $x/d \in M^\sharp$. Using $L/M = \mathbb{Z}/d y$, $|\text{res } M| = d^2$ and $x \cdot y \equiv 1 \pmod{d}$, we obtain $\text{res } M = \mathbb{Z}/d y \oplus \mathbb{Z}/d \frac{x}{d}$.

The d -neighbors N of L with $L \cap N = M$ are in bijection with the Lagrangians $I \subset \text{res } M$ which are *transversal* to L/M , *i.e.* with $I \cap (L/M) = \{0\}$. As $|I| = d$, each such Lagrangian is necessarily $\simeq \mathbb{Z}/d$ and generated by a unique element of the form $x/d - ry$ with $r \in \mathbb{Z}/d$. But such an element is isotropic if, and only if, we have $\frac{x.x}{d} \equiv 2r \pmod{d\mathbb{Z}}$. For d odd, there is a unique possibility for r , and for d even there are none if $x.x \not\equiv 0 \pmod{2d}$, and exactly two otherwise, namely the $r_\epsilon = \frac{x.x}{2d} + \epsilon \frac{d}{2} \pmod{d}$ with $\epsilon \in \mathbb{Z}/2$. In the latter case, note that we have

$$x.(x/d - r_\epsilon y) \equiv \frac{x.x}{d} - r_\epsilon \equiv \frac{x.x}{2d} + \epsilon \frac{d}{2} \pmod{d\mathbb{Z}}.$$

This proves the first assertion of (i), as well as the second once we observe that for any $\tilde{x} \in x + dL$ we have $\tilde{x}/d \in M^\sharp$, and thus $\tilde{x}.\tilde{x} \equiv 0 \pmod{d^2}$ if and only if \tilde{x}/d is isotropic in $\text{res } M$. \square

Remark 3.3. (i) By the proof, we may define $\mathcal{N}_d(L, x)$ (resp. $\mathcal{N}_d(L; x; \epsilon)$) by taking $\tilde{x} = x + rdy$ in (3.1) with $r = \frac{d+1}{2d} x.x$ (resp. $r = -\frac{x.x}{2d} + \epsilon \frac{d}{2}$). Here y denotes any element of L with $x.y \equiv 1 \pmod{d}$; it is unique modulo M , so those lattices do not depend on this choice of y .

(ii) For d odd, the lattice $\mathcal{N}_d(L; x)$ only depends on M , hence on the \mathbb{Z}/d -line ℓ of L/dL generated by x , so we may also denote it by $\mathcal{N}_d(L; \ell)$. This fails for d even. Indeed, if we set $x' = x + dv$ with $v \in L$, a simple computation using *e.g.* the formula given in (i) shows $\mathcal{N}_d(L; x'; \epsilon) = \mathcal{N}_d(L; x; \epsilon + v.v)$. In the case L is even, and only in this case, the lattice $\mathcal{N}_d(L; x; \epsilon)$ only depends on $(L; \ell; \epsilon)$ (see Proposition 3.8 (ii)).

We denote by $\mathcal{N}_d(L)$ the set of all d -neighbors of L . As the sublattice $L \cap N$, for $N \in \mathcal{N}_d(L)$, plays an important role in this paper, we give a name to it.

Definition 3.4. *If N is a d -neighbor of L , we call the lattice $M = L \cap N$ the visible part of N . By (Na), this is an integral lattice with $\text{res } M \simeq \mathbb{H}(\mathbb{Z}/d)$.*

3.5. The quadric C_L

Fix $L \in \mathcal{L}_n$, $d \geq 1$ and set again $e = 1$ if d is odd and $e = 2$ otherwise. Consider the finite quadric

$$(3.2) \quad C_L(\mathbb{Z}/d) = \{\ell \subset L \otimes \mathbb{Z}/d \mid \ell \simeq \mathbb{Z}/d \ \& \ \ell.\ell \equiv 0 \pmod{ed}\}.$$

For N in $\mathcal{N}_d(L)$, we denote by $l(N)$ the unique $\ell \in C_L(\mathbb{Z}/d)$ satisfying $M_d(\ell) = N \cap L$. Alternatively, we have⁹ $l(N) = (dN + dL)/dL$. The map

$$(3.3) \quad l : \mathcal{N}_d(L) \rightarrow C_L(\mathbb{Z}/d), \quad N \mapsto l(N)$$

⁹Indeed, the image ℓ' of dN in L/dL is isomorphic to $N/(N \cap L) \simeq \mathbb{Z}/d$ and satisfies $dN.dN \equiv 0 \pmod{ed\mathbb{Z}}$, so we have $\ell' \in C_L(\mathbb{Z}/d)$. But we have $dN.M \equiv 0 \pmod{d\mathbb{Z}}$ and thus $M = M(\ell) \subset M(\ell')$, hence an equality and $\ell = \ell'$.

will be called *the line map*. Proposition 3.2 (i) asserts:

Corollary 3.6. *The line map is $e : 1$ and surjective.*

It would be easy to give a close formula for $|C_L(\mathbb{Z}/d)|$, hence for $|\mathcal{N}_d(L)|$.

3.7. Parity of a d -neighbor

We now discuss the *parity* of a neighbor, and the related notion of characteristic vectors. Recall that an integral lattice L is called *even* if we have $v.v \in 2\mathbb{Z}$ for all $v \in L$, and *odd* otherwise. If L is unimodular, the map $L \rightarrow \mathbb{Z}/2, v \mapsto v.v \pmod 2$ is \mathbb{Z} -linear, hence of the form $v \mapsto \xi \cdot v \pmod 2$ for some vector $\xi \in L$, uniquely determined modulo $2L$. Such vectors $\xi \in L$ are called the *characteristic vectors* of L ; they form a coset in $L/2L$ that we will denote by $\text{Char}(L)$. As an example, we have

$$(3.4) \quad \text{Char}(\mathbb{I}_n) = \{(\xi_1, \dots, \xi_n) \in \mathbb{Z}^n \mid \xi_i \equiv 1 \pmod 2, 1 \leq i \leq n\}.$$

We have $\text{Char}(L) = 2L$ if and only if L is even, and if L is odd and ξ is in $\text{Char}(L)$, then $M_2(L; \xi)$ coincides with the largest even sublattice of L . It is clear that if L and L' are d -neighbors with d odd, then L is even if and only if L' is even. The case d even is more interesting:

Proposition 3.8. *Assume L is an integral unimodular lattice, d is even and $x \in L$ is d -primitive with $x.x \equiv 0 \pmod{2d}$.*

(i) *If L is odd, the d -neighbor $\mathcal{N}_d(L; x; \epsilon)$ is even if, and only if, x is a characteristic vector of L and satisfies $\frac{x.x}{2d} \equiv (1 + \frac{d}{2})\epsilon \pmod 2$.*

(ii) *If L is even, then $\mathcal{N}_d(L; x; \epsilon)$ is even if and only if $\epsilon = 0$.*

Proof. By definition, $\mathcal{N}_d(L; x; \epsilon) = M + \mathbb{Z}\frac{\tilde{x}}{d}$ is even if and only if the lattice $M = M_d(L; x)$ and the integer $\tilde{x}.\tilde{x}/d^2$ are even, with \tilde{x} as in Remark 3.3 (i).

Assume first L is odd. As L/M is cyclic of even order, there is a unique lattice $M \subset H \subset L$ with $L/H = \mathbb{Z}/2$, namely $H = M_2(L; x)$. It follows that M is even if and only if $M_2(L; x)$ is the largest even sublattice of L , i.e. if x is a characteristic vector of L . A trivial computation, using $y.y \equiv x.y \equiv 1 \pmod 2$, then shows $\frac{\tilde{x}.\tilde{x}}{d^2} \equiv \frac{x.x}{2d} + (1 + \frac{d}{2})\epsilon \pmod 2$, and concludes the proof of (i).

If L is even, a simple computation shows $\tilde{x}.\tilde{x}/d^2 \equiv \epsilon \pmod 2$, hence (ii). \square

3.9. Orbits

As emphasized in the introduction, it is a difficult question in general to understand the isometry classes of the d -neighbors of a given L . A standard observation though is that the isometry group $O(L)$ of L naturally acts on $\mathcal{N}_d(L)$, and so two neighbors in a same orbit are isometric. This group also acts on $C_L(\mathbb{Z}/d)$ and we have the following obvious proposition.

Proposition 3.10. *The line map (3.3) is $O(L)$ -equivariant.*

Equivalently, we have $g(M_d(x)) = M_d(g(x))$ for all $x \in L$ and $g \in O(L)$. For d odd, the isometry class of a neighbor N depends thus only on the $O(L)$ -orbit of its line $l(N)$. For d even, the same holds up to the ϵ ambiguity; more precisely, by assertion (i) of Remark 3.3 we have for all $g \in O(L)$, all d -primitive $x \in L$ with $x.x \equiv 0 \pmod{2d}$, and all $\epsilon \in \mathbb{Z}/2$, the formula

$$(3.5) \quad g(\mathcal{N}_d(L; x; \epsilon)) = \mathcal{N}_d(L; g(x); \epsilon).$$

3.11. d -neighbors of I_n

We finally specify the previous considerations to the standard odd unimodular lattice $L = I_n$, and relates the general definitions in this case to the notations already introduced in § 1.2. Fix $d \geq 1$. The finite bilinear space $I_n \otimes \mathbb{Z}/d$ is just the standard \mathbb{Z}/d -valued inner product on the space $(\mathbb{Z}/d)^n$, and we set $C_n(\mathbb{Z}/d) = C_{I_n}(\mathbb{Z}/d)$. The element $x \in \mathbb{Z}^n$ is d -primitive if, and only if, we have $\gcd(x_1, \dots, x_n, d) = 1$. The line $l(x) \subset I_n \otimes \mathbb{Z}/d$ it generates is in $C_n(\mathbb{Z}/d)$ if and only if Formula (1.3) holds, *i.e.* x is d -isotropic. We ease the notations by denoting by

$$M_d(x), M_d(\ell), N_d(x), N_d(\ell), N_d(x; \epsilon)$$

the lattices $M_d(I_n; x), M_d(I_n; \ell), N_d(I_n; x), N_d(I_n; \ell), N_d(I_n; x; \epsilon)$. For d even, we also denoted $N_d(x)^\pm$ the lattices $N_d(x; \epsilon)$ in the introduction. A characteristic vector of I_n is 1^n , so Proposition 3.8 reads:

Corollary 3.12. *The lattice $N_d(x; \epsilon)$ is even if and only if x_i is odd for each i and we have $\sum_i x_i^2 \equiv d(2+d)\epsilon \pmod{4d}$ (which forces $n \equiv 0 \pmod{8}$).*

The isometry group $O(I_n)$ is unusually large: this is the group $\{\pm 1\}^n \rtimes S_n$ acting on \mathbb{Z}^n by all possible permutations and sign changes of coordinates. The $O(I_n)$ -orbits on $C_n(\mathbb{Z}/d)$, which are of great interest by Proposition 3.10, will thus be in manageable quantity for small d and n . An element $x \in \mathbb{Z}^n$ will be called d -ordered if it satisfies (1.9), *i.e.* $0 \leq x_1 \leq x_2 \leq \dots \leq x_d \leq d/2$.

Fact 3.13. *For any $O(I_n)$ -orbit $\Omega \subset (\mathbb{Z}/d\mathbb{Z})^n$, there is a unique d -ordered element $x \in \mathbb{Z}^n$ with $x \pmod{d} \in \Omega$.*

This obvious fact explains why we always choose our d -isotropic elements x to be d -ordered in our lists. Note however that two distinct d -isotropic and d -ordered elements of \mathbb{Z}^n may generate the same line in $(\mathbb{Z}/d)^n$, hence give birth to the same d -neighbors (see [AC] for a discussion about this).

Remark 3.14. Assume $x \in \mathbb{Z}^n$ is d -isotropic, d is even and $g \in \text{O}(\mathbb{I}_n)$. We have $g(\text{N}_d(x; \epsilon)) = \text{N}_d(g(x); \epsilon)$ by (3.5). Beware however that, by assertion (ii) of Remark 3.3, if we choose some i and define $x' \in \mathbb{Z}^n$ by $x'_j = x_j$ for $j \neq i$, and $x'_i = x_i \pm d$, then x' generates obviously the same line as x in $(\mathbb{Z}/d)^n$, but we have $\text{N}_d(x'; \epsilon) = \text{N}_d(x; \epsilon + 1)$.

Corollary 3.15. Assume $x \in \mathbb{Z}^n$ is d -isotropic, with d even and $x_i \equiv d/2 \pmod{d}$ for some $i \in \{1, \dots, n\}$. Then we have $\text{N}_d(x; 0) \simeq \text{N}_d(x; 1)$.

Proof. Set $x' = x - 2x_i \varepsilon_i$. Using Formula (3.5) for $g \in \text{O}(\mathbb{I}_n)$ defined by $g(\varepsilon_i) = -\varepsilon_i$ and $g(\varepsilon_j) = \varepsilon_j$ for $j \neq i$, we deduce $\text{N}_d(x; 0) \simeq \text{N}_d(x'; 0)$. By Remark 3.14 and $2x_i \equiv d \pmod{2d}$, we also have $\text{N}_d(x'; 0) = \text{N}_d(x; 1)$. \square

4. Some invariants of lattices

4.1. Configuration of vectors of given norm

Define an *Euclidean set* as a set X equipped with an injection $X \xrightarrow{j} V$ into some Euclidean space V . We then denote by $\mathbb{R}X$ the Euclidean subspace of V generated by $j(X)$. Euclidean sets form a category if we define a morphism $(X, j) \rightarrow (X', j')$ as a map $f : X \rightarrow X'$ induced by a linear isometric embedding $\tilde{f} : \mathbb{R}X \rightarrow \mathbb{R}X'$, i.e. verifying¹⁰ $\tilde{f} \circ j = j' \circ f$. Note that it make sense to talk about the *scalar product* $x.y$ of two elements x, y of an Euclidean set (X, j) (namely $x.y = j(x).j(y)$), about the *rank* of X (i.e. $\dim \text{Vect}_{\mathbb{R}}(j(X))$), about the *lattice generated by X* (i.e. $\mathbb{Z}j(X)$) ...

Let L be an integral lattice in V . The *configuration of vectors of norm i* of L , already introduced in (1.7), is the Euclidean set

$$(4.1) \quad \mathbf{R}_i(L) = \{v \in L \mid v.v = i\}$$

with understood embedding $\mathbf{R}_i(L) \subset V$. Its isomorphism class is an invariant of the isometry class of L . Recall the notation $r_i(L) := |\mathbf{R}_i(L)|$. A natural variant of $\mathbf{R}_i(L)$ is the Euclidean set $\mathbf{R}_{\leq i}(L)$ defined by replacing $v.v = i$ with $v.v \leq i$ in (4.1). It is obvious that for two integral Euclidean lattices L and L' , if we choose i big enough so that $\mathbf{R}_{\leq i}(L)$ and $\mathbf{R}_{\leq i}(L')$ generate L and L' , then L is isometric to L' if and only if $\mathbf{R}_{\leq i}(L)$ isomorphic to $\mathbf{R}_{\leq i}(L')$.

For $i = 1$, the isomorphism class of $\mathbf{R}_1(L)$ is obviously nothing more than the even integer $r_1(L)$, since we have $w.v = 0$ for $v \neq \pm w$ and $w, v \in \mathbf{R}_1(L)$.

¹⁰Note that given f , such an \tilde{f} is unique if it exists.

For $i = 2$ this is the same as the **ADE** root system of L , an important invariant that we review in §4.3 below. We are not aware of any general study or classification for the possible isomorphism classes of $R_i(L)$ for $i \geq 3$. This question for $i = 3$ is of great importance here, as experiments show that the unimodular lattices of dimension n in our range are almost always generated over \mathbb{Z} by their $R_{\leq 3}$: see Table 4.1 for the *a posteriori* statistics, which explains much of Corollary D.

$n \setminus d$	1	2	3	4	≥ 5
26	1857	38	2	4	0
27	14425	64	1	3	0

Table 4.1: The number of isometry classes of rank n unimodular lattices L such that $R_{\leq 3}(L)$ generates a sublattice of index d in L (including $d = \infty$).

Remark 4.2. (Computation of the set $R_{\leq i}(L)$) If L is an integral Euclidean lattice (given by a Gram matrix G), we use the Fincke-Pohst algorithm [FP85] to compute the sets $R_{\leq i}(L)$ (function `qfminim(G, i)` in PARI/GP). As an indication, the average CPU time in ms on our machine to compute $R_{\leq i}(L)$, when L is our list of 17059 odd unimodular lattices of rank 27, is about 2.3 ms for $i = 1$, 2.8 ms for $i = 2$ and 20 ms for $i = 3$. By comparison, computing a Gram matrix for such a lattice in neighbor form takes about 0.4 ms.

4.3. ADE root systems and root lattices

By a *root* in an Euclidean space V , we mean an element α in V with $\alpha \cdot \alpha = 2$. The orthogonal symmetry about a root α is given by $s_\alpha(v) = v - (v \cdot \alpha)\alpha$. An **ADE root system** is a finite Euclidean set R consisting of roots such that for all α, β in R we have $\alpha \cdot \beta \in \mathbb{Z}$ and $s_\alpha(R) = R$. In other words, R is a root system in $\mathbb{R}R$ in the sense of [BOU81b, Ch. VI] satisfying $\alpha^\vee = \alpha$ for all $\alpha \in R$. Any such R generates an even Euclidean lattice, called the *associated root lattice*, and denoted $Q(R)$ following Bourbaki. We also set $\text{res } R = \text{res } Q(R)$. A *morphism* $R \rightarrow R'$ of root systems, also called an *embedding*, is a morphism of Euclidean sets, or equivalently, a linear isometric embedding $Q(R) \rightarrow Q(R')$. We talk about *sub root systems* for embeddings defined by an inclusion.

For any integral Euclidean lattice L , then $R := R_2(L)$ trivially is an **ADE** root system, called the *root system of L* . We say that L is a *root lattice* if we have $Q(R) = L$. The (non obvious but true) general equality $R = R_2(Q(R))$ shows that the functors $R \mapsto Q(R)$ and $L \mapsto R_2(L)$ define inverse equivalences between the category of **ADE** root systems and that

of root lattices (for linear isometries). We trivially have $R_2(L_1 \perp L_2) = R_2(L_1) \amalg R_2(L_2)$, where \amalg denotes *orthogonal disjoint union* of Euclidean sets, and $Q(R_1 \amalg R_2) = Q(R_1) \perp Q(R_2)$. In particular, *irreducible* root systems correspond to *indecomposable* root lattices.

Recall the standard lattices A_n ($n \geq 0$), D_n ($n \geq 0$) and E_n ($6 \leq n \leq 8$) from Sect. 2 (v). All but D_1 are root lattices, and all are indecomposable for $n \geq 1$ but D_2 . We denote respectively by \mathbf{A}_n , \mathbf{D}_n and \mathbf{E}_n their root systems. By the **ADE** classification, any irreducible root system is isomorphic such a root system, and the unique coincidences between them are $\mathbf{A}_0 = \mathbf{D}_0 = \mathbf{D}_1 = \emptyset$, $\mathbf{D}_2 \simeq \mathbf{A}_1 \amalg \mathbf{A}_1$ and $\mathbf{D}_3 \simeq \mathbf{A}_3$.

Remark 4.4. (An algorithm for computing root systems) If L is an integral lattice in the Euclidean space V , the structure of its root system $R_2(L)$ may be efficiently computed as follows. Determine first the set $R = R_2(L)$ as in Remark 4.2, choose a linear form φ on V with $0 \notin \varphi(R)$, and set $R^+ = \{\alpha \in R, \varphi(\alpha) > 0\}$ (*positive roots*). (Actually PARI's `qfminim(G,2)` function directly returns such an R^+ rather than R). Compute then the *Weyl vector* $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ and the *basis* $B = \{\alpha \in R^+ \mid \rho \cdot \alpha = 1\}$ of R associated to R^+ . Compute the scalar products $b \cdot b'$ for $b, b' \in B$ and view B as the vertices of the undirected graph with an edge between b and b' if and only if $b \neq b'$ and $b \cdot b' \neq 0$ (*Dynkin diagram of R*). The connected (*irreducible*) components of this union of trees are easily computed recursively, and identified as of type \mathbf{A}_n , \mathbf{D}_n or \mathbf{E}_n by simply looking at their unique vertice x with valence > 2 (if exists), and the sum of the valences of the 3 neighbors of x . See [CHE20b] for our concrete implementation. As an indication, the average CPU time for computing the isomorphism class of the root system of a unimodular lattice of rank 27 in our list is 5.9 ms. As $|B| \leq \dim V$ is very small in practice, naive graphs algorithms are perfectly suitable for the last part above: 99.8 % of the CPU time is used for the computation of the sets R^+ and B .

Let L be an integral Euclidean lattice with ADE root system $R := R_2(L)$. For each $\alpha \in R$ the orthogonal reflexion $s_\alpha(x) = x - (x \cdot \alpha)\alpha$ lies in $O(L)$. The *Weyl group* of L is the subgroup $W(L)$ of $O(L)$ generated by those s_α with $\alpha \in R$. This is a normal subgroup of $O(L)$ isomorphic to $W(R) := W(Q(R))$. Moreover, if we choose a positive root system R^+ of R , and denote by ρ the associated Weyl vector as in Remark 4.4, we have another subgroup

$$(4.2) \quad O(L; \rho) := \{\gamma \in O(L) \mid \gamma(\rho) = \rho\}.$$

As is well-known, $W(L)$ acts simply transitively on the set of Weyl vectors of L (and on the set of positive root systems). We have thus

$$(4.3) \quad O(L) = W(L) \rtimes O(L; \rho).$$

Also, for $w \in W(L)$ we obviously have $O(L; w(\rho)) = wO(L; \rho)w^{-1}$. It follows that the $W(L)$ -conjugacy class of the subgroup $O(L; \rho)$ in $O(L)$ is canonical, and does not depend on the choice of ρ . Moreover, each $O(L; \rho)$ is naturally isomorphic to $O(L)/W(R)$.

Definition 4.5. *Let L be integral Euclidean lattice L with root system R . Define the reduced isometry group of L as the group $O(L)^{\text{red}} = O(L)/W(R)$, and the reduced mass of L by the formula $\text{rmass}(L) := \frac{1}{|O(L)^{\text{red}}|}$.*

A first important application of this notion is the following remark. As the structure of $W(R)$ is well-known, it follows that in order to determine generators of $O(L)$, or simply its order, it is enough to do so for $O(L; \rho)$.

Remark 4.6. (*Computation of the reduced isometry group*) Let L be an integral Euclidean lattice. Choose a Weyl vector ρ of L as in Remark 4.4. As was already observed and used in [CHE20a], it turns out that the Plesken-Souvignier algorithm [PS97] directly allows to compute generators and the order of $O(L; \rho)$. Indeed, is enough to apply it to a pair consisting of a Gram matrix of L and of a Gram matrix of the bilinear form $(x, y) \mapsto 4(\rho \cdot x)(\rho \cdot y)$, in a same basis. This actually returns order and generators for $\pm O(L; \rho)$, but the similar information for $O(L; \rho)$ easily follows.¹¹ This computation of $O(L; \rho)$ is usually much faster than that of $O(L)$ (the bigger $R_2(L)$ is, the faster). As an indication, the average CPU time for the computation of $O(L)$ for our 14493 rank 27 unimodular lattices with no norm 1 vector using this method (and PARI's `qfauto`) is 8.9 s.

We end this section with a few more definitions, for a later use.

- An embedding $f : R' \hookrightarrow R$ is called *saturated* if the subgroup $f(Q(R'))$ is saturated in $Q(R)$ (see Sect. 2 (vi)). The saturated sub root systems of an **ADE** root system $R \subset V$ are those obtained by intersecting R with a subspace of V ; they are sometimes called *parabolic* and their Dynkin diagram are obtained from that of R by removing a finite set of vertices.

- We denote by **RS** the set of isomorphism classes of **ADE** root systems. We often write $n_1 R_1 \ n_2 R_2 \ \dots \ n_k R_k$ for the orthogonal disjoint union of n_i copies of R_i , for $i = 1, \dots, k$.

4.7. Vectors of norm ≤ 3

We start with some information on the number of vectors of norm 3 of the lattices we are interested in. The statement of the following proposition uses the notation $\text{Exc}(L)$ for $L \in \mathcal{L}_n$, that will only be introduced in §9.1 when discussing exceptional lattices.

¹¹Souvignier's code is available in PARI/GP as `qfauto(G)` with G a Gram matrix of L . For all the computations of $O(L; \rho)$ in this paper, we use the the LLL-algorithm (PARI's `qf111gram`) to find suitable a G . See [AC] for more clever choices of bases.

Proposition 4.8. *For all $L \in \mathcal{L}_n$ with $24 \leq n \leq 28$ and $r_1(L) = 0$ we have $r_3(L) = \frac{4}{3}n(n^2 - 69n + 1208) + 2(n - 24)r_2(L) - 2^{36-n} |\text{Exc}(L)|$.*

Proof. Assume first we have $L \in \mathcal{L}_{28}$ (and possibly norm 1 vectors) and let M denote the even part of L . A simple computation of the coefficient in q of the theta series of M^\sharp , following the arguments in [BV01] §4 (or [CS99, Ch. 4 §4]), shows the relation $256 r_1(M^\sharp) = -r_3(L) + 8r_2(L) + 468 r_1(L) + 2240$. Since $28 \equiv 4 \pmod{8}$ we also have $r_1(M^\sharp) = r_1(L) + |\text{Exc}(L)|$ by Formula (9.2). Choose now $L_0 \in \mathcal{L}_n$ with $n \leq 28$ and $r_1(L) = 0$. We apply above relation to $L = L_0 \perp I_m$ with $m = 28 - n$. We conclude the equality of the statement (for L_0) by the equalities $r_1(L) = 2m$, $r_2(L) = r_2(L_0) + 2^2 \binom{m}{2}$, $r_3(L) = r_3(L_0) + 2m r_2(L_0) + 2^3 \binom{m}{3}$, and $|\text{Exc}(L)| = 2^m |\text{Exc}(L_0)|$ for $n \geq 24$. \square

Example 4.9. We deduce $r_3(L) = 3120 + 4r_2(L) - 1024 |\text{Exc}(L)|$ in the case $n = 26$, and $r_3(L) = 2664 + 6r_2(L) - 512 |\text{Exc}(L)|$ for $n = 27$. In both cases, we have $|\text{Exc}(L)| = 2$ if L is exceptional, and $|\text{Exc}(L)| = 0$ otherwise, by Proposition 9.5.

Our aim now is to discuss a few invariants of $R_{\leq 3}$ that we have used during our simultaneous proofs of Theorem A and Corollary D. We sincerely apologize that what follows is mostly empirical. We mostly relates facts (namely Propositions 4.11 and 4.13) that we observed during our search and only proved by case by case computations. In each case, it is an open problem to find conceptual explanations for our computations.

A *component of size s* of a root system R is a union of s distinct irreducible components of R . We denote by $\underline{R} \in \mathbf{RS}$ the isomorphism class of R .

Definition 4.10. *For L an integral Euclidean lattice and $s \geq 0$ an integer, we denote by $\delta_s(L) \in \mathbb{Z}[\mathbf{RS} \times \mathbb{N}]$ the sum of (\underline{C}, m) of $\mathbf{RS} \times \mathbb{Z}_{\geq 0}$, where C runs among the components of size s of $R_2(L)$ and $m = |R_3(C^\perp \cap L)|$.*

The invariant $\delta_0(L)$ is just $r_3(L)$, a weak information by Example 4.9. The invariants $\delta_k(L)$ have already been used for instance by Megarbané in his study of the rank 26 even lattices of determinant 3 [MEG18] (see Example 9.12). As an example, let us consider again the 7 isometry classes of unimodular lattices of rank 26 with root system $10\mathbf{A}_1$, listed in Table 1.3. The invariants δ_1 and δ_2 of these lattices (ordered as in that table) are given by Table 4.2:

δ_1	δ_2
10 ($\mathbf{A}_1, 2578$)	$4(2\mathbf{A}_1, 1968) + 6(2\mathbf{A}_1, 2000) + 20(2\mathbf{A}, 2032) + 15(2\mathbf{A}_1, 2064)$
10 ($\mathbf{A}_1, 2578$)	$13(2\mathbf{A}_1, 2000) + 16(2\mathbf{A}_1, 2032) + 16(2\mathbf{A}, 2064)$
10 ($\mathbf{A}_1, 2578$)	$11(2\mathbf{A}_1, 2000) + 24(2\mathbf{A}_1, 2032) + 10(2\mathbf{A}, 2064)$
10 ($\mathbf{A}_1, 2578$)	$20(2\mathbf{A}_1, 2000) + 25(2\mathbf{A}_1, 2064)$
10 ($\mathbf{A}_1, 2578$)	$4(2\mathbf{A}_1, 1936) + 16(2\mathbf{A}_1, 2000) + 25(2\mathbf{A}, 2064)$
10 ($\mathbf{A}_1, 2578$)	$20(2\mathbf{A}_1, 1936) + 25(2\mathbf{A}_1, 2064)$
$(\mathbf{A}_1, 18) + 9(\mathbf{A}_1, 1042)$	$9(2\mathbf{A}_1, 16) + 36(2\mathbf{A}_1, 912)$

Table 4.2: The invariants δ_1 and δ_2 of the 7 lattices of Table 1.3.

In particular, all those lattices are distinguished by δ_2 (but not by δ_1). It follows from our computations that this is a general fact in rank 26.

Proposition 4.11. *Two unimodular lattices of rank 26 are isometric if, and only if, they have the same root system and the same invariants δ_1 and δ_2 .*

The invariants δ_s are however not strong enough in dimension 27. We now discuss a second invariant of $\mathbf{R}_{\leq 3}$.

Definition 4.12. *For an Euclidean integral lattice L , we define $G(L)$ as the undirected graph with vertices the nonzero pairs $\{\pm x\}$ with $x \in \mathbf{R}_{\leq 3}(L)$, and with $|x.y|$ arrows between $\{\pm x\}$ and $\{\pm y\}$.*

Of course, we have $|x.y| \leq 3$ for all $x, y \in \mathbf{R}_{\leq 3}(L)$. The isomorphism class of this graph $G(L)$ only depends on that of $\mathbf{R}_{\leq 3}(L)$. In Table 4.3 below, we give the *a posteriori* information for the number of vertices of $G(L)$ for L in \mathcal{L}_n with $r_1(L) = 0$ for $n = 26, 27$ (agreeing Example 4.9).

n	min	max	average
26	556	2850	$\simeq 1776$
27	820	3277	$\simeq 1573$

Table 4.3: The minimum, maximum, and average, number of vertices of the graph $G(L)$ for $L \in \mathcal{L}'_n$.

Any of invariant of graphs can be applied to study $G(L)$. From a computational point of view, an especially simple one that we can consider is the rank $h_p(G)$, of the adjacency matrix mod p of a graph G . Here p is any given prime. So for $s \geq 0$ and a prime p , we can define a variant

$$\delta_{s,p}(L) \in \mathbb{Z}[\mathbf{RS} \times \mathbb{N} \times \mathbb{N}]$$

of δ_s by replacing each (\underline{C}, m) in Definition 4.10 by (\underline{C}, m, r) with $r = h_p(\mathrm{G}(C^\perp \cap L))$. It turns out that these $\delta_{s,p}$ do suffice to distinguish all unimodular lattices in rank 27. More precisely, our computations show:

Proposition 4.13. *Two unimodular lattices of rank 27 are isometric if, and only if, they have the same root system and the same invariants $\delta_{s,p}$ with $s \leq 3$ and $p = 5$, unless their root system is in the following list :*

$$3\mathbf{A}_1, 6\mathbf{A}_1, 7\mathbf{A}_1, 3\mathbf{A}_1\mathbf{A}_2, 5\mathbf{A}_1\mathbf{A}_2, 7\mathbf{A}_1\mathbf{A}_2, 4\mathbf{A}_12\mathbf{A}_2, 6\mathbf{A}_12\mathbf{A}_2, 8\mathbf{A}_12\mathbf{A}_2, 5\mathbf{A}_13\mathbf{A}_2.$$

For such root systems, we need the invariants $\delta_{s,p}$ for $s \leq 7$ and $p = 5, 7$.

We will not insist much on these invariants $\delta_{s,p}$ here, but rather refer to the companion work [AC] in which we will define another invariant of $\mathrm{G}(L)$ (inspired by the work of Bacher and Venkov [BV01]) which will turn out to be fine enough to distinguish all unimodular lattices of rank ≤ 28 .

5. The visible root system of a d -neighbor of I_n

5.1. The root system

Let $x \in \mathbb{Z}^n$ be d -isotropic, and let $N = \mathrm{N}_d(x')$ be an associated d -neighbor of I_n . Recall $\mathrm{I}_n \cap N = \mathrm{M}_d(x)$. In this section, we study the visible root system of N in the sense of Definition 1.9, namely the subroot system $\mathrm{R}_2(\mathrm{M}_d(x))$ of $\mathrm{R}_2(N)$. The term *visible* reflects the fact that this root system is immediately seen on the shape of x . The recipe is as follows.

To any $x \in \mathbb{Z}^n$ and $d \geq 1$ we have an equivalence relation \sim on $\{1, \dots, n\}$ defined by

$$(5.1) \quad i \sim j \Leftrightarrow x_i \equiv \pm x_j \pmod{d}.$$

There are two *distinguished* subsets D and D' , defined respectively as the subset of $i \in \{1, \dots, n\}$ with $x_i \equiv d/2 \pmod{d}$ or $x_i \equiv 0 \pmod{d}$ (so $D = \emptyset$ for d odd). We set $m(x) = |D|$, $m'(x) = |D'|$ and we denote by $\mathbf{a}(x)$ the integer partition $\mathbf{a}_1(x) \geq \mathbf{a}_2(x) \geq \dots$ of $n - m(x) - m'(x)$ defined by the sizes of the equivalence classes of \sim different from D and D' . All of \sim , $\mathbf{a}(x)$, $m(x)$ and $m'(x)$ only depend on the line $\ell = l(x) \subset (\mathbb{Z}/d)^n$ generated by x , and we also use the notations \sim_ℓ , $\mathbf{a}(\ell)$, $\mathbf{a}_i(\ell)$, $m(\ell)$ and $m'(\ell)$ for them.

Proposition 5.2. *For all x in \mathbb{Z}^n and $d \geq 1$ we have an isomorphism*

$$\mathrm{R}_2(\mathrm{M}_d(x)) \simeq \mathbf{D}_{m(x)} \mathbf{D}_{m'(x)} \mathbf{A}_{\mathbf{a}_1(x)-1} \mathbf{A}_{\mathbf{a}_2(x)-1} \dots$$

Proof. Recall $\mathrm{R}_2(\mathrm{I}_n) = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$ from (1.10). By definition of $\mathrm{M}_d(x)$, the root $\varepsilon_i - \varepsilon_j$ (resp. $\varepsilon_i + \varepsilon_j$) of I_n belongs to $\mathrm{M}_d(x)$ if and only if $x_i \equiv x_j \pmod{d}$ (resp. $x_i \equiv -x_j \pmod{d}$). Up to applying an element of $\mathrm{O}(\mathrm{I}_n)$ if necessary, we may assume x is d -ordered. In this case,

the equivalence classes of the relation \sim clearly are intervals. Also, the visible roots are the $\varepsilon_i - \varepsilon_j$ whenever $x_i = x_j$, and for d even the $\pm(\varepsilon_i + \varepsilon_j)$ whenever $x_i \equiv x_j \equiv 0 \pmod{d/2}$ (with $i \neq j$ in both cases): we recognize the root system of the statement. \square

5.3. Saturation properties

As noticed in §1.8, we will usually assume $x_i \not\equiv 0 \pmod{d}$ for all i , or equivalently $m'(x) = 0$, in order to have $r_1(M_d(x)) = 0$. Our aim until the end of this §5 is to discuss the relations between the visible root system R^\vee and the actual root system of a d -neighbor N of I_n , as well as the constraints on the embedding $R^\vee \hookrightarrow N$.

Lemma 5.4. *Let $x \in \mathbb{Z}^n$ and $d \geq 1$ with $x_i \not\equiv 0 \pmod{d}$ for each i . Set $M = M_d(x)$ and $R^\vee = R_2(M)$. Then $Q(R^\vee)$ is saturated in M .*

Proof. Applying an element of $O(I_n)$ if necessary, we may assume x is d -ordered. As observed in Proposition 5.2, the classes I of the equivalence relation \sim on $\{1, \dots, n\}$ associated to x and d are intervals in this case, and simply determine R^\vee : each $I \neq D$ gives rise to an $\mathbf{A}_{|I|-1}$ component (note $x_i \not\equiv 0 \pmod{d}$ for $i \in I$), and $I = D$ to a $\mathbf{D}_{|I|}$ component. For each class I , define an abelian group Δ_I as follows: set $\Delta_I = \mathbb{Z}$, unless $I = D$, $|D| \geq 2$ and $\Delta_I = \mathbb{Z}/2$. Define also $\varphi_I : \mathbb{Z}^n \rightarrow \Delta_I$ by $\varphi_I(v) \equiv \sum_{i \in I} v_i$. We have a surjective linear map $\varphi = \prod_I \varphi_I : \mathbb{Z}^n \rightarrow \prod_I \Delta_I$, where I runs among the equivalence classes of \sim . We clearly have

$$\varphi(M) = \{(w_I) \in \prod_I \Delta_I \mid \sum_I x_I w_I \equiv 0 \pmod{d}\},$$

where $x_I \in \mathbb{Z}/d$ denotes the common class of x_i for $i \in I$. (Note that if $D \neq \emptyset$ the product $x_D w_D$ is well-defined in \mathbb{Z}/d as we have d even and $x_D \equiv d/2$.) The description of R^\vee recalled above shows $\ker \varphi = Q(R^\vee)$, hence $\varphi(M) \simeq M/Q(R^\vee)$. The group $\prod_I \Delta_I$ is torsion-free, hence so is its subgroup $\varphi(M)$, unless d is even and $|D| \geq 2$. In this case, the unique torsion element (w_I) is defined by $w_I \equiv 0$ for $I \neq D$, and $w_D \equiv 1$. But this element does not belong to $\varphi(M)$ as $x_D \equiv d/2 \not\equiv 0 \pmod{d}$. \square

Definition 5.5. *Let L be an integral Euclidean lattice, $R = R_2(L)$ and R' a sub root system of R . We say that R' is a d -kernel of L if there is a surjective linear map $\varphi : L \rightarrow \mathbb{Z}/d$ with $R_2(\ker \varphi) = R'$. In the case $L = Q(R)$ we also say that R' is a d -kernel of R .*

In other words, the d -kernels of L are the $R' = R_2(M)$ for M a sublattice of L with $L/M \simeq \mathbb{Z}/d$. In such a situation, and if we set $R = R_2(L)$, we also have $\text{Im}(Q(R) \rightarrow L/M) \simeq \mathbb{Z}/d'$ for some $d' \mid d$, hence R' is a d' -kernel of R .

Corollary 5.6. *Assume $x \in \mathbb{Z}^n$ is d -isotropic with $x_i \not\equiv 0 \pmod{d}$ for each i . Set $M = M_d(x)$, $R^\vee = R_2(M)$ and let N be a d -neighbor of I_n with line $l(x)$. Then R^\vee is a d -kernel of N . If furthermore d is prime to $a_i(x)$ for all i , and odd in the case $m(x) > 1$, then $Q(R^\vee)$ is saturated in N .*

Proof. The first assertion is clear. As $Q(R^\vee)$ is saturated in M by Lemma 5.4, the saturation S of $Q(R^\vee)$ in N satisfies $S \cap M = Q(R^\vee)$. The isotropic subgroup $S/Q(R^\vee)$ of $\text{res } R^\vee$ embeds thus in \mathbb{Z}/d . So its order s satisfies ¹² $s \mid d$ and $s \mid |\text{res } R^\vee|$. We conclude as by Proposition 5.2 and Table 2.1 we have $|\text{res } R^\vee| = f \prod_i a_i(x)$ with $f = 4$ (case $m(x) > 1$) or $f = 1$ (otherwise). \square

Remark 5.7. *As a consequence, if d is a sufficiently big prime then $Q(R^\vee)$ is saturated in N , which is a strong constraint. Theorem E shows that generically this is actually the only constraint.*

Of course, there are many examples of unimodular lattices N with root system R such that if S denotes the saturation of $Q(R)$ in L then $S/Q(R)$ is not a cyclic group. We will thus often have $R^\vee \subsetneq R_2(N)$ in practice. By the remark following Definition 5.5, and still in the notations of Corollary 5.6, the visible root system R^\vee is also a d' -kernel of $R_2(N)$ for some $d' \mid d$. We postpone to §5.15 the description of all the d -kernels of a given root system. We content ourselves here with the following simple observation (see Remark 5.19 for a more precise statement).

Proposition 5.8. *Let R' be a saturated sub root system of the **ADE** root system R with $\text{rk } R' < \text{rk } R$. Then R' is a d -kernel of R for all d big enough.*

Proof. We have an abelian group decomposition $Q(R) = Q(R') \oplus P$ with $P \simeq \mathbb{Z}^r$ and $r = \text{rk } R - \text{rk } R' > 0$. As R is finite, we may find a surjective linear map $\varphi : Q(R) \rightarrow \mathbb{Z}$ with $\varphi(R') = 0$ and $\varphi(r) \neq 0$ for $r \notin R'$. We conclude by using $\varphi \otimes \mathbb{Z}/d$ with $d > \varphi(r)$ for all $r \in R$. \square

5.9. An example : the *safe* case

For the purpose of unimodular hunting, we are led to the following definition.

Definition 5.10. *Let R and S be **ADE** root systems. We say that (R, S) is safe if for any integral lattice L with $R_{\leq 2}(L) = R$, there is an isometric embedding $Q(S) \rightarrow L$ whose image is saturated in L .*

¹²We even have $s^2 \mid |\text{res } R^\vee|$. Indeed, for any isotropic subgroup I of a finite bilinear abelian group A we have $|A| = |A/I^\perp| \cdot |I^\perp/I| \cdot |I|$ and $A/I^\perp \simeq \text{Hom}(I, \mathbb{Q}/\mathbb{Z}) \simeq I$.

By definitions, if (R, S) is safe then S is isometric to a saturated sub root system of R . Moreover, (R, S') is also safe for any saturated sub root system S' of S . Note also that if L is an integral lattice with root system R , and if V is the Euclidean space generated by R , then $L \cap V$ is saturated in L , so that we may actually assume $L \subset V$ in the definition above. In this case, $L \subset \mathbb{Q}(R)^\sharp$ is uniquely determined by the isotropic subspace $I = L/\mathbb{Q}(R)$ of $\text{res } R$, by Sect. 2 (iii). In terms of the Venkov map recalled *loc. cit.* (iv), the assumption $R_{\leq 2}(L) \simeq R$ is equivalent to $\text{qm}(x) \neq \frac{1}{2}, 1$ for all $x \in I$.

Definition 5.11. *We call an ADE root system S detecting if for all integral lattices $\mathbb{Q}(S) \subsetneq L \subset \mathbb{Q}(S)^\sharp$ we have $S \subsetneq R_{\leq 2}(L)$. Equivalently, S is detecting if for all x in $\text{res } S$ with $\text{qm}(x) \in \frac{1}{2}\mathbb{Z}$ we have $\text{qm}(x) \leq 1$.*

The interest of this notion for us is the following proposition.

Proposition 5.12. *Assume R is the orthogonal disjoint union of its sub root systems S and T . If S is detecting then (R, S) is safe.*

Proof. Write $R = S \perp T$ and let L be an integral lattice with $R_{\leq 2}(L) = R$. The saturation L' of $\mathbb{Q}(S)$ in L is orthogonal to T , so we must have $R_2(L') = S$. We also have $R_1(L') \subset R_1(L) = \emptyset$. As S is detecting, we deduce $L' = \mathbb{Q}(S)$. \square

Here are a few examples of detecting root systems. For instance, it follows from Table 2.1 that \mathbf{A}_m is detecting if, and only if, there is no integer $1 \leq i \leq m$ such that $m+1 \mid i^2$ and $i(m+1-i) > 2(m+1)$. This holds in particular if $m+1$ is square free or for all $m \leq 10$. Similarly, \mathbf{D}_m is detecting unless we have $m \equiv 4 \pmod{8}$ and $m \neq 4$, and \mathbf{E}_m is detecting for $6 \leq m \leq 8$. Here is another example.

Example 5.13. *$m\mathbf{A}_1 n\mathbf{A}_2$ is detecting if, and only if, we have $0 \leq m, n \leq 5$ and $(m, n) \neq (2, 3), (4, 3)$. Indeed, the only nonzero value of qm on $\text{res } \mathbf{A}_1$ (resp. $\text{res } \mathbf{A}_2$) is $\frac{1}{4}$ (resp. $\frac{1}{3}$).*

For a given R , we will often have to find a sub system $S \subset R$ as large as possible such that (R, S) is safe, which usually reduces to a problem in coding theory. For instance for $R \simeq n\mathbf{A}_1$, it amounts to ask for the maximal integer m such that for any even linear binary code I in $(\mathbb{Z}/2)^n$ with minimal distance ≥ 6 , there is a partition $\{1, \dots, n\} = S \amalg T$ with $|S| = m$ and such that the natural projection $(\mathbb{Z}/2)^n \rightarrow (\mathbb{Z}/2)^T$ is injective on I . We leave as an exercise to the reader to check the following assertion.

Example 5.14. *Assume either $n \leq 8$ and $m \leq n-1$, or $n \leq 10$ and $m \leq n-2$, then $(n\mathbf{A}_1, m\mathbf{A}_1)$ is safe.*

5.15. Classification of the d -kernels of **ADE** root systems

We first reduce to the irreducible case.

Lemma 5.16. *Let R be an **ADE** root system, $R = \coprod_i R_i$ its irreducible decomposition, $d \geq 1$ an integer, S a sub root system of R , and $S_i := S \cap R_i$. Then S is a d -kernel of R if, and only if, there are divisors d_i of d with $\text{lcm}\{d_i\}_{i \in I} = d$ and such that S_i is a d_i -kernel of R_i for each i .*

Proof. Just use $R = \coprod_i R_i$ and $Q(R) = \bigoplus_i Q(R_i)$. □

Lemma 5.2 can be viewed as a classification of the d -kernels of I_n . It would be easy to classify the d -kernels of \mathbf{A}_n and \mathbf{D}_n using a similar same method: see Remark 5.21 for the result. We follow a different approach, which works in all cases including type **E**, inspired by classical works of Borel-de-Siebenthal and Dynkin.

Fix an irreducible **ADE** root system R of rank $n \geq 1$. Choose a positive root system $R^+ \subset R$, with associated basis $\{\alpha_i\}_{1 \leq i \leq n}$ and dual basis $\{\varpi_i\}_{1 \leq i \leq n}$ in the weight lattice $Q(R)^\sharp$ (*fundamental weights*). In all examples below, we choose the same numbering of simple roots as [BOU81b] to fix ideas. For each $1 \leq i \leq n$, set $h_i = \max\{\varpi_i \cdot \alpha \mid \alpha \in R\}$. Recall that $\tilde{\alpha} = \sum_{i=1}^n h_i \alpha_i$ is in R^+ (*highest root*). We also set $\alpha_0 = -\tilde{\alpha}$, $h_0 = 1$, $\varpi_0 = 0$ and $I = \{0, 1, \dots, n\}$. For each subset $J \subset I$, we define a sub root system $R_J \subset R$ by the formulas

$$(5.2) \quad Q_J = \sum_{i \in I-J} \mathbb{Z} \alpha_i \quad \text{and} \quad R_J = R_2(Q_J).$$

As Q_J is a subgroup of $Q(R)$ generated by roots we have $Q_J = Q(R_J)$.

Lemma 5.17. *Assume we have $J \subset I$ with $J \neq \emptyset$. Then $\{\alpha_i \mid i \in I - J\}$ is a basis of the root system R_J , and we have*

$$(5.3) \quad R_J = \{\alpha \in R \mid \exists n \in \{-1, 0, 1\}, \forall i \in J, \varpi_i \cdot \alpha = nh_i, \}.$$

Proof. The sum in (5.2) is direct since $J \neq \emptyset$. Note that for $\alpha \in R$ and $n \in \mathbb{Z}$, the condition $\varpi_i \cdot \alpha = nh_i$ for all $i \in J$ implies $n \in \{-1, 0, 1\}$, and even $n = 0$ in the case $0 \in J$, by definition of the h_i . All assertions are then simple consequences of the fact that for any $\alpha \in R^+$, both α and $\tilde{\alpha} - \alpha$ are a finite sum of α_i for $i \neq 0$. □

The Dynkin diagram of R_J is thus obtained from the affine Dynkin diagram of R by removing each α_j for $j \in J$. Familiar cases include the *parabolic* case $0 \in J$, and the case $|J| = 1$ (see [BOU81b, Ch. VI, §4, Ex. 4]).

Proposition 5.18. For $d \geq 1$, the d -kernels of R are the $w(R_J)$ with $w \in W(R)$ and $J \subset I$ such that there exists integers $x_j > 0$ for all $j \in J$ with $\sum_{j \in J} x_j h_j = d$ and $\gcd(d, \{x_j \mid j \in J\}) = 1$.

Proof. Any linear form $\varphi : Q(R) \rightarrow \mathbb{Z}/d$ has the form $\varphi(x) = \xi.x \pmod{d}$ for some $\xi \in Q(R)^\sharp$. We also have $\varphi(w(x)) = (w^{-1}\xi).x \pmod{d}$ for $w \in W(R)$. Applying the affine Weyl group $Q(R) \rtimes W(R)$ to ξ/d , we may assume that ξ is in the closed alcove defined by $0 \leq \xi.\alpha \leq d$ for all $\alpha \in R^+$. In particular,

$$(5.4) \quad R' := R_2(\ker \varphi) = \{\alpha \in R \mid \xi.\alpha \in \{-d, 0, d\}\}.$$

On the other hand, we may write $\xi = \sum_{i \in I} x_i \varpi_i$ in a unique way such that $\sum_{i \in I} x_i h_i = d$, namely $x_i = \xi.\alpha_i \in \mathbb{Z}$ for $i \neq 0$ and $x_0 = d - \sum_{i=1}^n x_i h_i$ (Kac coordinates). The alcove inequalities are equivalent to $x_i \geq 0$ for all $i \in I$. Set $J = \{i \in I \mid x_i \neq 0\}$, so that $\sum_{i \in J} x_i h_i = d$ and $x_i \geq 1$ for all $i \in J$. For $\alpha \in R^+$, the element $\xi.\alpha = \sum_{i \in J} x_i \varpi_i.\alpha$ is 0 (resp. d) if and only if we have $\varpi_i.\alpha = 0$ (resp. $\varpi_i.\alpha = h_i$) for all $i \in J$. Formulas (5.3) and (5.4) show then $R_J = R'$. Also, φ is surjective if and only if the set of x_j with $j \in J$ is coprime with d . Conversely, any R_J as in the statement is a d -kernel by this same analysis, setting $\xi = \sum_{i \in J} x_i \varpi_i$ and $\varphi(x) = \xi.x \pmod{d}$. \square

Remark 5.19. Assume $J \subset I$ contains some element i with $h_i = 1$ (e.g. $0 \in J$) and $|J| > 1$. The existence of x_j as in Proposition 5.18 is then equivalent to $\sum_{i \in J} h_i \leq d$, setting $x_j = 1$ for all $j \neq i$.

Remark 5.20. If we omit the condition that the set of x_j is coprime to d , we obtain a description of all the d' -kernels of R for some d' dividing d .

Example 5.21. (\mathbf{A}_n) The affine diagram of \mathbf{A}_n ($n \geq 1$) is a “circle” with $h_i = 1$ for all i . By removing $s \geq 2$ vertices, we see that the d -kernels of \mathbf{A}_n with $d \geq 2$ are isomorphic to $\mathbf{A}_{a_1} \mathbf{A}_{a_2} \dots \mathbf{A}_{a_s}$ with $2 \leq s \leq d$, $a_i \geq 0$ and $n + 1 = s + \sum_{i=1}^s a_i$.

(\mathbf{D}_n) The affine diagram of \mathbf{D}_n ($n \geq 4$) is a “bone”, with boundary heights $h_0 = h_1 = h_n = h_{n-1} = 1$ and $h_i = 2$ otherwise. Removing for instance $s \geq 2$ inner vertices, we see that for d even, the root systems of the form $\mathbf{A}_{a_1} \mathbf{A}_{a_2} \dots \mathbf{A}_{a_{s-1}} \mathbf{D}_m \mathbf{D}_{m'}$ with $2 \leq s \leq d/2$, $a_i \geq 0$, $m, m' \geq 2$ and $n + 1 = m + m' + s + \sum_{1 \leq i \leq s-1} a_i$, are d -kernels of \mathbf{D}_n .

Example 5.22. (2-kernels) The 2-kernels of \mathbf{A}_n with $n \geq 1$ are the $\mathbf{A}_p \mathbf{A}_{n-1-p}$ with $0 \leq p \leq n-1$. Those of \mathbf{D}_n with $n \geq 4$ are \mathbf{A}_{n-1} and the $\mathbf{D}_p \mathbf{D}_{n-p}$ for $1 \leq p \leq n-1$. The 2-kernels of \mathbf{E}_n are $\mathbf{A}_1 \mathbf{A}_5$ and \mathbf{D}_5 for $n = 6$, $\mathbf{A}_1 \mathbf{D}_6$, \mathbf{A}_7 and \mathbf{E}_6 for $n = 7$, $\mathbf{A}_1 \mathbf{E}_7$ and \mathbf{D}_8 for $n = 8$.

6. King's mass formulas

Our aim in this section is to discuss various *mass formulas*. It will be convenient to recall first a few elementary but useful concepts about *groupoids*.

6.1. Preliminaries on groupoids

A groupoid is a category X all of whose arrows are isomorphisms. We say that such an X is *finite* if it has finitely many objects up to isomorphism, and if $\text{Aut}(x)$ is a finite group for all objects x of X . We then denote by \underline{X} the finite set of isomorphism classes of objects in X . The *class number* of a finite groupoid X is $h(X) = |\underline{X}|$, and its *mass* is the rational number

$$(6.1) \quad \text{mass}(X) = \sum_x 1/|\text{Aut}(x)|,$$

where x runs among representatives of \underline{X} . Two *equivalent* finite groupoids (in the sense of categories) have the same class numbers and masses. Assume now $F : X \rightarrow Y$ is any *morphism* of groupoids, which simply means that F is a functor. If y is an object of Y , we define the (naive) *fiber* of F at y as the full subcategory $F^{-1}y$ of X whose objects x satisfy $F(x) \simeq y$. This is a groupoid which only depends on the isomorphism class of y . If X is finite, then so are the fibers of F , and we trivially have

$$(6.2) \quad \text{mass}(X) = \sum_y \text{mass}(F^{-1}y),$$

where y runs among representatives of \underline{Y} .

A typical finite groupoid is obtained as follows. Let G be a finite group acting on a finite set S . We denote by $[S/G]$ the finite groupoid with set of objects S , and with morphisms $s \rightarrow s'$ the set of elements $g \in G$ with $gs = s'$ (with obvious compositions). Then $h([S/G])$ is the number of G -orbits in S and we have

$$(6.3) \quad \text{mass}([S/G]) = |S|/|G|$$

by the *orbit-stabilizer* formula. In the proofs below, we will usually compute the mass of a given X fiberwise, using (6.2) for a suitable morphism $F : X \rightarrow Y$ and (6.3) by identifying each fiber of F with a concrete $[S/G]$.

6.2. Minkowski-Siegel-Smith mass formulas

Recall that the *mass* of an integral lattice L is defined by

$$\text{mass}(L) = \frac{1}{|\mathcal{O}(L)|}.$$

More generally, if \mathcal{L} is any collection of integral lattices (embedded in some fixed Euclidean space or not), but consisting of finitely many isometry classes L_1, \dots, L_h , the *mass* of \mathcal{L} may be defined and $\text{mass}(\mathcal{L}) = \sum_i \frac{1}{|\text{O}(L_i)|}$. Alternatively, this is the mass of the finite groupoid (that we will still denote by \mathcal{L}) with objects \mathcal{L} and with morphisms given by isometries.

When \mathcal{L} is a genus of integral lattices, the famous *Minkowski-Siegel-Smith mass formula* gives a closed formula for $\text{mass}(\mathcal{L})$. Recall that \mathcal{L}_n is the disjoint union of the two genera $\mathcal{L}_n^{\text{odd}}$ and $\mathcal{L}_n^{\text{even}}$ consisting of odd or even lattices: see *e.g.* [CS88] for concrete formulas for their respective masses.

6.3. King's mass formulas

For any groupoid \mathcal{L} of integral Euclidean lattices L , and any **ADE** root system R , we denote by $\mathcal{L}(R)$ the groupoid of objects L of \mathcal{L} whose root system $R_2(L)$ is isometric to R . King explains in [KIN03] an algorithm to compute, for each root system R , the mass

$$(6.4) \quad \text{mass}(\mathcal{L}_n^{\text{even}}(R)).$$

He uses for this the expression for the “mass-weighted” sum of the Siegel theta series of rank n even unimodular lattices as a Siegel Eisenstein series (Siegel-Weil formula), Katsurada's formula for its Fourier coefficients, as well as elementary properties of root lattices. Using the computer, he was then able to determine the quantity (6.4) for all R up to $n = 32$.

As explained in §4 of [KIN03], using ideas of Conway and Sloane in [CS99] Chap. 16 one can deduce from this computation the mass of $\mathcal{L}_n^{\text{odd}}(R)$ for all R and $n \leq 30$. As the argument is only sketched *loc. cit.*, and with tables not freely available, we now give more details about how this computation can be made. Actually, we follow a different method and give more general formulas.

Assume $m > 1$ is an integer, and R and R' are **ADE** root systems. Denote by $N_m(R, R')$ the number of root systems $S \subset R$ which are isomorphic to \mathbf{D}_m , saturated¹³ in R , and with $S^\perp \cap R$ isometric to R' (see §4.3). This integer only depends on the isomorphism class of R and R' . Its concrete determination is easily deduced from Table 4 in [KIN03, §8]. Recall the root lattice \mathbf{D}_n for $n > 1$ (see Sect. 2 (v)). The integer $|\text{O}(\mathbf{D}_n)|$ is $|\text{O}(\mathbf{I}_n)| = 2^n n!$ for $n \neq 4$, and $3 \cdot 2^4 4!$ for $n = 4$ (*triality*).

Proposition 6.4. *Fix integers $n, m > 1$ with $m + n \equiv 0 \pmod{8}$. We have*

$$\text{mass}(\mathcal{L}_n^{\text{odd}}(R)) = \frac{|\text{O}(\mathbf{D}_m)|}{2} \sum_{R'} N_m(R, R') \text{mass}(\mathcal{L}_{n+m}^{\text{even}}(R')),$$

¹³This condition is empty for $m \neq 8$, and for $m = 8$ it means that the irreducible component of R containing S is not of type \mathbf{E}_8 .

where R is any **ADE** root system of rank $\leq n$, and R' runs among all the isomorphism classes of **ADE** root systems of rank $\leq n + m$.

Our aim now is to prove this proposition. Denote by \mathcal{D}_n the groupoid of (abstract) even lattices D of rank n such that the finite quadratic space $\text{res } D$ is isomorphic to $\text{res } \mathcal{D}_n$. Set $f_n = 1$ for $n \not\equiv 4 \pmod{8}$, and $f_n = 3$ otherwise.

Lemma 6.5. *For any $n > 1$ and any **ADE** root system R we have*

$$\text{mass}(\mathcal{L}_n^{\text{odd}}(R)) = f_n \text{mass}(\mathcal{D}_n(R)).$$

Proof. If L is an integral lattice, it has a largest even sublattice denoted L^{even} , and which is of index 2 if L is odd. As $\mathcal{D}_n = (\mathcal{I}_n)^{\text{even}}$, and $\mathcal{L}_n^{\text{odd}}$ is the genus of \mathcal{I}_n , it follows that $L \mapsto L^{\text{even}}$ defines a morphism of groupoids $\text{ev}_n : \mathcal{L}_n^{\text{odd}} \rightarrow \mathcal{D}_n$. As $\mathcal{R}_2(L) = \mathcal{R}_2(L^{\text{even}})$ for all integral lattices L , and by Formula (6.2), it is enough to show that for all D in \mathcal{D}_n we have

$$(6.5) \quad \text{mass}(\text{ev}_n^{-1}D) = f_n \text{mass}(D).$$

Fix D in \mathcal{D}_n and let S be the set of odd unimodular lattices L in $V = D \otimes \mathbb{R}$ with $L^{\text{even}} = D$. Any object in $\text{ev}_n^{-1}D$ is isomorphic to some element of S . Better, any isometry $L \rightarrow L'$ with L, L' in S is induced by an isometry of $D = L^{\text{even}} = (L')^{\text{even}}$. The functor $L \mapsto L, [S/\text{O}(D)] \rightarrow \text{ev}_n^{-1}D$, is thus an equivalence. By Formula (6.3) it only remains to show $|S| = f_n$. By Sect. 2 (iii) and (iv), the map $\beta_D : L \mapsto L/D$ defines a bijection between S and the set of order 2 subgroups of $\text{res } D$ generated by an element v with $v.v \equiv 0$ but $q(v) \not\equiv 0$. An inspection of $\text{res } D \simeq \text{res } \mathcal{D}_n$ (Table 2.1) concludes. \square

Proposition 6.4 is a special case of the following more general proposition. It will be convenient to make first a definition.

Definition 6.6. *Let R be an **ADE** root system and set $Q = \mathcal{Q}(R)$. We say that R satisfies (M) if the finite quadratic space $\text{res } Q$ is anisotropic, or equivalently, if Q is saturated in any even lattice containing it. We say that R satisfies (M') if we have $\text{qm}(x) = 1$ for all isotropic $x \in \text{res } \mathcal{Q}(R)$, or equivalently, if all the even integral lattices between Q and Q^\sharp are root lattices.*

Example 6.7. It is clear that (M) implies (M'). The irreducible root systems satisfying (M) are the following : \mathbf{A}_m in the case $m + 1$ is either squarefree or of the form $4q$ with q odd and squarefree, \mathbf{D}_m for $m \not\equiv 0 \pmod{8}$, \mathbf{E}_6 , \mathbf{E}_7 and \mathbf{E}_8 . The root systems \mathbf{D}_8 , \mathbf{A}_m for $m \leq 14$, and $m\mathbf{A}_1$ for $m \leq 7$, satisfy (M').

For **ADE** root systems R_1, R_2, R_3 we denote by $N(R_1, R_2, R_3)$ the number of saturated sub root systems T of R_2 satisfying $T \simeq R_1$ and $T^\perp \cap R_2 \simeq R_3$.

When R_1 satisfies (M'), then $N(R_1, R_2, R_3)$ can be deduced¹⁴ from Table 4 in [KIN03, §8].

Proposition 6.8. *Let R_0 be a rank m ADE root system satisfying (M'), let $n > 1$ an integer with $m + n \equiv 0 \pmod{8}$, and \mathcal{G} the groupoid¹⁵ of rank n even Euclidean lattices L with $\text{res } L \simeq -\text{res } Q(R_0)$ (as quadratic spaces). Then for any ADE root system R we have*

$$\text{mass}(\mathcal{G}(R)) = \frac{|\text{O}(Q(R_0))|}{|\text{O}(\text{res } R_0)|} \sum_{R'} N(R_0, R', R) \text{mass}(\mathcal{L}_{m+n}^{\text{even}}(R'))$$

where R' runs over all isomorphism classes of ADE root systems of rank $\leq m + n$.

In the statement above, $\text{O}(\text{res } R_0)$ denotes the isometry group of the finite quadratic space $\text{res } R_0 = \text{res } Q(R_0)$ (equipped with the form q of §2 (iv)).

Proof. (Proposition 6.8 \Rightarrow Proposition 6.4). Observe that we have $\text{res } D_n \simeq -\text{res } D_m$, so we are in the situation of Proposition 6.8 with $R_0 \simeq \mathbf{D}_m$, $Q(R_0) \simeq D_m$, and $\mathcal{G} = \mathcal{D}_n$. As the isometry group of $\text{res } D_m$ is isomorphic to S_3 for $m \equiv 4 \pmod{8}$, and to $\mathbb{Z}/2$ otherwise, we have $|\text{O}(\text{res } D_m)| = 2f_m$. We conclude by Lemma 6.5, $f_n = f_m$ and $N(\mathbf{D}_m, R', R) = N_m(R', R)$. \square

Proof. We now prove Proposition 6.8. Denote by \mathcal{U}_{n, R_0} the groupoid with objects the pairs (U, Q) where U is an even unimodular lattice of rank $m + n$ and Q is a saturated subgroup of U isomorphic to the root lattice $Q(R_0)$, and with morphisms $(U, Q) \rightarrow (U', Q')$ the isometries $U \rightarrow U'$ sending Q onto Q' . For (U, Q) in \mathcal{U}_{n, R_0} the orthogonal Q^\perp of Q in U is an object of \mathcal{G} since we have isomorphisms of finite quadratic spaces $\text{res } Q^\perp \simeq -\text{res } Q$ by Prop. 1.6.1 in [NIK79]. The map $U \mapsto Q^\perp$ induces a morphism of groupoids $\text{orth} : \mathcal{U}_{n, R_0} \rightarrow \mathcal{G}$.

Lemma 6.9. *For all D in \mathcal{G} we have $\text{mass}(\text{orth}^{-1}D) = \frac{|\text{O}(\text{res } R_0)|}{|\text{O}(Q(R_0))|} \text{mass}(D)$.*

Proof. Fix D in \mathcal{G} . Set $M = Q(R_0) \perp D$, $V = M \otimes \mathbb{R}$, and let S be the set of even unimodular lattices U of V containing M and with $Q(R_0)$ saturated in U . This set has a natural action of the group $G = \text{O}(Q(R_0)) \times \text{O}(D)$, and the map $L \mapsto (L, Q(R_0))$ trivially induces an equivalence $[S/G] \xrightarrow{\sim} \text{orth}^{-1}D$. We have thus $|\text{mass}(\text{orth}^{-1}D)| = |S|/|G|$. By Sect. 2 (iii) and (iv), the map $\beta_M : U \mapsto U/M$ induces a bijection between S and the set of quadratic Lagrangians in $\text{res } M = \text{res } Q(R_0) \perp \text{res } D$ which are transversal to $\text{res } Q(R_0)$. But in a finite quadratic space $A \perp B$ with $B \simeq -A$, there are $|\text{O}(A)|$ Lagrangians which are transversal to A , namely the $I_\varphi = \{x + \varphi(x), x \in A\}$, with φ any isomorphism $A \xrightarrow{\sim} -B$, thus $|S| = |\text{O}(\text{res } R_0)|$. \square

¹⁴This is especially simple if R_1 satisfies (M), since this number is just the number of sub root systems T of R_2 satisfying $T \simeq R_1$ and $T^\perp \cap R_2 \simeq R_3$.

¹⁵We shall not use it, but \mathcal{G} is actually a single genus, by [NIK79, Cor. 1.9.4].

Denote by $\mathcal{U}_{n,R_0,R}$ the full subcategory of \mathcal{U}_{n,R_0} whose objects (U, Q) satisfy $R_2(Q^\perp) \simeq R$, i.e. Q^\perp is an object of $\mathcal{G}(R)$. This is the disjoint union of all the fibers of orth over $\mathcal{G}(R)$, so Lemma 6.9 and Formula (6.2) imply

$$(6.6) \quad \text{mass}(\mathcal{U}_{n,R_0,R}) = \frac{|\text{O}(\text{res } R_0)|}{|\text{O}(\text{Q}(R_0))|} \text{mass}(\mathcal{G}(R)).$$

Consider now the forgetful functor $\text{forg} : \mathcal{U}_{n,R_0,R} \rightarrow \mathcal{L}_{m+n}^{\text{even}}, (U, Q) \mapsto U$. Fix U in $\mathcal{L}_{m+n}^{\text{even}}$, and consider the set $\text{S}(R_0, U, R)$ of saturated subgroups Q of U satisfying $Q \simeq \text{Q}(R_0)$ and $R_2(Q^\perp) \simeq R$. Then $\text{S}(R_0, U, R)$ has a natural $\text{O}(U)$ -action, and the fiber $\text{forg}^{-1}U$ is trivially equivalent to the groupoid $[\text{S}(R_0, U, R)/\text{O}(U)]$.

Lemma 6.10. *We have $|\text{S}(R_0, U, R)| = \text{N}(R_0, R', R)$.*

Proof. Set $R' = R_2(U)$ and let $Q = \text{Q}(S)$ be a sub root-lattice of U , hence of $\text{Q}(R')$. If Q is saturated in U , then Q is obviously saturated in $\text{Q}(R')$. Assume conversely that Q is saturated in $\text{Q}(R')$. Let Q' be the saturation of Q in U . As Q' is even, we have $Q' = \text{Q}'(S')$ for $S' = R_2(Q')$ since R_0 satisfies (M'). But then we have $S' \subset R_2(U) = R'$ and thus $Q' \subset \text{Q}(R')$, and then $Q = Q'$ as Q is saturated in $\text{Q}(R')$, so Q is saturated in U . \square

We have thus $\text{mass}(\text{forg}^{-1}U) = \text{N}(R_0, R', R) \text{mass}(U)$ by Formula (6.3) and Lemma 6.10. Using Formula (6.2), we deduce

$$\text{mass}(\mathcal{U}_{n,R_0,R}) = \sum_{R'} \text{N}(R_0, R', R) \text{mass}(\mathcal{L}_{m+n}^{\text{even}}(R')),$$

the sum being over all isomorphism classes of **ADE** root systems R' , and conclude by (6.6). \square

As a consequence of King's results and Proposition 6.8, we obtain new lower bounds for the number of isometry classes in the genera of even lattices of rank $32-r$ and residue isometric to $-\text{res } \text{Q}(R_0)$ with R_0 an **ADE** root system of rank $r \geq 1$ satisfying (M'). This will be a useful ingredient in forthcoming work with O. Taïbi. As a very simple example, let us consider the genus \mathcal{G} of even lattices of rank 31 and determinant 2. We are in the case $R_0 \simeq \mathbf{A}_1$, $m = 1$ and $n = 31$. The numerical application of the Proposition above shows that there are exactly 18 437 root systems in this genus and using the same argument as in [KIN03, §6] we obtain :

Corollary 6.11. *There are at least 6 678 411 375 even lattices of determinant 2 in dimension 31. Moreover, the mass of those lattices having an empty root system is $11882632915662079/591224832 \simeq 20098331.92$.*

6.12. Odd unimodular lattices without norm 1 vector

We denote by \mathcal{L}'_n the groupoid of all L in \mathcal{L}_n with $r_1(L) = 0$. Note that in the canonical decomposition $L = A \perp B$ recalled in Formula (1.8) of the introduction, we obviously have $\text{mass}(L) = \frac{\text{mass}(B)}{2^m m!}$. Using the convention $\text{mass}(\mathcal{L}_0) = \text{mass}(\mathcal{L}'_0) = 1$ (that is, $\text{Aut}(0) = \{1\}!$) we deduce

$$(6.7) \quad \sum_{n \geq 0} \text{mass}(\mathcal{L}'_n) x^n = e^{-x/2} \sum_{n \geq 0} \text{mass}(\mathcal{L}_n) x^n.$$

The relevant numerical values are given in Table 6.1 below.

n	$\text{mass}(\mathcal{L}'_n)$	n	$\text{mass}(\mathcal{L}'_n)$
0	1	20	4060488226549/11479871952566228090880000
8	1/696729600	21	138813595637/54497004983156736000000
12	1/980995276800	22	1475568922019/45471119389159682211840
14	1/16855282483200	23	21569773276937492389/28590262351867673365708800000
15	1/41845579776000	24	4261904533831299496396870055017/129477933340026851560636148613120000000
16	5213041/277667181515243520000	25	103079509578355844357599/37291646545914356563968000000
17	1/49662885888000	26	15661211867944570315962162816169/34253421518525622105988399104000000
18	1073351/32780153373327360000	27	18471746857358122138056975582390629/121385562506275173338096389324800000
19	37813/450541700775936000	28	1722914776839913679032185321786744287148737/16573175467523436999761427022479360000000

Table 6.1: The nonzero $\text{mass}(\mathcal{L}'_n)$ for $0 \leq n \leq 28$

Note also that still in the canonical decomposition $L = A \perp B$ above, we have $r_1(B) = 0$ and thus $R_2(L) = R_2(A) \amalg R_2(B)$ and $R_2(A) \simeq \mathbf{D}_m$. If $\mathcal{L}_n^{\text{odd}'}$ denotes the groupoid of rank n odd unimodular lattices with no norm 1 vector, and if R is an **ADE** root system, we have thus

$$(6.8) \quad \text{mass}(\mathcal{L}_n^{\text{odd}}(R)) = \sum_{(m,S)} \frac{1}{2^m m!} \text{mass}(\mathcal{L}_{n-m}^{\text{odd}'}(S))$$

where (m, S) runs among the pairs with m an integer ≥ 0 and S an isomorphism class of root systems such that $\mathbf{D}_m \amalg S \simeq R$. Of course, the term $\text{mass}(\mathcal{L}_n^{\text{odd}'}(R))$ appears in the sum (6.8) for $m = 0$, so we obtain an expression for it in terms of King's $\text{mass}(\mathcal{L}_{32}^{\text{even}}(R'))$ for $n \leq 30$: see [CHE20b] for tables and Table 1.2 in the introduction for a few information.

7. Visible isometries

7.1. The visible isometry group of a d -neighbor

Fix a $L \in \mathcal{L}_n$, $d \geq 1$ an integer, an isotropic line $\ell \in C_L(\mathbb{Z}/d)$. The group $O(L)$ naturally acts on $C_L(\mathbb{Z}/d)$ and we denote by $O(L; \ell)$ the stabilizer of ℓ . As it is equivalent to stabilize ℓ and its orthogonal $M_d(\ell)/dL$, we also have

$$(7.1) \quad O(L; \ell) = O(L) \cap O(M_d(\ell)).$$

Observe that the action of any $g \in \mathcal{O}(L; \ell)$ on the line $\ell \simeq \mathbb{Z}/d$ is \mathbb{Z}/d -linear, hence given by the multiplication by some element $\nu(g) \in (\mathbb{Z}/d)^\times$. Then

$$(7.2) \quad \nu : \mathcal{O}(L; \ell) \rightarrow (\mathbb{Z}/d)^\times, g \mapsto \nu(g)$$

is a group homomorphism. For instance we have $\nu(-\text{id}_L) = -1$. Our aim now is to determine $\mathcal{O}(L; \ell)$ when L is the lattice I_n . For this we choose $x \in \mathbb{Z}^n$ such that $x \bmod d$ generates ℓ , denote by $X(\ell)$ the multiset $\{\{\pm x_1, \dots, \pm x_n\}\}$ of elements of $(\mathbb{Z}/d)/\{\pm 1\}$, and set¹⁶

$$(7.3) \quad \mathcal{H}(\ell) = \{\lambda \in (\mathbb{Z}/d)^\times \mid \lambda X(\ell) = X(\ell)\}.$$

Then $\mathcal{H}(\ell)$ is a subgroup of $(\mathbb{Z}/d)^\times$ not depending on the choice of x . Recall also from Sect. 5 that attached to ℓ is a natural partition of the integer n

$$n = m(\ell) + m'(\ell) + \sum_{i \in I} a_i(\ell).$$

Proposition 7.2. *For $\ell \in \mathcal{C}_n(\mathbb{Z}/d)$ and ν as in Formula (7.2) we have*

$$\text{Im } \nu = \mathcal{H}(\ell) \text{ and } \text{Ker } \nu \simeq \mathcal{O}(I_{m(\ell)}) \times \mathcal{O}(I_{m'(\ell)}) \times \prod_{i \in I} \mathcal{S}_{a_i(\ell)}.$$

Proof. Recall that $\mathcal{O}(I_n) \simeq \{\pm 1\}^n \rtimes S_n$ acts on $I_n = \mathbb{Z}^n$ as arbitrary signed permutations of the coordinates. We have thus $\nu(g) \in \mathcal{H}(\ell)$ for all $g \in \mathcal{O}(L; \ell)$. Conversely, assume we have $\lambda \in \mathcal{H}(\ell)$. By definition, there are $\sigma \in S_n$ and $e \in \{\pm 1\}^n$ such that for all $1 \leq i \leq n$, $\lambda x_i = e_i x_{\sigma(i)}$. In other words, there is $g \in \mathcal{O}(I_n)$ with $g(x) = \lambda x \bmod d$, i.e. with $\nu(g) = \lambda$. We have proved $\text{Im } \nu = \mathcal{H}(\ell)$.

An element $\sigma \in \mathcal{O}(L; \ell)$ is in $\text{ker } \nu$ if, and only if, we have $\sigma(x) \equiv x \bmod d$. The natural action of the signed permutation σ on $\{1, \dots, n\}$ preserves thus the partition of $\{1, \dots, n\}$ defined in §5, with sign changes only allowed on the subsets D and D' . Indeed, for $i \in \mathbb{Z}/d$ we have $-i \equiv i \bmod d$ if, and only if, d is even and $i \equiv 0$ or $i/2 \bmod d$. This proves the assertion about $\text{ker } \nu$. \square

We now go back to the case of a general $L \in \mathcal{L}_n$. Set $e = 1$ for d odd, $e = 2$ otherwise. Then $\mathcal{O}(L; \ell)$ preserves the lattice $M_d(\ell)$, and thus permutes the e cyclic d -neighbors of L with line ℓ , which defines a natural group morphism

$$(7.4) \quad \mathcal{O}(L; \ell) \rightarrow S_e.$$

Definition 7.3. *We denote by $\mathcal{O}(L; \ell)^0$ the kernel of the morphism (7.4). If N is any of the 1 or 2 cyclic d -neighbors of L with line ℓ , then $\mathcal{O}(L; \ell)^0$ coincides with the subgroup $\mathcal{O}(N)^\vee := \mathcal{O}(N) \cap \mathcal{O}(L)$ of $\mathcal{O}(N)$. We call it the visible isometry group of the d -neighbor N of L .*

¹⁶For a multiset X over $(\mathbb{Z}/d)/\{\pm 1\}$ and $\lambda \in (\mathbb{Z}/d)^\times$ we set $\lambda X = \{\{\lambda x \mid x \in X\}\}$.

The idea behind these definitions is that visible isometries can be concretely determined from an inspection of the line ℓ . Of course, in the case d odd we always have $O(L; \ell) = O(L; \ell)^0 = O(N)^\vee$. This also holds for d even unless $O(L; \ell)$ permutes the 2 cyclic d -neighbors with line ℓ . An element of $O(N)^\vee$ will also be called a *visible isometry* of N .

Remark 7.4. For d odd and $M = L \cap N$, we have $O(N) \cap O(M) \subset O(L)$ since L is the unique cyclic d -neighbor of N with visible part M by Proposition 3.2. This shows the equality $O(N)^\vee = O(N) \cap O(M)$ for d odd.

For the purpose of unimodular hunting, the most basic application of visible isometries is that if we fix $\sigma \in O(L)$ and restrict to isotropic lines $\ell \in C_L(\mathbb{Z}/d)$ satisfying $\sigma(\ell) = \ell$, then the corresponding d -neighbors of L necessarily contain σ as a visible isometry (or perhaps σ^2 , if d is even and σ permutes the 2 neighbors with line ℓ). In what follows we explain some efficient construction of such lines for $L = I_n$.

7.5. Some stable isotropic lines for I_n

Let us focus on the case $L = I_n$. Fix two integers q and k with $qk \leq n$, and consider an element $\sigma \in S_n \subset O(I_n)$ which is a product of k cycles of length q with disjoint supports $C_1, \dots, C_k \subset \{1, \dots, n\}$. Set $C = \coprod_{j=1}^k C_j$. Choose any odd prime $p \equiv 1 \pmod{q}$, and for such a p , choose an element $\omega \in (\mathbb{Z}/p)^\times$ of order q . Choose any $x \in (\mathbb{Z}/p\mathbb{Z})^n$ such that

$$x_i \neq 0 \iff i \in C, \quad \text{and} \quad x_{\sigma^{-1}(i)} = \omega x_i \quad \text{for all } i \in C.$$

There are exactly $(p-1)^k$ such elements x . Note that line $\ell = \mathbb{Z}/p x$ generated by x is automatically in $C_n(\mathbb{Z}/p)$ for $q \neq 2$, as we have $\sum_{s=0}^{q-1} \omega^{2s} = 0$. In the case $q = 2$ (so $\omega = -1$) we add the assumption that ℓ is isotropic. By construction, we have $\sigma(x) = \omega x$ and thus

$$\sigma \in O(L; \ell) \quad \text{and} \quad \nu(\sigma) = \omega.$$

So σ is a visible isometry of the p -neighbor $N_p(x)$. This construction is promising, but too restrictive in practice. Indeed, as we (must) have $x_i = 0$ for $i \notin C$ the visible root system of $N_p(x)$ necessarily contains \mathbf{A}_{n-qk} , which is too restrictive to find the most interesting lattices. We can circumvent this problem using the following trick. Consider an extra integer d prime to p , and choose any d -isotropic vector $y \in C_n(\mathbb{Z}/d)$ such that

$$y_{\sigma(i)} = y_i \quad \forall i \in C.$$

We clearly have $\sigma(y) = y$, so $\sigma \in O(L; \mathbb{Z}/d y)$, and $\nu(\sigma) = 1$. By the chinese remainder theorem, there is a unique $z \in (\mathbb{Z}/pd)^n$ with $z \equiv x \pmod{p}$ and $z \equiv y \pmod{d}$. Then z is pd -isotropic and generates a line ℓ' such that

$$\sigma \in O(L; \ell'), \quad \nu(\sigma) \equiv \omega \pmod{p} \quad \text{and} \quad \nu(\sigma) = 1 \pmod{d}.$$

Note that for q odd, σ necessarily belongs to the subgroup $\sigma \in O(L; \ell')^0$.

Using such lines, we may even produce lattices with empty visible root systems ! Indeed, it is enough to ensure that we have $y_i \not\equiv \pm y_j \pmod{d}$ for i and j not in the same C_k .

7.6. An example : The Bacher-Venkov unimodular lattices with no root in dimension 28

Let us consider the problem of finding representatives of X_{28}^\emptyset , which is the most difficult computation in [BV01]. For this computation, we will use the variant BV of an invariant used by Bacher-Venkov, which is defined in the companion paper [AC]. Using either the arguments *loc. cit.* or King's results, we know that the mass of X_{28}^\emptyset is $17924389897/26202009600$. The denominator of this mass factors as $2^{12} \cdot 3^9 \cdot 5^2 \cdot 13$.

(a) This suggests to look first for lattices with a visible automorphism of order 13. By enumerating isotropic lines of the form above for $q = 13$, the prime $p = 53$, $k = 2$ and odd $d \leq 17$, we find for $d = 17$ (so $pd = 901$) two unimodular lattices with no element of norm ≤ 2 and a visible isometry of order 13, after running only over 12 isotropic lines ! Their masses turn out to be respectively $1/18341406720$ and $1/116480$. For instance the first one is $N_{901}(x)$ with $x \in (\mathbb{Z}/901)^{28}$ defined by the following formula, with $t \equiv 16 \pmod{53}$ (of order 13):

$$\begin{aligned} x \pmod{53} &= (1, t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12}, 1, t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12}, 0, 0) \\ x \pmod{17} &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 6, 7) \end{aligned}$$

This is one of the last lattices found in [BV01], whereas this method finds it instantly. The remaining mass is then $17021999/24883200$, whose denominator is $2^{12} \cdot 3^5 \cdot 5^2$.

(b) Now we use $q = 5$, with the prime $p = 11$. For $k = 5$, we find 6 new lattices in X_{28}^\emptyset in the first thousand of lines, namely 3 for $d = 35$, with masses $1/400$, $1/7680$ and $1/15360$, and 3 others for $d = 37$, with masses $1/696729600$, $1/7680$ and $1/3317760$. The remaining mass is $474647137/696729600$, but other tries do not seem to find new lattices. On the other hand, we have $696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, and the prime 7 now appears.¹⁷

(c) So we try $q = 7$, with the prime $p = 43$. For $k = 4$ and $d \leq 15$, we find 4 new lattices in the first two hundreds of lines for $d = 15$ (hence $pd = 645$), with respective mass $1/344064$, $1/112$ and $1/96768$ and $1/9676800$. The remaining mass is then $836459/1244160$, with denominator $2^{10} \cdot 3^5 \cdot 5$ (no more 7).

(d) Trying $q = 3$, $p = 7$ and $k = 8$ we immediately find 7 lattices for $d = 45$ with masses $1/96$, $1/1728$, $1/96$, $1/15552$, $1/55296$, $1/6144$ and $1/192$. The remaining mass is then $4957/7680$, with denominator $2^9 \cdot 3 \cdot 5$ (luckily the exponent of

¹⁷Actually, 7 already appeared actually in the two masses found in (a), but disappeared in the remaining mass there by an unlucky cancelation.

7 is now 0 and that of 5 is 1). For $k = 9$ we find two more lattices for $d = 41$, with mass $1/24$ and $1/192$, hence the remaining mass $4597/7680$, with denominator $2^9 \cdot 3 \cdot 5$.

(e) At this point, we have actually found all the lattices with smallest mass, so the most efficient (and lazy) method is to run the general algorithm described in the introduction, biased with an empty visible root system. This way we do quickly find the 17 remaining lattices as d -neighbors of I_{28} with $61 \leq d \leq 70$.

7.7. A constraint : the *type* of a prime order isometry of a unimodular lattice

Let $L \in \mathcal{L}_n$ and $\gamma \in O(L)$ of odd prime order q . The characteristic polynomial of γ has the form¹⁸

$$\Phi_q^k \Phi_1^l, \quad \text{with } n = l + k(q - 1).$$

As explained in [NEB13, §4], the rank l sublattice $L_1 = \ker(\gamma - 1) \subset L$ satisfies $L_1^\sharp/L_1 \simeq (\mathbb{Z}/q)^s$ with $s \leq k, l$ and $s \equiv k \pmod{2}$. The pair (k, s) is called the *type* of γ ; it only depends on the $\mathbb{Z}_q[\gamma]$ -module $L \otimes \mathbb{Z}_q$.

In the case $L = I_n$, then σ is necessarily a product of k cycles of length q , and the orthogonal of L_1 in L is clearly isometric to A_{q-1}^k . So σ is of type (k, k) . As a consequence, for any stable isotropic line $\ell \in C_n(\mathbb{Z}/d)$ with d prime to q , the visible isometry σ of the associated d -neighbors will also have type (k, k) . This is an important restriction in the above method, although types (k, k) seems to be the most common ones in practice. Another restriction, more obvious, is the fact that we must have $kq \leq n$, or equivalently $l \geq k$, instead of the most general case $k(q - 1) \leq n$. This excludes for instance, in the case $n \equiv 0 \pmod{q - 1}$, the unimodular lattices defined by *Hermitian* $\mathbb{Z}[\zeta_q]$ -lattices of rank $\frac{n}{q-1}$.

Example 7.8. Going back to the Example of § 7.6, there are 11 elements in X_{28}^\emptyset having an isometry of order 5. Two of them, with masses $1/320$ and $1/160$, have not been found in step (b) (nor in steps (a), (c) and (d) as we have $160 = 2^5 \cdot 5$). Indeed, using [GAP] we can check that these two lattices have a single conjugacy class of order 5 isometry, and whose type is $(6, 4)$. But there is no isometry of I_{28} with characteristic polynomial $\Phi_5^6 \Phi_1^4$. Actually, a third lattice also has a single conjugacy class of order 5 isometry, which is of type $(6, 4)$. But this lattice has an order 13 isometry of type $(2, 2)$, and we found it in step (a).

7.9. “The neighbors of a lattice with small mass likely have a small mass”

Here is an alternative method that we used in many instances to find lattices of small mass, and whose slogan is the title of this subsection. The ideas is

¹⁸Here Φ_d denotes the d -th cyclotomic polynomial.

simply that if we have $L_0 \in \mathcal{L}_n$ with a large isometry group, then $C_{L_0}(\mathbb{Z}/d)$ will usually contain many points with a non-trivial stabilizer in $O(L_0)$, hence leading to d -neighbors of L_0 with non-trivial visible isometry groups.

In practice we often take $d = 2$, and assume L_0 given as a d_0 -neighbor of I_n with d_0 odd, say $L_0 = N_{d_0}(x_0)$. So we expect to find lattices with non-trivial isometry groups of the form $N_{2d_0}(x)$ with $x \equiv x_0 \pmod{d_0}$ by Lemma 11.2. In practice we often combine this idea with that of the visible root system, by imposing that $N_{d_0}(x)$ and $N_{2d_0}(x)$ have the same visible root systems (we talk about "strict 2-neighbors" of $N_{d_0}(x)$).

Example 7.10. *Consider for instance the problem of finding the elements $L \in X_{27}$ with $r_1(L) = 0$ and $R_2(L) \simeq 12A_1$. From King's results, the total reduced mass of those lattices is $368401/138240$. Using the visible root system $9A_1$, and an already lengthy enumeration of the corresponding d -neighbors for odd d from $d = 37$ to $d = 45$, we find 26 such lattices, with remaining reduced mass $731/276480$. The one with smallest mass is $N_{45}(x)$ with*

$$x = (1, 1, 2, 4, 4, 6, 6, 7, 9, 9, 10, 11, 12, 13, 13, 14, 16, 16, 17, 17, 18, 18, 19, 20, 21, 21, 22) \in \mathbb{Z}^{27}$$

whose reduced mass is $1/2048$. By considering "solely" the $\simeq 2^{16}$ strict 2-neighbors of this lattice we quickly find the three remaining ones, with respective reduced mass $1/384$, $1/46080$ and $1/55296$: see Table 7.1.

reduced mass	1/4	1/8	1/16	1/32	1/48	1/64	1/384	1/640	1/720	1/2048	1/46080	1/55296
#lattices	8	2	4	3	2	1	4	1	1	1	1	1

Table 7.1: The 29 lattices in X_{27} with no norm 1 elements and root system $12A_1$.

8. An example : the lattice $N_{2n+1}(1, 2, \dots, n)$

In this section we give an example of study of the non-visible part of a neighbor. More precisely, we fix an integer $n \geq 1$ with $n \not\equiv 1 \pmod{3}$ and set $x = (1, 2, \dots, n)$. As already explained in the introduction Formula (1.6), we have a unimodular lattice

$$N_n := N_{2n+1}(x) \in \mathcal{L}_n$$

defined by $N_n = M_n + \mathbb{Z} \frac{1}{2n+1} x'$ with $M_n := M_{2n+1}(x)$ and $x' = x + \frac{(2n+1)n^2(n+1)}{6} \epsilon_1$. As we have $\pm i, \pm i \pm j \not\equiv 0 \pmod{2n+1}$ for $1 \leq i < j \leq n$, we have no visible element of norm ≤ 2 , i.e. $R_2(M_n) = \emptyset$. Better :

Proposition 8.1. *We have $r_2(N_n) = 0$ for all $n \geq 23$.*

Proof. Fix $z \in N_n \setminus M_n$. For some divisor b of $2n + 1$, we may write $z = m + \frac{k}{b}x'$ with $m \in M$ and $1 \leq k < b$ coprime with b . Write

$$b = 2s + 1 \text{ and } 2n + 1 = (2t + 1)(2s + 1).$$

We have $n = (2s + 1)t + s$, so the coordinates of x are \equiv modulo b to $\pm 1, \pm 2, \dots, \pm s, 0$ (t times) and then to $1, 2, \dots, s$, modulo b . Observe that if S is a subset of $X := \mathbb{Z}/(2s + 1) \setminus \{0\}$ satisfying $X = S \amalg -S$, then the same holds for kS for all $k \in (\mathbb{Z}/(2s + 1))^\times$. It follows that the coordinates mod b of the element $bz = bm + kx'$, also are $2t + 1$ times $\pm 1, \pm 2, \dots, \pm s$, and t times 0. A trivial coordinatewise inequality shows then

$$(8.1) \quad z.z \geq \frac{1}{b^2}(2t + 1) \sum_{i=1}^s i^2 = \frac{2n + 1}{24} \left(1 - \frac{1}{(2s + 1)^2} \right).$$

As $2n + 1 \not\equiv 0 \pmod{3}$ we have $s \geq 2$, and thus $z.z \geq \frac{53}{24} \frac{24}{25} > 2$ for $n \geq 26$. The two remaining cases $n = 23$ and 24 (Short Leech and Odd Leech lattices) could be further analyzed in this style (or checked with a computer), but they are classical so we omit them. \square

Proposition 8.2. *For $n \geq 5$ the lattice M_n is generated by $R_3(M_n)$, and for $n \geq 36$ we have $R_3(M_n) = R_3(N_n)$.*

Proof. Let L_n be the orthogonal of $x = (1, 2, \dots, n)$ in I_n . We have $L_n \subset M_n$. A simple computation shows that L_5 is generated by its norm 3 vectors (there are 8 such vectors). Using $L_n = L_{n-1} \times \{0\} + \mathbb{Z}e$ with $e = (-1, 0, \dots, 0, -1, 1)$, the same holds for L_n with $n \geq 5$. The vector $e' = (0, 1, 0, \dots, 0, 1, 1)$ satisfies $e'.e' = 3$ and $e'.x = 2n + 1$, so we have $M_n = L_n + \mathbb{Z}e'$ and the first assertion holds.

A non-visible vector z in $N_{2n+1}(1, 2, \dots, n)$ satisfies the inequality (8.1), for some integer $s \geq 2$ such that $2s + 1$ divides $2n + 1$. For $n > 37$, the right hand side of (8.1) is $\geq \frac{77}{24} \frac{24}{25} > 3$. For $n = 36$, then $2n + 1 = 73$ is prime, so we have $s = n$ and $z.z \geq \frac{222}{73} > 3$. \square

We finally determine the isometry group of N_n . Recall from Sect. 7 the visible subgroup $O(N_n)^\vee = O(N_n) \cap O(I_n)$ as well as the morphism

$$\nu : O(N_n)^\vee \rightarrow (\mathbb{Z}/(2n + 1))^\times.$$

Proposition 8.3. *For all $n \geq 1$, the morphism ν defines an isomorphism $O(N_n)^\vee \xrightarrow{\sim} (\mathbb{Z}/(2n + 1)\mathbb{Z})^\times$. Moreover, for $n \geq 32$ we have $O(N_n)^\vee = O(N_n)$.*

Proof. The first assertion is a consequence of Prop. 7.2. Indeed, in the notations of this proposition, and given the shape of x , we have $m(\ell) =$

$m'(\ell) = 0$ and $a_i(\ell) = 1$ for all $i \in I$, so ν is injective, as well as $X(\ell) = (\mathbb{Z}/(2n+1) \setminus \{0\})/\{\pm 1\}$, so we have $H(\ell) = (\mathbb{Z}/(2n+1))^\times$ and ν is surjective.

For $n \geq 36$, we have $O(N_n) \subset O(M_n)$ by Proposition 8.2, hence the last assertion by Remark 7.4. In the remaining cases $n = 32, 33, 35$, it is enough to check $|O(N_n)| = \varphi(2n+1)$, which follows from a computation using the Plesken-Souvignier algorithm (actually, we have $R_3(M_n) \subsetneq R_3(N_n)$ for those n). \square

Remark 8.4. For $n = 23, 24, 26, 27, 29$ and 30 , then $|O(N_n)|/\varphi(2n+1)$ is respectively 1 839 366 144 000, 23 876 075 520, 360 000, 192, 4 and 2.

9. Exceptional lattices and visible characteristic vectors

As already observed by Bacher and Venkov in their study of lattices with no root [BV01], certain unimodular lattices that they term *exceptional* tend to be harder to find. For this same reason they played an important role in our search, which explains this section.

9.1. Exceptional lattices

Let L be a unimodular lattice of rank n . Recall that $\text{Char}(L)$ denotes the set of characteristic vectors of L (see § 3.7). We have

$$(9.1) \quad \forall \xi \in \text{Char}(L), \quad \xi \cdot \xi \equiv n \pmod{8}.$$

Indeed, if L is even this holds as $\xi \in 2L$, and if L is odd it also holds as L and I_n are isometric over \mathbb{Z}_2 . The following definition is a generalization of a terminology of Bacher and Venkov [BV01, §3].

Definition 9.2. A unimodular lattice L of rank n is called *exceptional* if there exists $\xi \in \text{Char}(L)$ with $\xi \cdot \xi < 8$.

As an example, it follows from Formula (3.4) that the lattice I_n is exceptional only for $n < 8$, in which case it has exactly 2^n characteristic vectors of norm < 8 . Observe that for $L = A \perp B$, we have $\text{Char}(L) = \{a + b \mid a \in \text{Char}(A), b \in \text{Char}(B)\}$. We deduce the following fact, where $\rho(n)$ denotes the unique integer r satisfying $n \equiv r \pmod{8}$.

Proposition 9.3. Let $m, n \geq 0$ and L be a unimodular lattice of rank n . Then $I_m \perp L$ is exceptional if, and only if, L is exceptional and $m + \rho(n) < 8$.

The exceptional unimodular lattices of rank $n \equiv 0 \pmod{8}$ are just the even unimodular lattices. Also, there is clearly no exceptional unimodular lattice of rank n with $\rho(n) = 1$ and $r_1(L) = 0$. We will say more about the cases $\rho(n) > 1$ below. For $L \in \mathcal{L}_n$ we denote by $\text{Exc}(L)$ the set of $e \in \text{Char}(L)$

with $e.e = \rho(n)$. Proposition 9.5 below gives information on $\text{Exc}(L)$ when $2 \leq \rho(n) \leq 4$. In order to prove it we recall a classical specificity of the case $n \equiv 4 \pmod{8}$.

Proposition 9.4. *Assume $L \in \mathcal{L}_n$ with $n \equiv 4 \pmod{8}$, and let M denote the largest even lattice in L (the "even part" of L). There are exactly two other unimodular lattices in \mathcal{L}_n with same even part M .*

Indeed, this immediately follows from $\text{res } M \simeq \text{res } D_4$ and Table 2.1. The two other lattices of the statement will be called the *companions* of L .

Proposition 9.5. *Assume $L \in \mathcal{L}_n$ is exceptional with $r_1(L) = 0$.*

- (i) *For $\rho(n) \in \{2, 3\}$, or $\rho(n) = 4$ and $r_2(L) = 0$, we have $|\text{Exc}(L)| = 2$.*
- (ii) *For $\rho(n) = 4$ we have $|\text{Exc}(L)| \leq 2n$.*

The ideas in the proof below are inspired from the proof of Prop. 4.1 in [BV01], which contains the special case $r_2(L) = 0$.

Proof. Assume $L \in \mathcal{L}_n$ with $\rho(n) = 4$ (but do not assume $r_1(L) = 0$ for the moment). Define M as even part of L , namely $M = M_2(L; \xi)$ for any $\xi \in \text{Char}(L)$, and denote by L' and L'' the two companions of L . Then we have $M^\sharp = L \cup L' \cup L''$, and the map $M^\sharp \rightarrow L, v \mapsto 2v$, defines bijections

$$(9.2) \quad R_1(M^\sharp) \xrightarrow{\sim} R_1(L) \coprod \text{Exc}(L) \quad \text{and} \quad R_1(L') \coprod R_1(L'') \xrightarrow{\sim} \text{Exc}(L).$$

Assume furthermore L exceptional. Up to renaming L' and L'' , we may thus assume $r_1(L') \neq 0$. But if any of L, L' and L'' has a norm 1 vector, then the orthogonal symmetry about this vector defines an isometry between the two other lattices (see [BV01, Prop. 2.3]). This shows $L \simeq L''$, and also that we have $L' \simeq L''$ in the case L has a norm 1 vector.

Assume first $L = I_r \perp L_0$ with $r \in \{1, 2\}$ and $r_1(L_0) = 0$. Then we have $L \simeq L' \simeq L''$ and thus $|\text{Exc}(L)| = 2r + 2r = 4r$. But the discussion before Prop. 9.3 shows $|\text{Exc}(L)| = 2^r \cdot |\text{Exc}(L_0)|$. This proves $|\text{Exc}(L_0)| = 4r/2^r = 2$, hence part (i) (for the lattice L_0) in the case $2 \leq \rho(n) \leq 3$.

Assume finally $r_1(L) = 0$. Then we have $r_1(L'') = 0$ and $L' \simeq I_k \perp N$ for some $1 \leq k \leq n$ and $r_1(N) = 0$. We have $|\text{Exc}(L)| = 2k$. Note that the norm 2 vectors of L, L' and M are the same. This shows $k = 1$ for $r_2(L) = 0$, and proves assertions (ii) and (i) for $\rho(n) = 4$. \square

Remark 9.6. (The case $\rho(n) = 4$) *Let $n \geq 1$ be an integer $\equiv 4 \pmod{8}$. Let \mathcal{A}_n be the groupoid of exceptional lattices $A \in \mathcal{L}_n$ with $r_1(A) = 0$, and let \mathcal{B}_n be the groupoid of non-exceptional lattices $B \in \mathcal{L}_n$ with $r_1(B) \neq 0$. The last paragraph in the proof above shows that each L in \mathcal{A}_n has a unique companion $L' \in \mathcal{L}_n$ with $r_1(L') \neq 0$, necessarily non-exceptional. Better, $L \mapsto L'$ defines a natural functor $\mathcal{A}_n \rightarrow \mathcal{B}_n$ which induces a bijection on isometry classes on both sides, and satisfies the following properties : $|\text{Exc}(L)| = r_1(L')$, $R_2(L) = R_2(L')$, and $O(L)$ is an index 2 subgroup of $O(L')$.*

9.7. Mass formulae for $2 \leq \rho(n) \leq 7$.

In the cases $2 \leq \rho(n) \leq 7$, the exceptional unimodular lattices are related to the genus¹⁹ \mathcal{G}_n of even Euclidean lattices of rank $n - 1$ and determinant $\rho(n)$. This is presumably quite classical, but we briefly recall how in this section, and use this to derive a few interesting mass formulae.

For $L \in \mathcal{L}_n$ and $e \in \text{Exc}(L)$ we denote by $L(e)$ the orthogonal of e in L . This is an even lattice as $e \in \text{Char}(L)$, with determinant $e.e = \rho(n)$ as L is unimodular, so we have $L(e) \in \mathcal{G}_n$. Also, $\mathbb{Z}e$ is saturated in L . Set $\delta(n) = 1$ if $\rho(n) = 2$, and 2 otherwise.

Lemma 9.8. *Assume $2 \leq \rho(n) \leq 7$, let $N \in \mathcal{G}_n$ and set $M = N \perp \mathbb{Z}e$ with $e.e = \rho(n)$. There are $\delta(n)$ integral unimodular lattices L containing M and in which $\mathbb{Z}e$ is saturated, and they are permuted transitively by $1 \times \text{O}(\mathbb{Z}e) \simeq \mathbb{Z}/2$.*

Proof. The finite bilinear space $V := \text{Res } \mathbb{Z}e$ is isometric to \mathbb{Z}/r equipped with the \mathbb{Q}/\mathbb{Z} -valued bilinear form $(i, j) \mapsto \frac{ij}{r} \pmod{\mathbb{Z}}$, and is isometric to $-\text{res } N$. By Sect. 2 (iii), the unimodular lattices L of the statement naturally correspond to the bilinear Lagrangians in $\text{res } M \simeq -V \perp V$ which are transversal to $0 \perp V$. We conclude as V is cyclic of order $r = \rho(n)$ and the only solutions to $x^2 = 1$ in \mathbb{Z}/r are the $\delta(n)$ elements $x = \pm 1$ for $2 \leq r \leq 7$. \square

For $L \in \mathcal{L}_n$ the group $\text{O}(L)$ naturally acts on $\text{Exc}(L)$, as well as on the quotient set $\text{Exc}(L)^\pm := \text{Exc}(L)/\{\pm \text{id}_L\}$. Let \mathcal{E}_n be the natural groupoid whose objects are the pairs $(L, \pm e)$ with $L \in \mathcal{L}_n$ and $\pm e \in \text{Exc}(L)^\pm$. (For $\rho(n) = 2, 3$, Proposition 9.5 shows that the datum of $\pm e$ is unique, hence superfluous). Lemma 9.8 shows:

Proposition 9.9. *The natural functor $F : \mathcal{E}_n \rightarrow \mathcal{G}_n$, $(L, \pm e) \mapsto L(e)$, is essentially surjective, and for all $(L, \pm e)$ in \mathcal{E}_n the natural morphism $\text{O}(L, \pm e) \rightarrow \text{O}(L(e))$ is surjective and $\delta(n)$ to 1.*

For an (isomorphism class of) ADE root system R , let us denote by \mathcal{E}_n^R the subgroupoid of pairs (L, e) in \mathcal{E}_n with $\text{R}_2(L(e)) \simeq R$.

Corollary 9.10. *Assume $2 \leq \rho(n) \leq 7$. Let L_1, \dots, L_h be representatives for the isometry classes of exceptional unimodular lattices of rank n .*

(i) *We have $\frac{1}{\delta(n)} \text{mass } \mathcal{G}_n = \text{mass } \mathcal{E}_n = \sum_{i=1}^h |\text{Exc}(L_i)^\pm| \text{mass}(L_i)$.*

(ii) *For each ADE root system R , we have $\frac{1}{\delta(n)} \text{mass } \mathcal{G}_n(R) = \text{mass } \mathcal{E}_n^R$.*

¹⁹These lattices form a single genus by [CS99, Chap. 15 p. 387]. An example is given by the root lattice $A_{r-1} \perp E_8^{\frac{n-r}{8}}$, with $r = \rho(n)$. For all such lattices L , we have in particular $\text{res } L \simeq \text{res } A_{r-1}$ (a cyclic group of order r).

Proof. The first equality in (i), as well as (ii), follow from Proposition 9.9. The second equality in (i) follows from the obvious equivalence of groupoids $\mathcal{E}_n \simeq \coprod_{i=1}^h [\text{Exc}(L_i)^\pm / \text{O}(L_i)]$ (see Sect. 6 for several similar arguments). \square

Remark 9.11. (Determination of the terms $\text{mass } \mathcal{G}_n(R)$) *Assume $\rho(n) = 2, 3, 4, 5$ or 6 , and set $R_0 = \mathbf{E}_7, \mathbf{E}_6, \mathbf{D}_5, \mathbf{A}_4$ or $\mathbf{A}_1\mathbf{A}_2$ respectively. Then we have $\text{res } \mathbf{A}_{\rho(n)-1} \simeq -\text{res } R_0$. In particular, \mathcal{G}_n coincides with the genus \mathcal{G} in the statement of Proposition 6.8, with $m := 9 - \rho(n)$. Assuming furthermore $n \leq 30$, we may thus deduce $\text{mass } \mathcal{G}_n(R)$ from that proposition and [KIN03].*

Example 9.12. (The cases $n = 26$ and 27) *For such an n and an exceptional lattice $L \in \mathcal{L}_n$ with $r_1(L) = 0$, we have $|\text{Exc}(L)| = 2$ by Prop. 9.5.*

(i) *The genus \mathcal{G}_{26} of even lattices of rank 25 and determinant 2 has been determined by Borcherds in [BOR00]. There are 121 such lattices: 24 of the form $\mathbf{A}_1 \perp N$ with N a Niemeier lattice, and 97 with a dual lattice with minimum $> 1/2$. Accordingly, the corresponding 26 dimensional exceptional lattice L is either $\mathbf{I}_1 \perp N$, or satisfies $r_1(L) = 0$ and $\mathbf{R}_2(L(e)) \coprod \{\pm e\} = \mathbf{R}_2(L)$.*

(ii) *The genus \mathcal{G}_{27} of even lattices of rank 26 and determinant 3 has also been determined by Borcherds [BOR84], up to a few indeterminacies that were settled by Mégarbané in [MEG18]. They are 678 such lattices. Also, for $(L, \pm e)$ in \mathcal{E}_{26} , we easily check the equivalences between²⁰: (a) $r_1(L) = 0$, (b) there is no $v \in L(e)^\sharp$ with $v.v = 2/3$, (c) $\mathbf{R}_2(L(e)) = \mathbf{R}_2(L)$.*

(iii) *The genus \mathcal{G}_{28} of even lattices of rank 27 and determinant 4 is also denoted \mathcal{D}_{28} in Sect. 6.3 and thus is easily deduced from \mathcal{L}_{27} . See also Remark 9.6 for yet another approach.*

9.13. Visible exceptional characteristic vectors

Our main aim now is to explain how to produce d -neighbors of \mathbf{I}_n which are exceptional.

Proposition 9.14. (Visible characteristic vectors) *Let L be an odd unimodular lattice of rank n , d be an even integer, $x \in L$ be a d -isotropic vector, $\epsilon \in \{0, 1\}$, and $N := \mathbf{N}_d(L; x; \epsilon)$ the associated d -neighbor of L . An element $e \in L$ is in $\text{Char}(N)$ if, and only if, the following properties are satisfied:*

(i) $x.e \equiv 0 \pmod{d}$,

(ii) either $e \in \text{Char}(L)$, or $\exists \xi \in \text{Char}(L)$ such that $x \equiv \xi - e \pmod{2L}$,

²⁰For $v \in L(e)^\sharp$ with $v.v = 2/3$, then $\pm v \pm e/3$ has norm 1 and $\pm v \pm 2e/3$ has norm 2.

$$(iii) \frac{x.e}{d} \equiv \frac{x.x}{2d} + \epsilon(1 + \frac{d}{2}) \pmod{2}.$$

Proof. Set $M = L \cap N$. By definition, we have $e \in M$ if and only if (i) holds. We first check that for $e \in M$, condition (ii) is equivalent to $m.m \equiv e.m \pmod{2}$ for all $m \in M$. Fix $\xi \in \text{Char}(L)$. For $m \in M$ we have $m.m \equiv \xi.m \pmod{2}$. We may thus assume $e \notin \text{Char}(L)$. The condition $(\xi - e).m \equiv 0 \pmod{2}$ for all $m \in M$ amounts to ask that the kernel $M = M_d(x)$ of the linear form $L \rightarrow \mathbb{Z}/d, v \mapsto v \cdot x \pmod{d}$, is included in the kernel H of the nonzero linear form $L \rightarrow \mathbb{Z}/2, v \mapsto v \cdot (\xi - e) \pmod{2}$. As L/M is cyclic, the unique index 2 lattice of L containing M is $M_2(x)$, so we have $H = M_2(x)$, so that condition is equivalent to $x \equiv \xi - e \pmod{2}$.

By definition of N , we have $N = M + \mathbb{Z}\frac{\tilde{x}}{d}$ with $\tilde{x} = x + dr y$, $r = -\frac{x.x}{2d} + \epsilon d/2 \in \mathbb{Z}$ and $y \in L$ with $y.x \equiv 1 \pmod{d}$ (see Remark 3.3). It only remains to check that condition (iii) is equivalent to $e \cdot \frac{\tilde{x}}{d} \equiv \frac{\tilde{x}}{d} \cdot \frac{\tilde{x}}{d} \pmod{2}$. We have

$$\tilde{x}.\tilde{x} \equiv x.x + 2dr x.y + d^2 r^2 y.y \equiv \epsilon d^2 + d^2 r y.y \pmod{2d^2},$$

$$d\tilde{x}.e \equiv dx.e + d^2 r y.e \pmod{2d^2}.$$

Using $y.y - y.e \equiv (\xi - e).y \equiv x.y \equiv 1 \pmod{2}$ we obtain

$$\tilde{x}.\tilde{x} - d\tilde{x}.e \equiv \epsilon d^2 + d^2 r + dx.e \equiv \epsilon d^2 (1+d/2) - d \frac{x.x}{2} + dx.e \pmod{2d^2},$$

and conclude the proof. \square

Remark 9.15. Part (iii) implies $2x.e \equiv x.x \pmod{8}$, and then $(x - 2e).x \equiv 0 \pmod{8}$. If we write $x = e + \xi$ with $\xi \in \text{Char}(L)$, we deduce the congruence $e.e \equiv \xi.\xi \pmod{8}$, in agreement with Formula (9.1).

9.16. The case $L = I_n$

We now discuss the special case $L = I_n$. Fix an integer $1 \leq r < n$ with $n \equiv r \pmod{8}$ and consider the element $e \in \mathbb{Z}^n$ defined by

$$e = (\underbrace{0, \dots, 0}_{n-r}, \underbrace{1, \dots, 1}_r), \text{ with } e.e = r.$$

Fix an even integer d , a d -isotropic $x \in \mathbb{Z}^n$ and $\epsilon \in \{0, 1\}$. Then conditions (i), (ii) and (iii) on (x, ϵ) in Proposition 9.14 take the following forms :

$$(i) \sum_{n-r < i \leq n} x_i \equiv 0 \pmod{d},$$

$$(ii) x_i \text{ is odd for } i \leq n - r, \text{ and } x_i \text{ is even for } i > n - r,$$

$$(iii) 2 \sum_{n-r < i \leq n} x_i \equiv \sum_{i=1}^n x_i^2 + d\epsilon(2 + d) \pmod{4d}.$$

(For (ii), just use that $\xi = (1, 1, \dots, 1)$ is in $\text{Char}(\mathbb{I}_n)$). These equations do have many solutions (x, ϵ) for sufficiently large d , and for such an (x, ϵ) the associated d -neighbor $N_d(x; \epsilon)$ has the concrete vector e as a characteristic vector of norm r . The special case $r < 8$ do lead to constructions of exceptional lattices. Note that these choices of x also allow to prescribe the visible root system, as well as visible isometries, to some extent. If we furthermore impose that the x_i are distinct mod d for $n - r < i$, the visible root system of $N_d(x; \epsilon)$ is a subroot system of e^\perp .

Example 9.17. *In the introduction, we considered the problem of finding all unimodular lattices of rank 26 with root system $10 \mathbf{A}_1$. Among the 7 such lattices, a single one is exceptional, namely the last, and most complicated to find there!, of Table 1.3. This lattice L is straightforward to find by the method above. For instance, using the visible root system $2\mathbf{A}_1 \mathbf{D}_2 \simeq 4\mathbf{A}_1$ we immediately find $L \simeq N_{92}(x)$ with $x = (1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 46, 46)$.*

9.18. Application : the exceptional unimodular lattices of rank ≤ 27

Let E_n denote the set of isometry classes of exceptional unimodular lattices L of rank n with $r_1(L) = 0$. As explained after Prop. 9.3, we have $|E_{24}| = 24$ (the Niemeier lattices) and $E_{25} = \emptyset$. Moreover, by Example 9.12 we know:

Proposition 9.19. *We have $|E_{26}| = 97$ and $|E_{27}| = 557$.*

Most of these exceptional lattices are found as d -neighbors of \mathbb{I}_n using our general algorithms without any specific efforts, but for a few of them it is much more efficient to use the isotropic lines described in §9.13, and prescribing suitable visible root systems. We already gave an example of such a situation in dimension 26 (Example 9.17). In Table 9.1 below, we give a few examples of exceptional unimodular lattices of dimension 27 with no norm 1 vectors obtained by this method:

R_2	d	$x \in \mathbb{Z}^{27}$	ϵ	reduced mass
$6\mathbf{A}_1$	70	$(1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 19, 21, 23, 23, 25, 25, 27, 27, 29, 31, 33, 33, 35, 4, 32, 34)$	1	1/23040
$3\mathbf{A}_1 \mathbf{A}_2$	74	$(1, 1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 17, 19, 19, 21, 23, 25, 27, 29, 29, 31, 33, 35, 37, 8, 32, 34)$	0	1/483840
$3\mathbf{A}_1$	82	$(1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 29, 31, 31, 33, 35, 37, 39, 41, 20, 24, 38)$	0	1/1512000
\mathbf{A}_2	100	$(1, 1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37, 39, 41, 43, 45, 2, 48, 50)$	0	1/277136640
\mathbf{A}_3	94	$(1, 1, 1, 1, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 47, 4, 44, 46)$	0	1/489646080
\emptyset	96	$(1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 30, 32, 34)$	1	1/1268047872

Table 9.1: A few rank 27 exceptional unimodular lattices of the form $N_d(x; \epsilon)$.

10. A few more examples in dimension 26

In this section, we illustrate our methods by studying a few more examples. We consider first, in dimension 26, the root systems in Table 10 below. In the end, they will turn out to be exactly those with at least 7 isometry classes of lattices.

R	$2\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 2\mathbf{A}_4$	$4\mathbf{A}_1 4\mathbf{A}_2 2\mathbf{A}_3$	$3\mathbf{A}_1 3\mathbf{A}_2 2\mathbf{A}_3 \mathbf{A}_4$	$5\mathbf{A}_1 3\mathbf{A}_2 2\mathbf{A}_3$	$7\mathbf{A}_1 3\mathbf{A}_2 \mathbf{A}_3$
reduced mass	6	77/16	6	4	8/3
#lattices	16	15	15	13	12
R	$6\mathbf{A}_1 4\mathbf{A}_2 \mathbf{A}_3$	$2\mathbf{A}_1 2\mathbf{A}_2 3\mathbf{A}_3 \mathbf{A}_4$	$6\mathbf{A}_1 4\mathbf{A}_2$	$4\mathbf{A}_1 2\mathbf{A}_2 4\mathbf{A}_3$	$3\mathbf{A}_1 3\mathbf{A}_2 3\mathbf{A}_3$
reduced mass	161/48	5	545/576	31/16	15/4
#lattices	11	11	10	10	10
R	$\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 \mathbf{A}_4 \mathbf{A}_5$	$8\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3$	$6\mathbf{A}_1 2\mathbf{A}_2 3\mathbf{A}_3$	$5\mathbf{A}_1 5\mathbf{A}_2 \mathbf{A}_3$	$5\mathbf{A}_1 3\mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4$
reduced mass	17/4	85/64	23/12	5/2	13/4
#lattices	9	9	9	9	9
R	$4\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 \mathbf{A}_4$	$8\mathbf{A}_1 2\mathbf{A}_2$	$4\mathbf{A}_1 4\mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4$	$3\mathbf{A}_1 5\mathbf{A}_2 2\mathbf{A}_3$	$3\mathbf{A}_1 2\mathbf{A}_2 3\mathbf{A}_3 \mathbf{A}_5$
reduced mass	4	491/1344	7/2	9/4	7/4
#lattices	9	8	8	8	7
R	$8\mathbf{A}_1 4\mathbf{A}_2$	$6\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3$	$5\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 \mathbf{A}_5$	$2\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 2\mathbf{A}_4 \mathbf{A}_5$	$2\mathbf{A}_1 4\mathbf{A}_2 2\mathbf{A}_3 \mathbf{A}_4$
reduced mass	41/32	15/8	7/4	3	3
#lattices	8	7	7	7	7
R	$2\mathbf{A}_1 3\mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4 \mathbf{A}_5$	$10\mathbf{A}_1$	$\mathbf{A}_1 2\mathbf{A}_2 \mathbf{A}_3 2\mathbf{A}_4 \mathbf{A}_5$	$9\mathbf{A}_1 3\mathbf{A}_2$	$6\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 \mathbf{A}_4$
reduced mass	13/4	4424507/116121600	11/4	15/16	7/4
#lattices	7	7	6	6	6

Table 10.1: The root systems R such that there are at least 7 isometry classes in X_{26} with root system R and no norm 1 vectors (and a few others).

As a first example, consider the root system $R := 2\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 2\mathbf{A}_4$. In this case the reduced mass is 6, hence a priori R is the root system of at least 12 lattices in X_{26} . By searching for isotropic lines with visible root system $\mathbf{A}_1 2\mathbf{A}_2 2\mathbf{A}_3 2\mathbf{A}_4$ (one \mathbf{A}_1 less!), we do quickly find 16 lattices for $16 \leq d \leq 25$: 8 with reduced mass 1/2 and 8 with reduced mass 1/4, the last one being²¹

$$N_{25}(x) \text{ with } x = (1^5, 2^3, 4^4, 5^2, 6^5, 11^3, 12^4).$$

As already said, all these 16 lattices are distinguished by their invariant δ_2 . However, they are not distinguished by δ_1 . In fact, if we rather choose the invariant δ_1 in the search above, we stop finding new lattices for $d \geq 26$ despite running several thousands of isotropic lines whose associated neighbor has root system R . At this point, the remaining mass is 3/4, which represents $(3/4)/6 = 12.5\%$ of the total mass, so if we believe in Theorems B & E (say,

²¹That is, $x := (1, 1, 1, 1, 1, 2, 2, 2, 4, 4, 4, 4, 5, 5, 6, 6, 6, 6, 6, 11, 11, 11, 12, 12, 12, 12)$.

ignoring the unknown terms $\text{emb}(R, -)$) we should have found instead hundreds of times the missing lattices. This is a very strong argument that the chosen invariant is wrong and has to be refined. We have used this strategy several times during our search before we discover fine enough invariants !

Most of the isometry classes in X_{26} with root system as in Table 10 are not especially hard to find. The most complicated is the case $R = 10\mathbf{A}_1$, which is why we discussed it at length in the introduction. One lattice with root system $8\mathbf{A}_1 2\mathbf{A}_2$ has reduced mass $1/1344$ and is not immediate to find. We immediately find it using a visible isometry of order 7 and visible root system $7\mathbf{A}_1$, and more precisely the $29 \cdot 27$ -isotropic vector $x \in \mathbb{Z}^{26}$ with

$$x \equiv (1^{14}, 2^7, 3, 5, 9, 10, 11) \pmod{27}$$

$$x \equiv (1, t, t^2, t^3, t^4, t^5, t^6, 1, t, t^2, t^3, t^4, t^5, t^6, 1, t, t^2, t^3, t^4, t^5, t^6, 0, 0, 0, 0, 0) \pmod{29},$$

for $t \equiv 16 \pmod{29}$ (an element of order 7).

We now give another example, namely the case of the root system $R = 22\mathbf{A}_1 \mathbf{D}_4$. The reduced mass is $53/60480$. This is a typical case where we cannot choose anything very close to R as a visible root system. Nevertheless, for even d then $11\mathbf{A}_1 \mathbf{D}_4$ is a possible visible root system a priori. It amounts to choose d -isotropic vectors $x \in \mathbb{Z}^{26}$ having 4 coordinates equal to $d/2$, as well as 11 other pairs of equal coordinates. This forces $d \geq 24$. And indeed, we quickly find the two following lattices, with respective masses $1/1152$ and $1/120960$, and conclude :

$$N_{30}(1^2, 2^2, 3^2, 4^2, 7^2, 8^2, 9^2, 10^2, 11^2, 13^2, 14^2, 15^4)$$

$$N_{46}(1^2, 5^2, 7^2, 9^2, 11^2, 13^2, 15^2, 17^2, 19^2, 20^2, 21^2, 23^4).$$

Both lattices can also be understood with the help of X_{22} . For instance, the orthogonal of the \mathbf{D}_4 in the latter is actually the even sublattice of the unique lattice in X_{22} with root system $22\mathbf{A}_1$ (and whose even part is also the orthogonal of some $2\mathbf{A}_1$ inside the Niemeier lattice with root system $24\mathbf{A}_1$!).

11. A few constructions of lattices in neighbor form

First, it is straightforward to “add I_m ” to unimodular lattices in neighbor forms.

Lemma 11.1. *Let $x \in \mathbb{Z}^n$ be d -isotropic and define $y \in \mathbb{Z}^{n+m}$ by $y_i = x_i$ for $i \leq n$, $y_i = 0$ otherwise. Then y is d -isotropic, and we have $N_d(y) = N_d(x) \perp I_m$ if d is odd, and $N_d(y; \epsilon) = N_d(x; \epsilon) \perp I_m$ if d is even and $\epsilon \in \{0, 1\}$.*

Proof. We have $M_d(y) = M_d(x) \perp I_m$. Consider a d -neighbor $N_d(x')$ associated with x , and define $y' \in \mathbb{Z}^{n+m}$ by $y'_i = x'_i$ for $i \leq n$, $y'_i = 0$ otherwise. As x (resp. x') coincides with y (resp. y') inside \mathbb{Z}^{n+m} , we have $N_d(y') = N_d(x') \perp I_m$. \square

Lemma 11.2. *Let $L, N \in \mathcal{L}_n$. Assume that N is a d -neighbor of \mathbf{I}_n associated to the d -isotropic vector $x \in \mathbb{Z}^n$, and that L is a d' -neighbor of N , with $\gcd(d, d') = 1$. Then L is a dd' -neighbor of \mathbf{I}_n associated to a dd' -isotropic vector $y \in \mathbb{Z}^n$ satisfying $y_i \equiv x_i \pmod{d}$ for all $i = 1, \dots, n$.*

Proof. Using $\gcd(d, d') = 1$, and localizing at primes dividing dd' , one readily observes $L \cap \mathbf{I}_n \subset N$, as well as $\mathbf{I}_n / (L \cap \mathbf{I}_n) \simeq \mathbb{Z}/dd'$. Let $\ell \subset \mathbf{I}_n \otimes \mathbb{Z}/dd'$ be the dd' -isotropic line satisfying $L \cap \mathbf{I}_n = M_{dd'}(\ell)$. By the inclusion $M_{dd'}(\ell) \subset N \cap \mathbf{I}_n = M_d(x)$, the reduction mod d of ℓ has to be $l(x)$, and the statement follows. \square

Our aim now is to address the following more interesting :

Problem : (Addition of \mathbf{D}_m) *Assume we know all (isometry classes of) unimodular lattices $L \in \mathcal{L}_n$ with given root system R , and in neighbor form. Choose $m \geq 2$. Find neighbor forms for the unimodular lattices $U \in \mathcal{L}_{n+m}$ with root system $R \amalg \mathbf{D}_m$.*

Lemma 11.3. *Assume $U \in \mathcal{L}_{n+m}$ contains $D := \{0\} \times \mathbf{D}_m$ as a saturated subgroup ($m \geq 2$), and set $L_0 = D^\perp \cap U$. Then there is a unique²² $L \in \mathcal{L}_n$ containing L_0 with index 2 and such that U is a 2-neighbor of $L \perp \mathbf{I}_m$.*

Proof. By §2 (iii), the lattice U is the inverse image of some (order 4) Lagrangian $I \subset \text{res } L_0 \perp \text{res } \mathbf{D}_m$ that is transversal to both summands. In particular, I contains a unique element of the form $a + b$, where b is a generator of $\mathbf{I}_m / \mathbf{D}_m \simeq \mathbb{Z}/2$ and $a \in \text{res } L_0$, necessarily satisfying $a \cdot a \equiv 0 \pmod{\mathbb{Z}}$ and $2a = 0$. The inverse image of $\mathbb{Z}/2 a$ in L_0^\sharp is thus a unimodular lattice containing L_0 that we denote by L . By construction, $U \cap (L \perp \mathbf{I}_m)$ contains $L_0 \cap \mathbf{D}_m$ with index 2, hence has index 2 in $L \perp \mathbf{I}_m$. \square

Proposition 11.4. *Assume $U \in \mathcal{L}_{n+m}$ contains $D := \{0\} \times \mathbf{D}_m$ as a saturated subgroup ($m \geq 2$). Let L be the associated rank n unimodular lattice given by Lemma 11.3. Assume we have $L \simeq \mathbf{N}_d(x)$ for some odd integer d and a d -isotropic $x \in \mathbb{Z}^n$. Then we have $U \simeq \mathbf{N}_{2d}(y)$ for some $2d$ -isotropic $y \in \mathbb{Z}^{n+m}$ satisfying $y_i \equiv x_i \pmod{d}$ for $i = 1, \dots, n$, and $y_i \equiv d \pmod{2d}$ for $i = n+1, \dots, n+m$.*

Proof. By Lemma 11.3, U is isometric to some 2-neighbor U' of $\mathbf{N}_d(x) \perp \mathbf{I}_m$ in which the natural \mathbf{D}_m is saturated. By Lemma 11.1, we have $\mathbf{N}_d(x) \perp \mathbf{I}_m = \mathbf{N}_d(\xi)$ with $\xi_i = x_i$ for $i = 1, \dots, n$, and $\xi_i = 0$ for $i = n+1, \dots, n+m$. By Lemma 11.2, we have thus $U' = \mathbf{N}_{2d}(y)$ with $y_i \equiv \xi_i$ for $i = 1, \dots, n+m$. As U' contains the natural \mathbf{D}_m , we also have, for all $i, j > n$, $y_i \equiv y_j \pmod{2d}$ and $y_i \equiv 0 \pmod{d}$. If we have $y_i \equiv 0 \pmod{2d}$ for $i > n$, then U' contains the natural \mathbf{I}_m , contradicting the saturation of \mathbf{D}_m . \square

²²Actually, we have $\text{res } L_0 \simeq -\text{res } \mathbf{D}_m$ so there is a unique unimodular lattice L containing L_0 of index 2, unless we have $m \equiv 0 \pmod{4}$ in which case there are three.

Remark 11.5. *If x as above is given, there are at most 2^n choices for $y \bmod 2d$, which may be a lot. In practice, it is often useful to restrict the search for y with visible root system $R^\vee \coprod D_m$, where R^\vee denotes the visible root system of $N_d(x)$. A concrete example of such a search was given in the introduction : this is how the last two lattices of Table 1.3 have been found (case $m = 2$, we have $\mathbf{D}_2 \simeq 2\mathbf{A}_1$.) We have used this method in very many cases during our proof of Theorem A.*

We end with a proposition the case $n \equiv 0 \pmod 4$ (see Proposition. 9.4).

Proposition 11.6. *Assume $n \equiv 4 \pmod 8$ and $x \in \mathbb{Z}^n$ is d -isotropic with d odd. Assume $y \in \mathbb{Z}^n$ satisfies $y_i \equiv x_i \pmod d$ and $y_i \equiv 1 \pmod 2$ for all $i = 1, \dots, n$. Then the two companions of $N_d(x)$ are the $N_{2d}(y)^\pm$.*

Proof. Each companion L of $N_d(x)$ is a $2d$ -neighbors of I_n by Lemma 11.2, so it is enough to show $L \supset M_{2d}(y)$. But the even part of I_n is $M_2(1^n)$, so that of $N_d(x)$ (hence of L) is $N_d(x) \cap M_2(1^n)$ since d is odd. We deduce $L \supset N_d(x) \cap M_2(1^n) \supset M_d(x) \cap M_2(1^n) = M_{2d}(y)$. \square

12. The Jordan-Hölder factors of $O(L)$ for $L \in \mathcal{L}_n$ and $n \leq 27$

In this section, we discuss the Jordan-Hölder factors of $O(L)$ for $L \in \mathcal{L}_n$ and $n \leq 27$. It is enough to study the (usually smaller) reduced isometry groups $O(L)^{\text{red}}$ (see §4.5). For this, we view $O(L)^{\text{red}}$ as a finite permutation groups of suitable small norm vectors of L , of which the Plesken-Souvignier algorithm provides generators (see Remark 4.6), and then use the permutation groups algorithms in [GAP]. For $n \leq 25$, the output is that the non-abelian simple groups appearing as a Jordan-Hölder factor of some $O(L)^{\text{red}}$ are

$\text{Co}_1, \text{Co}_2, \text{Co}_3, \text{HS}, \text{M}_{24}, \text{M}_{22}, \text{M}_{12}, \text{U}_6(2), \text{A}_8, \text{A}_7, \text{A}_6, \text{A}_5$ and $\text{L}_2(7)$.

In dimension 26, there are only 39 lattices L in X_{26} with $r_1(L) = 0$ and such that $|O(L)^{\text{red}}|$ is both $\equiv 0 \pmod 4$ and not of the form $p^a q^b$ with p, q primes. We checked that in each of these cases $O(L)^{\text{red}}$ is indeed non solvable²³. For exactly 6 lattices of these 39 lattices, we obtain the following “new” Jordan-Hölder factors, not appearing for unimodular lattices of rank < 26 : $\text{A}_9, \text{O}_5(5), \text{L}_3(4), \text{L}_3(3)$ and $\text{L}_2(8)$ (twice).

²³With a unique non abelian simple factor, except for the lattice with root system $10\mathbf{A}_1$ and reduced mass $1/7372800$, which contains two factors isomorphic to A_5 .

$R_2(L)$	$10\mathbf{A}_1$	\emptyset	$22\mathbf{A}_1 \mathbf{D}_4$	$26\mathbf{A}_1$	$\mathbf{A}_1 9\mathbf{A}_2$ & $9\mathbf{A}_1 \mathbf{A}_2$
$ O(L)^{\text{red}} $	92897280	18720000	120960	11232	3024
$O(L)^{\text{red}}$	$C_2.(C_2^8.A_9)$	$O_5(5) : C_4$	$(L_3(4) : C_3) : C_2$	$L_3(3) : C_2$	$C_2 \times (L_2(8) : C_3)$

Table 12.1: The 6 lattices in X_{26} whose reduced isometry group has a “new” Jordan-Hölder factor, as described by GAP’s `StructureDescription` function.

In a similar manner, there are only 74 lattices in X_{27} with no norm 1 vectors and whose reduced isometry group has order both $\equiv 0 \pmod{4}$ and not of the form $p^a q^b$ with p, q primes. Their reduced isometry groups are indeed non solvable in all cases, with a unique non abelian factor. Again, for exactly 6 lattices, a new Jordan-Hölder factor appears, namely : $U_6(2)$, ${}^3D_4(2)$, $U_4(2)$, $O_5(3)$, $U_3(5)$, M_{11} and $PSL_2(11)$.

\mathbf{D}_4	\emptyset	\emptyset	$3\mathbf{A}_1$	$\mathbf{A}_1 \mathbf{A}_3$	$11\mathbf{A}_1 \mathbf{A}_4$
55180984320	634023936	1658880	756000	7920	660
$C_2 \times ((U_6(2) : C_3) : C_2)$	$C_2 \times ({}^3D_4(2) : C_3)$	$C_2 \times (C_2^6.O_5(3))$	$C_2 \times ((U_3(5) : C_3) : C_2)$	$C_2 \times M_{11}$	$C_2 \times L_2(11)$

Table 12.2: The 6 lattices L in X_{27} whose reduced isometry group has a “new” Jordan-Hölder factor (same format as Table 12.1).

13. Proof of Theorem B.

We now explain how to modify the proof of Thm. A in [CHE22] in order to deal as well with the case of cyclic d -neighbors with d non necessarily prime. Note that the result is obvious for $n \leq 4$, since we know $|X_n| = 1$ in this case. To keep the discussion short we freely use the notations in [CHE22].

Theorem 13.1. *Let L be an integral lattice in \mathbb{R}^n with $n > 4$. Assume $\mathcal{G} = \text{Gen}(L)$ is a single spinor genus, choose L' in \mathcal{G} , and for d prime to $2 \det L$ denote by $N_d(L, L')$ the number of cyclic d -neighbors of L which are isometric to L' . Fix $\epsilon > 0$. Then for d prime to $2 \det L$, we have*

$$(13.1) \quad \frac{N_d(L, L')}{|C_L(\mathbb{Z}/d)|} = \frac{1/|O(L')|}{m(\mathcal{G})} + O\left(\frac{1}{d^{1-\epsilon}}\right) \text{ when } d \rightarrow \infty.$$

For this, we first generalize the discussion in [CHE22, §4] to p^α -neighbors. Fix p an odd prime and V a nondegenerate quadratic space over \mathbb{Q}_p . We assume $\dim V \geq 3$ and that the set $\mathcal{U}(V)$ of *unimodular* integral \mathbb{Z}_p -lattices in V is non empty, fix $L \in \mathcal{U}(V)$. For any integer $\alpha \geq 1$, the cyclic p^α -neighbors of L are the lattices $N \in \mathcal{U}(V)$ such that $L/(L \cap N)$ is cyclic

of order p^α . They form a subset $\mathcal{N}_{p^\alpha}(L)$ of $\mathcal{U}(V)$. A similar argument as in Corollary 3.6 shows that $\mathcal{N}_{p^\alpha}(L)$ is in natural bijection with $C_L(\mathbb{Z}/p^\alpha)$, and form a single $O(L)$ -orbit. Assuming $L = (\mathbb{Z}_p e \oplus \mathbb{Z}_p f) \perp M$ with M unimodular, $e.e = f.f = 0$ and $e.f = 1$ (this is always possible), then

$$(13.2) \quad (\mathbb{Z}_p p^\alpha e \oplus \mathbb{Z}_p p^{-\alpha} f) \perp M \in \mathcal{N}_{p^\alpha}(L).$$

We denote by T_{p^α} the element of the Hecke ring H_V defined, for $L \in \mathcal{U}(V)$, by $T_{p^\alpha} L = \sum_{N \in \mathcal{N}_{p^\alpha}(L)} N$. The key estimate is the following.

Proposition 13.2. *Let V be a non-degenerate quadratic space over \mathbb{Q}_p of dimension ≥ 5 , $L \in \mathcal{U}(V)$, and $\alpha \geq 1$ an integer. Let U a unitary irreducible unramified $\mathbb{C}[O(V)]$ -module, and $\lambda \in \mathbb{C}$ the eigenvalue of T_{p^α} on the line $U^{O(L_p)}$. If $\dim U > 1$ then we have $|\lambda| \leq |C_L(\mathbb{Z}/p^\alpha)| (\alpha + 1)^2 p^{-\alpha}$.*

Proof. The proof is the same as that of [CHE22, Prop. 6.5], up to replacing the subset $C \subset O(V)$ *loc. cit.* with the double coset $O(L)cO(L)$ of elements $g \in O(V)$ such that $g(L)$ is a cyclic p^α -neighbor of L . In the notation of that proof, and by Formula (13.2), we may take $c = \varepsilon_1^*(p^\alpha)$. Applying [OH02, Thm. 1.2], we obtain the inequality $|\langle ce, e \rangle| \leq \Xi_{\mathrm{PGL}_2(\mathbb{Q}_p)}(p^\alpha)^2$ with $\Xi_{\mathrm{PGL}_2(\mathbb{Q}_p)}(p^\alpha) = \frac{1}{p^{\alpha/2}} \frac{\alpha(p-1)+p+1}{p+1}$. \square

We now follow the arguments given in the beginning of [CHE22, §6]. For the application to Theorem 13.1 (“lattice case”) the finite set S *loc. cit.* is the set of primes dividing $D := 2 \det L$. For each integer d prime to D we have a natural global Hecke operator T_d corresponding to the cyclic d -neighbors, and generalizing those above when d is a prime power. For $d = d'd''$ with coprime d' and d'' , it satisfies $T_d = T_{d'}T_{d''}$, and its degree $c_V(d) := |C_L(\mathbb{Z}/d)|$ of course satisfies $c_V(d) = c_V(d')c_V(d'')$ as well. These operators pairwise commute and act in a diagonalisable way on the space of automorphic forms denoted $M(K)$ *loc. cit.* Fix a common eigenvector v in the subspace $M(K)$ and denote by $\lambda(d)$ the eigenvalue of $T_d/c_V(d)$ on v . If v is in the subspace $M(K)^0$ *loc. cit.*, and p is a prime not dividing D , Prop. 13.2 implies that for all $\alpha \geq 1$ we have $\lambda(p^\alpha) \leq (\alpha + 1)^2 p^{-\alpha}$. It follows that for each d prime to D we have $|\lambda(d)| \leq \sigma_0(d)^2 d^{-1}$, where $\sigma_0(n)$ denotes the number of divisors of the integer $n \geq 1$. It is well-known that, for any $\epsilon > 0$, we have $\sigma_0(n) = O(n^\epsilon)$ for $n \rightarrow \infty$. This proves $\lambda(d) = O(d^{2\epsilon-1})$ for $d \rightarrow \infty$, and the result follows. \square

Remark 13.3. (i) *If $\det L$ is odd, L is even, and if we restrict to even cyclic d -neighbors, then the statement (and its proof) also holds with $2 \det L$ replaced by $\det L$.*

(ii) *The cases $n = 3, 4$ could be handled as well using similar methods as in the second proof of Theorem 6.3 in [CHE22] (see the end of §6).*

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