Automorphic representations and the global Langlands group

LMS Sheffield Summer School

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Three lectures

- **Lecture 1: From classical to adelic automorphic forms.**

  Introduce automorphic forms, with an emphasis on GL_n over $\mathbb{Q}$, first explaining the fundamental questions in the classical setting, and then preparing for the adelic setting.

- **Lecture 2: Automorphic representations of $\mathrm{GL}_n(\mathbb{A})$.**

  Define discrete and cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, as well as their collection of Satake parameters.

- **Lecture 3: The global Langlands group.**

  Discuss the global Langlands conjectures via the conjectural Langlands group $\mathrm{L}_{\mathbb{Q}}$. Not only does it organize the theory into a short set of axioms about $\mathrm{L}_{\mathbb{Q}}$, but it suggests precise statements about automorphic forms which can be formulated, tested, and sometimes proved without mentioning $\mathrm{L}_{\mathbb{Q}}$.

Lecture 1

From classical to adelic automorphic forms

Harmonic analysis on $\Gamma\backslash G$

Let G be a locally compact (second countable) topological group and $\Gamma \subset G$ *a lattice*:

$$\Gamma \text{ discrete subgroup} \qquad \& \qquad \mathrm{vol}(\Gamma\backslash G) < \infty.$$

(with vol=Haar/count) The right regular action of G on $\mathrm{L}^2(\Gamma\backslash G)$ leads to the

Problem : decompose $\mathrm{L}^2(\Gamma\backslash G)$ into irreducible representations of G.

First example:

$$G = \mathbb{R}, \quad \Gamma = \mathbb{Z}, \quad \mathrm{L}^2(\mathbb{Z}\backslash\mathbb{R})$$

is classical Fourier analysis : $\mathrm{L}^2(\mathbb{Z}\backslash\mathbb{R}) \;=\; \widehat{\bigoplus_{n\in\mathbb{Z}}}\, \mathbb{C}\, e^{2i\pi\, n x}$ (orthogonal direct sum of unitary characters).

The main geometric examples

For this course the model examples are

$$G = \mathrm{SL}_n(\mathbb{R}), \qquad \Gamma = \mathrm{SL}_n(\mathbb{Z})$$

or (with slight preference for GL_n), the closely related quotients

$$\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R})/\mathbb{R}_{>0} = \mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R}) = \{\textit{lattices in } \mathbb{R}^n\}/\mathbb{R}_{>0}.$$

$$g \mapsto g^{-1}(\mathbb{Z}^n).$$

Proposition (reduction theory, Minkowski–Siegel) *These quotients have finite volume.*

Discrete and continuous spectrum

There is a general tautological G-stable orthogonal decomposition

$$\mathrm{L}^2(\Gamma\backslash G) = \mathrm{L}^2_{\mathrm{disc}}(\Gamma\backslash G) \perp \mathrm{L}^2_{\mathrm{cont}}(\Gamma\backslash G),$$

where

$$\mathrm{L}^2_{\mathrm{disc}}(\Gamma\backslash G) := \overline{\sum \{\text{irreducible, closed, G-subrepresentations of } \mathrm{L}^2(\Gamma\backslash G)\}}.$$

(Think about entirely continuous example $\mathrm{L}^2(\mathbb{R}) = \int_{\mathbb{R}} \mathbb{C}\, e^{ixy}\, dy$ given by Fourier analysis on $\mathbb{R}$.)

Proposition (Gelfand–Graev–Piatetski-Shapiro) *If $\Gamma\backslash G$ is compact, then:*

$$\mathrm{L}^2_{\mathrm{cont}} = 0, \qquad \mathrm{L}^2(\Gamma\backslash G) = \mathrm{L}^2_{\mathrm{disc}}(\Gamma\backslash G) = \widehat{\bigoplus}_{\pi\in\mathrm{Irr}_u(G)} \mathrm{M}(\pi)\otimes\pi,$$

with finite-dimensional multiplicity spaces $\mathrm{M}(\pi)$.

In *arithmetic non-compact cases* such as $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ we have $\mathrm{L}^2_{\mathrm{cont}} \neq 0$, but this decomposition of $\mathrm{L}^2_{\mathrm{disc}}(\Gamma\backslash G)$ remains true, with deeper proofs (GGPS, Harish-Chandra, Langlands, Selberg...).

What representations can appear?

$\mathrm{Irr}_u(G)$ = set of isom. classes of unitary irrep. of G (*unitary dual*).

$$\text{Which } \pi \in \mathrm{Irr}_u(G) \text{ occurs in } \mathrm{L}^2_{\mathrm{disc}}(\Gamma\backslash G), \text{ and with what multiplicity } \mathrm{m}(\pi) = \dim \mathrm{M}(\pi) \text{ ?}$$

For real (non-compact) reductive groups like $\mathrm{GL}_n(\mathbb{R})$, unitary dual is difficult to understand : long series of works by Harish-Chandra, Langlands, Knapp-Zuckerman and many others...

Technical digression: (*Harish-Chandra modules*) Since Harish-Chandra, one usually rather works with the larger, *entirely algebraic*, class of irreducible *admissible* $(\mathfrak{g}, K)$*-modules*. They have a nicer classification; unitarity is then a further, more delicate question. Here $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ and K is a fixed maximal compact subgroup of G. The $(\mathfrak{g}, K)$-module of an irreducible unitary rep. π is its (dense) subspace of *smooth and K-finite* vectors.

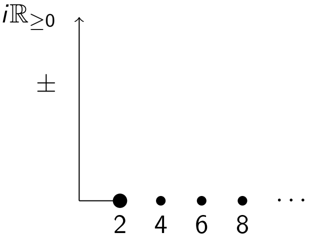
Unitary dual of $\mathrm{PGL}_2(\mathbb{R})$ (Bargmann)

Irreducible unitary representations of $\mathrm{PGL}_2(\mathbb{R})$:

$$\mathbf{1}, \qquad \mathrm{sgn} \circ \det,$$

$$\mathrm{PS}(s)^{\pm}, \qquad s \in i\mathbb{R}_{\geq 0} \cup [0,1[,$$

$$\mathrm{D}_k, \qquad k = 2, 4, 6, 8, \dots .$$



$\mathrm{PS}(s)^{\pm}$ = *principal/complementary* series, D_k = *discrete series*.

Other irreducible $(\mathfrak{g}, K)$-modules : $\mathrm{PS}(s)^{\pm}$ for all $s \in \mathbb{C} \setminus 2\mathbb{Z}+1$ (mod $s \mapsto -s$), finite dimensional rep.

Harmonic analysis on $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$ (Selberg)

In $\mathrm{L}^2_{\mathrm{disc}}(\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R}))$ we have:

- $\mathrm{M}(\mathbf{1}) \otimes \mathbf{1}$ is the space of constant functions, and $\mathrm{M}(\mathrm{sgn} \circ \det) = 0$;
- Modular cusp forms of weight k (viewed as lattice functions) generate D_k:
$$\mathrm{M}(D_k) \simeq S_k(\mathrm{SL}_2(\mathbb{Z}));$$
- $\mathrm{M}(\mathrm{PS}(s)^{\pm})$ identifies with the space of $\pm$-Maass forms with Laplace eigenvalue
$$\frac{1}{4} - \frac{s^2}{4}.$$

(WARNING : nonzero for only countably many (unknown) values of $s \to +i\infty$. Essentially the discrete spectrum of the Laplace operator on $\mathrm{L}^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$).

$\mathrm{L}^2_{\mathrm{cont}}(\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})) = \int_0^{\infty} \mathrm{PS}(it)^+ d\mu(t)$ (theory of *unitary Eisenstein series*).

Hecke correspondences on lattices 1

Arithmetic quotients come with extra symmetries: Hecke correspondences. Set

$$\mathcal{L}_n = \mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R})/\mathbb{R}_{>0} = \{\textit{lattices in } \mathbb{R}^n\}/\mathbb{R}_{>0}.$$

For a finite abelian group A and a *lattice function* $f : \mathcal{L}_n \longrightarrow \mathbb{C}$, then

$$T_A f(L) = \sum_{\substack{L' \subset L \\ L/L' \simeq A}} f(L')$$

is again a lattice function. Example : usual $\mathrm{T}_p = \mathrm{T}_{\mathbb{Z}/p\mathbb{Z}}$.

These operators T_A generate (additively) an abstract subring $\mathcal{H}_n$ of $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[\mathcal{L}_n])$.

Fact : $\mathcal{H}_n$ is a commutative ring.

Hecke correspondences on lattices 2

The ring $\mathcal{H}_n$ acts on $\mathrm{L}^2(\mathcal{L}_n)$ by normal endomorphisms, commuting with the right action of $\mathrm{GL}_n(\mathbb{R})$.

Hence every multiplicity space

$$\mathrm{M}(\pi) = \mathrm{Hom}_G(\pi, \mathrm{L}^2_{\mathrm{disc}}(\mathcal{L}_n))$$

becomes a (codiagonalizable) $\mathcal{H}_n$-module.

Refined problem: for $\pi \in \Pi_u(\mathrm{GL}_n(\mathbb{R}))$, understand the systems of Hecke eigenvalues in $\mathrm{M}(\pi)$.

– Trivial example (constant functions) : $\mathrm{T}_p(\mathbf{1}) = (1 + p + \cdots + p^{n-1})\mathbf{1}$.

– Well-known interesting eigenvalues in general : contains Fourier expansions of normalized eigenforms for $\mathrm{SL}_2(\mathbb{Z})$, such as Ramanujan's $\tau(n)$ and many more.

Remark : no discrete series for $\mathrm{PGL}_n(\mathbb{R})$ with $n > 2$. Situation closer to that of Maass forms for $n = 2$.

Why introduce adèles ?

Next step : switch to $\mathrm{L}^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/\mathbb{R}_{>0})$ as a rep. of the adèle group $\mathrm{GL}_n(\mathbb{A})$.

Why ?

- To get all the interesting systems of Hecke eigenvalues, need to allow (in the lattice function model) a generalization to lattices L equipped with level structures $L/NL \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^n$; cleaner in adelic setting as congruence subgroups and levels are built into the formalism;
- Hecke operators become a genuine group action at finite places and representations of p-adic groups $\mathrm{GL}_n(\mathbb{Q}_p)$ enter the picture;
- archimedean and non-archimedean places appear in the same object;
- the cuspidal subspace (not discussed so far) has its natural definition.

Drawback: slightly hides the elementary aspects of the main objects; this is why we started this way.

Brief review of the adele ring of $\mathbb{Q}$

The (commutative) ring of adeles of $\mathbb{Q}$ is

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f, \qquad \mathbb{A}_f = \left\{ (x_p) \in \prod_p \mathbb{Q}_p \ : \ x_p \in \mathbb{Z}_p \text{ for almost all } p \right\}.$$

Unique (additive) group topology on $\mathbb{A}_f$ such that the subring

$$\widehat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p$$

is open and equipped with its natural profinite topology. Then $\mathbb{A}_f$ and $\mathbb{A}$ are locally compact.

Proposition: The diagonal subgroup $\mathbb{Q} \subset \mathbb{A}$ is discrete and cocompact.

$$\mathbb{Q} \cap \Big( ]-1,1[\times \prod_p \mathbb{Z}_p \Big) \ = \ \{0\} \ \text{ and } \mathbb{A} = \mathbb{Q} + \Big([0,1] \times \prod_p \mathbb{Z}_p \Big)$$

Conclusion: $\mathbb{Q}$ is to $\mathbb{A}$ what $\mathbb{Z}$ is to $\mathbb{R}$.

Adelic groups

Let $G = \mathrm{GL}_n$ (or more generally, G any connected linear reductive group over $\mathbb{Q}$ defined over $\mathbb{Z}$),

$$G(\mathbb{A}_f) = \left\{ (g_p) \in \prod_p G(\mathbb{Q}_p) \ : \ g_p \in G(\mathbb{Z}_p) \text{ for almost all } p \right\}.$$

Unique group topology on $G(\mathbb{A}_f)$ such that

$$G(\widehat{\mathbb{Z}}) = \prod_p G(\mathbb{Z}_p)$$

is open and equipped with natural topology. Then $G(\mathbb{A}_f)$ and $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$ are l.c.

Basis of neighborhoods of $1 \in G(\mathbb{A}_f)$ given by congruence subgroups

$$K(N) = \{g \in G(\widehat{\mathbb{Z}}) : g \equiv 1 \pmod N\}, \ \ N \geq 1.$$

The diagonal subgroup

$$G(\mathbb{Q}) \subset G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$$

is discrete.

The case $n = 1$: ideles

For $G = \mathrm{GL}_1 = \mathbb{G}_m$, the quotient

$$\mathbb{Q}^\times \backslash \mathbb{A}^\times$$

is the *idele class group*.

Fact: We have $\mathbb{A}^\times = \mathbb{Q}^\times \times \left(\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times\right)$, hence $\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0} = \widehat{\mathbb{Z}}^\times$.

(a compact group) As a consequence, entirely discrete Fourier decomposition:

$$\mathrm{L}^2(\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0}) = \widehat{\bigoplus_{\chi}} \mathbb{C}\,\chi$$

indexed by *primitive* Dirichlet characters $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ (each occurring once).

Exercise: write down the character of $\mathbb{A}^\times$ associated to a Dirichlet character.

$\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$ as a space of lattices (Eichler, Weil...)

Proposition: We have $\mathrm{GL}_n(\mathbb{A}_f) = \mathrm{GL}_n(\mathbb{Q}) \cdot \mathrm{GL}_n(\widehat{\mathbb{Z}})$.

Idea : to give a lattice $L \subset \mathbb{Q}^n$ = to give a collection local lattices $L_p \in \mathbb{Q}_p^n$ with $L_p = \mathbb{Z}_p^n$ for a.a. p:

$$L_p = L \otimes \mathbb{Z}_p \subset \mathbb{Q}_p^n, \qquad L = \bigcap_p L_p.$$

So the natural (transitive) action of $\mathrm{GL}_n(\mathbb{Q})$ on the set of lattices in $\mathbb{Q}^n$ naturally extends to a (transitive!) action of $\mathrm{GL}_n(\mathbb{A}_f)$, with the stabilizer $\mathrm{GL}_n(\widehat{\mathbb{Z}})$.

Corollary: $\mathrm{GL}_n(\mathbb{R}) \stackrel{\mathrm{can}}{\to} \mathrm{GL}_n(\mathbb{A})$ induces a homeo $\mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R}) \stackrel{\sim}{\to} \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\widehat{\mathbb{Z}})$.

Level structures

Corollary: for all $N \geq 1$,

$$\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K(N) = \mathrm{GL}_n(\mathbb{Z})\backslash(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}))$$

$=$ set of all lattices L in $\mathbb{R}^n$ equipped with an isomorphism $L/NL \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^n$ (*level N structure*).

Remark: space on the right is a disjoint union of $|(\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}|$ copies of $\Gamma(N)\backslash\mathrm{GL}_n(\mathbb{R})$, with $\Gamma(N) = \ker \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$ the *principal congruence subgroup of level N*.

There is a natural harmonic analysis that we can make on

$$\mathrm{L}^2(\mathrm{GL}_n(\mathbb{Z})\backslash(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}))),$$

similar to the one discussed in the beginning of the lecture, with $\mathrm{GL}_n(\mathbb{R})$ acting as well as the subring of H_n generated by the T_A with A coprime to N.

It will be contained in the adelic point of view by Corollary.

$\mathrm{GL}_n(\mathbb{Q})$ is a lattice in $\mathrm{GL}_n(\mathbb{A})/\mathbb{R}_{>0}$

Theorem (Reduction theory, Borel–Harish-Chandra) $G(\mathbb{Q})\backslash G(\mathbb{A})/Z$ has finite volume.

- For $G = \mathrm{GL}_n$, Z is the central $\mathbb{R}_{>0}$ of the $\mathrm{GL}_n(\mathbb{R})$ factor. Theorem follows from previous slides + the fact that $\mathrm{PGL}_n(\mathbb{Z})$ is a lattice in $\mathrm{PGL}_n(\mathbb{R})$.
- For general G need to take $Z = S(\mathbb{R})^0$ with S be the maximal $\mathbb{Q}$-split central torus of G.

Lecture 2

Automorphic representations of $\mathrm{GL}_n(\mathbb{A})$

The space $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$

Let $G = \mathrm{GL}_n$ (most of what follows can be adapted to any connected linear reductive group G over $\mathbb{Q}$).

Recall $Z = \mathbb{R}_{>0} \subset \mathrm{GL}_n(\mathbb{R})$.

We have seen that $G(\mathbb{Q})$ is a lattice in the locally compact group $G(\mathbb{A})/Z$.

We are interested in the unitary $G(\mathbb{A})$-representation

$$\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z).$$

As before, we shall focus on the discrete part.

Following Harish-Chandra, it is convenient to work with a *decompletion* of this space, leading to a more algebraic and manageable theory.

Square integrable automorphic forms

Set $K_\infty = \mathrm{O}(n)$ (a maximal compact subgroup of $G(\mathbb{R})$),

$$\mathfrak{g} = \mathrm{Lie}\, G(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{M}_n(\mathbb{C}),$$

and let $\mathfrak{z}(\mathfrak{g})$ be the center of the enveloping algebra of $\mathfrak{g}$ (the $\mathrm{GL}_n(\mathbb{R})$-*invariant differential operators*).

Definition (Gelfand–Graev–Piatetski-Shapiro, Harish-Chandra, Borel–Jacquet).
The space $\mathcal{A}^2(G)$ of square integrable automorphic forms is the space of functions $f : G(\mathbb{Q})\backslash G(\mathbb{A})/Z \longrightarrow \mathbb{C}$ such that:

(i) f is right invariant under $K(N) \subset G(\mathbb{A}_f)$ for some $N \geq 1$,

(ii) f is smooth in the variable $G(\mathbb{R})$,

(iii) f is K_∞-finite and $\mathfrak{z}(\mathfrak{g})$-finite (explain),

(iv) $|f(g)|^2$ is integrable on $G(\mathbb{Q})\backslash G(\mathbb{A})/Z$.

Remarks on $\mathcal{A}^2(G)$

- $\mathcal{A}^2(G)$ is naturally a smooth $G(\mathbb{A}_f)$-module for right translations by (i).
- $\mathcal{A}^2(G)$ depends on the choice of K_∞; it is stable by K_∞, but not by $G(\mathbb{R})$.
- The $\mathbb{C}$-algebra $\mathfrak{z}(\mathfrak{g})$ is a polynomial algebra $\mathbb{C}[C_1, \ldots, C_n]$, with C_i of degree i (*generalized Laplace operators*). Thus $\mathfrak{z}(\mathfrak{g})$-finiteness means finitely many invariant differential equations.
- Due to (iv), stability of $\mathcal{A}^2(G)$ under $\mathfrak{g}$ is not obvious, but true. Intuitively, need some analogue of

$$|f'|_2^2 \leq |f|_2 |f''|_2, \text{ for } f \in \mathcal{C}^\infty(\mathbb{R}).$$

  The key is an elliptic differential equation satisfied by f here, using (ii),(iii) (Harish-Chandra).

$\Longrightarrow$ Thus $\mathcal{A}^2(G)$ is a module for $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$.

Remark: moderate growth variant ?

A preparation: irreducible admissible representations of $G(\mathbb{A})$

Definition: *Let $\Pi(G(\mathbb{A}))$ be the set of isomorphism classes of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules of the form*

$$\pi = \pi_\infty \otimes \pi_f,$$

where:

- *π_∞ is an irreducible admissible Harish-Chandra module,*
- *π_f is an irreducible admissible smooth representation of $G(\mathbb{A}_f)$ (in the sense of the local course).*

Standard abuse of language: *irreducible admissible representation of $G(\mathbb{A})$*. Note that π_∞ and π_f are uniquely determined up to isomorphism, and that π is $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-irreducible (exercise!)

Remark: Let $\pi \in \Pi(G(\mathbb{A}))$.

- By Schur's lemma, the algebra $\mathfrak{z}(\mathfrak{g})$ acts by scalars on π_∞ (*infinitesimal character*, say more ?).
- Later: decompose π_f as a restricted tensor product of π_p's, but unnecessary for the moment.

Discrete automorphic representations

Theorem (GGPS, HC, BJ). $\mathcal{A}^2(G)$ *has a natural action of*

$$(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f),$$

and we have an orthogonal direct sum decomposition

$$\mathcal{A}^2(G) = \bigoplus_{\pi \in \Pi(G(\mathbb{A}))} m(\pi)\,\pi, \qquad m(\pi) < \infty.$$

Moreover, $\mathcal{A}^2(G)$ *is dense in* $\mathrm{L}^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$.

Definition. *The discrete automorphic representations are the elements of*

$$\Pi_{\mathrm{disc}}(G) = \{\pi \in \Pi(G(\mathbb{A}))\,:\, m(\pi) \neq 0\}.$$

In particular, only countably many $\pi \in \Pi(G(\mathbb{A}))$ are discrete automorphic.

How to recover the classical setting?

We have $\mathcal{A}^2(G) = \cup_{N\geq 1}\mathcal{A}^2(G)^{K(N)}$, and for $N \geq 1$ a $(\mathfrak{g}, K_\infty) \times \mathcal{H}(G(\mathbb{A}_f), K(N))$-module isomorphism:

$$\mathcal{A}^2(G)^{K(N)} = \bigoplus_{\pi\in\Pi(G(\mathbb{A}))} m(\pi)\,\pi_\infty \otimes \pi_f^{K(N)}.$$

On the other hand, from Lecture 1, $\mathcal{A}^2(G)^{K(N)}$ is a subspace of functions on

$$G(\mathbb{Q})\backslash G(\mathbb{A})/(Z \times K(N)) = \mathrm{GL}_n(\mathbb{Z})\backslash(\mathrm{GL}_n(\mathbb{R})/Z \times \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}))$$

The action of the subalgebra $\bigotimes_{p\nmid N}\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p)) \subset \mathcal{H}(G(\mathbb{A}_f), K(N))$ on $\mathcal{A}^2(G)^{K(N)}$ is generated by the T_A with $|A|$ prime to N. In particular, we have $\mathcal{H}(G(\mathbb{A}_f), K(1)) = \mathcal{H}_n$ of lecture 1.

Remark: (*decompletion*) $\mathcal{A}^2(G)^{K(N)}$ coïncides with the subspace of smooth, K_∞-finite, $\mathfrak{z}(\mathfrak{g})$-finite functions in $\mathrm{L}^2_{\mathrm{disc}}(\mathrm{GL}_n(\mathbb{Z})\backslash(\mathrm{GL}_n(\mathbb{R})/\mathbb{R}_{>0} \times \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})))$. It is dense in the latter.

Exercise: Realize classical modular cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ as elements of $\mathcal{A}^2(\mathrm{GL}_2)$.

The subspace of cusp forms

Definition (GGPS, HC, BJ). *Define* $\mathcal{A}_{\mathrm{cusp}}(G) \subset \mathcal{A}^2(G)$ *as the subspace of functions satisfying*

$$(v) \quad \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)\, dn = 0, \ \forall g \in G(\mathbb{A}),$$

for each proper parabolic $\mathbb{Q}$*-subgroup* $P \subset G$*, with unipotent radical* N*.*

The integral makes sense because $N(\mathbb{Q})\backslash N(\mathbb{A})$ is compact. For $G = \mathrm{GL}_n$, it is enough to consider, for all $a + b = n$ with $a, b \geq 1$, the standard maximal parabolic of type (a, b), for which we have

$$N = \{\begin{pmatrix} 1_a & * \\ 0 & 1_b \end{pmatrix}\} \simeq (\mathrm{M}_{a,b}, +).$$

This is the natural generalization of cuspidal modular forms. Constant functions are cusp forms iff $n = 1$.

Geometric picture: The average of f is zero along any rational horosphere.

Cuspidal automorphic representations

Theorem (GGPS, HC, BJ). *$\mathcal{A}_{\mathrm{cusp}}(G)$ is a sub-$(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module of $\mathcal{A}^2(G)$. In particular,*

$$\mathcal{A}_{\mathrm{cusp}}(G) = \bigoplus_{\pi \in \Pi(G(\mathbb{A}))} m_{\mathrm{cusp}}(\pi)\, \pi,$$

with $m_{\mathrm{cusp}}(\pi) \le m(\pi) < \infty$ for all $\pi \in \Pi(G(\mathbb{A}))$.

Remark: We can directly define $\mathcal{A}_{\mathrm{cusp}}(G)$ as for $\mathcal{A}^2(G)$ but using conditions (i), (ii), (iii) and (v), without (iv). But then (iv) follows. Indeed, cusp forms are bounded on $G(\mathbb{Q})\backslash G(\mathbb{A})/Z$ (HC).

Definition. (*Cuspidal automorphic representations*) We set

$$\Pi_{\mathrm{cusp}}(G) = \{\pi \in \Pi(G(\mathbb{A})) \,:\, m_{\mathrm{cusp}}(\pi) \neq 0\} \subset \Pi_{\mathrm{disc}}(G).$$

Spherical or "unramified" representations

Let p be a prime.

Definition. *An irreducible admissible representation* π_p *of* $\mathrm{GL}_n(\mathbb{Q}_p)$ *is called unramified if* $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)} \neq 0$*.*

Then $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a finite-dimensional irreducible module over the spherical Hecke algebra

$$\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)).$$

But this algebra is commutative, hence $\dim \pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)} = 1$. Terminology *unramified* ?

Fact. *The map* $\pi_p \mapsto \pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ *induces a bijection between isom. classes of unramified representations of* $\mathrm{GL}_n(\mathbb{Q}_p)$ *and* $\mathbb{C}$*-algebra morphisms* $\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)) \longrightarrow \mathbb{C}$ *(*systems of Hecke eigenvalues*).*

We need a way to parameterise either side. Two related ways : spherical principal series or Satake isomorphism. I focus on the latter.

The Satake isomorphism

Need a concrete description of systems of Hecke eigenvalues.

Theorem: (Satake, Langlands, for $G = \mathrm{GL}_n$). *There is a natural $\mathbb{C}$-algebra isomorphism*

$$\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)) \simeq \mathbb{C}[\mathrm{GL}_n(\mathbb{C})]^{\mathrm{GL}_n(\mathbb{C})}$$

where the right hand side denotes regular functions on $\mathrm{GL}_n(\mathbb{C})$ invariant under conjugation.

Consequently (exercise?),

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p)), \mathbb{C}) \simeq \{\text{semisimple conjugacy classes in } \mathrm{GL}_n(\mathbb{C})\}.$$

Definition. *The Satake parameter of an unramified π_p is the corresponding semisimple conjugacy class*

$$\mathrm{c}(\pi_p) \subset \mathrm{GL}_n(\mathbb{C}).$$

The Satake parameter

We thus have a canonical bijection

$$\left\{ \begin{array}{c} \text{unramified irreducible admissible} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{semisimple conjugacy classes} \\ \text{in } \mathrm{GL}_n(\mathbb{C}) \end{array} \right\}.$$

See *e.g.* Gross, *On the Satake isomorphism*, for how to concretely compute the Satake isomorphism.

Example: let $\mathrm{T}_p = \mathrm{T}_{\mathbb{Z}/p\mathbb{Z}}$ be the Hecke operator of Lecture 1. If $0 \neq v \in \pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ and $T_p v = a_p v$, then

$$a_p = p^{(n-1)/2} \,\mathrm{tr}\, \mathrm{c}(\pi_p).$$

Remark: Generalizes *e.g.* to split reductive groups G over $\mathbb{Z}_p$, but the Satake parameter is then a semisimple conjugacy class in Langlands' dual group $\widehat{G}(\mathbb{C})$. We have used $\widehat{\mathrm{GL}_n} = \mathrm{GL}_n$.

Back to $G(\mathbb{A}_f)$: Flath's tensor product theorem

For each prime p, $G(\mathbb{Q}_p)$ is a closed normal subgroup of $G(\mathbb{A}_f)$. Also, $G(\mathbb{Q}_p)$ commutes with $G(\mathbb{Q}_q)$ for $p \neq q$, and the $G(\mathbb{Q}_p)$'s generate a dense subgroup in $G(\mathbb{A}_f)$.

Theorem (Flath). *It is equivalent to give:*

- *an irreducible admissible smooth representation π_f of $G(\mathbb{A}_f)$,*
- *a collection $(\pi_p)_p$ of irreducible admissible representations of $G(\mathbb{Q}_p)$ such that $\pi_p^{G(\mathbb{Z}_p)} \neq 0$ for almost all p.*

In this correspondence,

$$\pi_f \simeq \bigotimes_p{}' \pi_p,$$

the *restricted tensor product* being taken with respect to the lines $\pi_p^{G(\mathbb{Z}_p)}$ at the unramified primes.

Thus an element $\pi \in \Pi(G(\mathbb{A}))$ may be written

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p.$$

Summary : the collection of Satake parameters attached to π

Let $\pi \in \Pi(\mathrm{GL}_n(\mathbb{A}))$.

There is a finite set S of primes such that π_p is unramified for all $p \notin S$.

Hence π gives rise to a collection

$$\{\, \mathrm{c}(\pi_p) \subset \mathrm{GL}_n(\mathbb{C}) \,\}_{p \notin S}$$

of semisimple conjugacy classes.

Basic question. *Which such collections can occur for* $\pi \in \Pi_{\mathrm{disc}}(\mathrm{GL}_n)$ *?*

This is an extremely rigid problem, as the case of classical modular forms already shows.

A few classical results and conjectures

Strong multiplicity 1 theorem (Jacquet–Shalika). *If* $\pi, \pi' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ *and* $\mathrm{c}(\pi_p) = \mathrm{c}(\pi'_p)$ *for almost all* p*, then* $\pi \simeq \pi'$*. Moreover* $m_{\mathrm{cusp}}(\pi) = 1$*.*

Generalized Ramanujan conjecture. *If* $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$*, and if* π_p *is unramified, then all the eigenvalues of* $\mathrm{c}(\pi_p)$ *have absolute value* 1*.*

A huge generalization of Ramanujan's original conjecture on $\tau(p)$ (proved by Deligne).

Theorem (JS) *(same ass.) All the eigenvalues of* $\mathrm{c}(\pi_p)$ *have absolute in* $]\, p^{-1/2}, p^{1/2}\, [$*.*

By contrast, the Ramanujan conj. never holds for $\pi \in \Pi_{\mathrm{disc}}(\mathrm{GL}_n) \smallsetminus \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$. Clear picture for GL_n:

Theorem (Mœglin–Waldspurger). *For each* $\pi \in \Pi_{\mathrm{disc}}(\mathrm{GL}_n)$*, there is a unique* $d \mid n$ *and a unique* $\varpi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_{n/d})$ *such that for all* p *big enough,*

$$\mathrm{c}_p(\pi) \;=\; \mathrm{c}_p(\varpi) \otimes \mathrm{Sym}^{d-1}\mathrm{diag}(p^{-1/2}, p^{1/2})$$

Moreover we have $\mathrm{m}(\pi) = 1$*.*

An important tool: L-functions

Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ be unramified outside S. Its partial standard L-function is defined by

$$\mathrm{L}^S(s,\pi) = \prod_{p \notin S} \frac{1}{\det(1 - p^{-s}\mathrm{c}(\pi_p))}, \quad \mathrm{Re}\, s > 3/2.$$

Examples. For $n = 1$ this gives the Dirichlet L-functions, including the Riemann zeta function. For $n = 2$, it contains the usual L-functions of modular forms (in their unitary normalisation).

Theorem (Godement–Jacquet) $\mathrm{L}^S(s,\pi)$ *has a meromorphic continuation to* $\mathbb{C}$. *It is entire unless* $n = 1$ *and* $\pi = 1$.

However, $\mathrm{L}^S(s,\pi)$ mostly only sees the distribution of *traces* of Satake parameters.

Langlands L-functions

To study the whole Satake parameters, Langlands introduced (in *Euler Products*), for any $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and an arbitrary algebraic representation

$$\rho : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C}),$$

the partial L-function:

$$\mathrm{L}^S(s, \pi, \rho) = \prod_{p \notin S} \frac{1}{\det(1 - p^{-s}\rho(\mathrm{c}(\pi_p)))}.$$

This Euler product converges absolutely for $\mathrm{Re}\, s \gg 0$.

Conjecture (Langlands). *If $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, then $\mathrm{L}^S(s, \pi, \rho)$ extends holomorphically to $\mathrm{Re}\, s > 1$.*

An elementary argument of Langlands, using Landau's lemma, shows that *this conjecture applied to all ρ implies the Ramanujan conjecture for π*. A similar idea was used by Deligne in his proof of the Weil conjecture.

Lecture 3

The global conjectures

$$\boxed{\begin{array}{c}\text{cuspidal automorphic representations of } \mathrm{GL}_n(\mathbb{A}) \\ \Updownarrow \\ n\text{-dimensional irreducible representations of } \mathrm{L}_{\mathbb{Q}}\end{array}}$$

A local-global problem

Which collections of π_p, or of conjugacy classes $\{\mathrm{c}(\pi_p)\}_{p \notin S}$, arise from some $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$?

Langlands' original guess in *Ein Märchen* (pursued later by Kottwitz, Arthur), inspired by analogies between Artin and automorphic L-functions: the situation could be similar to the way decomposition groups, or Frobenius elements, interact with one another in global Galois groups.

The relevant, or "automorphic", Galois group has to be larger than $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, but should share similar properties: conjectural Langlands group $\mathrm{L}_{\mathbb{Q}}$.

This lecture: admit the existence of $\mathrm{L}_{\mathbb{Q}}$ satisfying a few explicit axioms and revisit several classical results or conjectures of the theory of automorphic forms from this perspective.

The compact group $\mathrm{L}_{\mathbb{Q}}$

We shall assume the existence of a *compact topological group* $\mathrm{L}_{\mathbb{Q}}$ satisfying 4 axioms:

- Axioms 1, 2, 3: local-global structure of $\mathrm{L}_{\mathbb{Q}}$;
- Axiom 4: the main "universal" property.

In our somewhat personal presentation, $\mathrm{L}_{\mathbb{Q}}$ is **compact**. This is because we use the harmless convention that a $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ has a central character trivial on $\mathbb{R}_{>0} \subset \mathrm{GL}_n(\mathbb{R})$. Without this convention, $\mathrm{L}_{\mathbb{Q}}$ should rather be of the form

$$\mathbb{R}_{>0} \times (\text{our compact}\, \mathrm{L}_{\mathbb{Q}}),$$

the first projection corresponding to the "norm". We keep our choice.

Challenging open problem: find a complete list of axioms for $\mathrm{L}_{\mathbb{Q}}$ which makes it unique, up to isomorphism.

Axiom 1: Local maps (decomposition groups)

Axiom 1. For each place v of $\mathbb{Q}$, there is a distinguished $\mathrm{L}_{\mathbb{Q}}$-conjugacy class of continuous morphisms

$$\mathrm{WD}_{\mathbb{Q}_v} \longrightarrow \mathrm{L}_{\mathbb{Q}}.$$

When v is a prime p, I use the convention

$$\mathrm{WD}_{\mathbb{Q}_p} = \mathrm{W}_{\mathbb{Q}_p} \times \mathrm{SU}(2).$$

(The nilpotent operator N of Weil–Deligne theory is replaced, via Jacobson–Morozov, by an $\mathrm{SU}(2)$.)

For $v = \infty$, the convention is $\mathrm{WD}_{\mathbb{R}} = W_{\mathbb{R}}$ (an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{C}^{\times}$).

Axiom 2: Restricted ramification and Cebotarev

Define $\mathrm{L}_{\mathbb{Q},S}$ as the quotient of $\mathrm{L}_{\mathbb{Q}}$ by the closed normal subgroup generated by the images of

$$\mathrm{I}_{\mathbb{Q}_p} \times \mathrm{SU}(2), \qquad p \notin S,$$

with $\mathrm{I}_{\mathbb{Q}_p} \subset \mathrm{W}_{\mathbb{Q}_p}$ the inertia subgroup. For $p \notin S$, we have thus a natural conjugacy class

$$\mathrm{Frob}_p \subset \mathrm{L}_{\mathbb{Q},S}.$$

Axiom 2. *For each finite set S of primes, the union of the conjugacy classes*

$$\mathrm{Frob}_p \subset \mathrm{L}_{\mathbb{Q},S}, \qquad p \notin S,$$

is dense in $\mathrm{L}_{\mathbb{Q},S}$.

This is an analogue of the Chebotarev density theorem. As we shall see later, this axiom actually follows from the others (see later for a strengthening), but we prefer to keep it for the discussion.

Axiom 3: Group of connected components

Axiom 3. *There is a distinguished continuous, surjective morphism*

$$\mathrm{L}_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

with connected kernel. Moreover, when restricted to $\mathrm{W}_{\mathbb{Q}_p}$ this morphism coincides with the usual local map induced by a choice of field embedding $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p$.

In particular, we have

$$\pi_0(\mathrm{L}_{\mathbb{Q}}) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

If $\mathbb{Q}_S \subset \overline{\mathbb{Q}}$ denotes the maximal algebraic extension of $\mathbb{Q}$ unramified outside S, we have a natural surjective map $\mathrm{L}_{\mathbb{Q},S} \longrightarrow \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ with connected kernel (exercise!).

Example: (Langlands group of $\mathbb{Z}$) This is $\mathrm{L}_{\mathbb{Z}} := \mathrm{L}_{\mathbb{Q},\emptyset}$. By Minkowski we have $\mathrm{Gal}(\mathbb{Q}_\emptyset/\mathbb{Q}) = 1$, so $\mathrm{L}_{\mathbb{Z}}$ is connected. Highly non trivial ! (I expect each connected, simply connected, semisimple compact Lie group should occur as a direct factor of $\mathrm{L}_{\mathbb{Z}}$.)

Axiom 4: a "universal" property

Define $\mathrm{Irr}_n(\mathrm{L}_\mathbb{Q})$ as the set of isomorphism classes of continuous irreducible $r : \mathrm{L}_\mathbb{Q} \to \mathrm{GL}_n(\mathbb{C})$.

(Main) Axiom 4. *For every $n \geq 1$, there is a given bijection*

$$\Pi_{\mathrm{cusp}}(\mathrm{GL}_n) \longrightarrow \mathrm{Irr}_n(\mathrm{L}_\mathbb{Q}), \qquad \pi \longmapsto r_\pi,$$

such that: if π is unramified outside S, then r_π factors through $\mathrm{L}_{\mathbb{Q},S}$ and satisfies

$$r_\pi(\mathrm{Frob}_p) \sim c(\pi_p), \qquad p \notin S.$$

- The representation r_π is called the *global Langlands parameter* of π. A general complex analogue of ℓ-adic Galois representations attached to *algebraic* automorphic representations.
- The surjectivity of $\pi \mapsto r_\pi$ is sometimes called *the reciprocity conjecture*.

First remarks

A finite-dimensional continuous representation of the compact group $\mathrm{L}_{\mathbb{Q}}$ is semisimple, hence factors through some $\mathrm{L}_{\mathbb{Q},S}$ by Axiom 4 (Exercise: $\mathrm{L}_{\mathbb{Q}} = \varprojlim_S \mathrm{L}_{\mathbb{Q},S}$).

A representation $r : \mathrm{L}_{\mathbb{Q},S} \to \mathrm{GL}_n(\mathbb{C})$ is thus determined by its character, hence by the traces

$$\mathrm{tr}(r(\mathrm{Frob}_p)), p \notin S$$

by Axiom 2. In particular, r_π is uniquely determined by π and the Frobenius compatibility above: the maps $\pi \mapsto r_\pi$ of Axiom 4 are unique if exist.

Corollary. (multiplicity one, JS) There is at most one cuspidal π with given π_p for almost all primes p.

Proof : Indeed, the π_p determine r_π by the above arguments, which determines π by Axiom 4.

Built-in Ramanujan conjecture

Corollary. Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$. For every unramified prime p, all eigenvalues of $c(\pi_p)$ have modulus 1.

Proof: if π_p is unramified then $\mathrm{c}(\pi_p)$ is conjugate to $r_\pi(\mathrm{Frob}_p)$ in $\mathrm{GL}_n(\mathbb{C})$. But $r_\pi(\mathrm{L}_{\mathbb{Q}})$ is a compact subgroup of $\mathrm{GL}_n(\mathbb{C})$, and every element of a compact subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a unitary matrix.

In real life, it is a wide open conjecture (even for GL_2).

Remark: (**Axiom 5**, local-global compatibility) We expect following strengthening of Axiom 4. For all $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and all places v of $\mathbb{Q}$,

$$r_\pi|_{\mathrm{WD}_{\mathbb{Q}_v}} = \text{local Langlands parameter of } \pi_v$$

under the (known) local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_v)$. It follows that $r_\pi|_{\mathrm{WD}_{\mathbb{Q}_v}}$ should have a bounded image ("temperedness of all π_v").

Isobaric uniqueness

Theorem. *Suppose that* $\pi_1, \ldots, \pi_n$ *and* $\pi'_1, \ldots, \pi'_m$ *are cuspidal automorphic representations of general linear groups of various sizes, and that for almost all* p

$$c(\pi_{1,p}) \oplus \cdots \oplus c(\pi_{n,p}) = c(\pi'_{1,p}) \oplus \cdots \oplus c(\pi'_{m,p})$$

as semisimple conjugacy classes.

Then

$$\{\{\pi_1, \ldots, \pi_n\}\} = \{\{\pi'_1, \ldots, \pi'_m\}\}.$$

Proof under $\mathrm{L}_{\mathbb{Q}}$: uniqueness of the irreducible summands in a semisimple representation.

In real life: this is JS theorem (*uniqueness of isobaric decomposition*).

Langlands' functoriality conjecture

Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and let

$$\rho : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$$

be a finite-dimensional continuous representation.

Since $\mathrm{L}_{\mathbb{Q}}$ is compact, $\rho \circ r_\pi$ is semisimple, so by Axiom 4:

$$\rho \circ r_\pi = \bigoplus_{i=1}^{s} r_{\pi_i}.$$

Corollary: *For all π and ρ as above, there are unique cuspidal representations*

$$\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_{m_i}), \qquad m = m_1 + \cdots + m_s,$$

such that, for almost all p,

$$\rho(c(\pi_p)) = \bigoplus_i c((\pi_i)_p).$$

Wide open in real life, even for $n = 2$ and $\rho = \mathrm{Sym}^m$ (but Newton-Thorne for modular forms).

L-functions become generalized Artin L-functions

For any cont. representation $r : \mathrm{L}_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$, we may define *à la Artin*

$$\mathrm{L}(s, r) = \prod_v \mathrm{L}(s, r_{|\mathrm{WD}_{\mathbb{Q}_v}}).$$

For $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, we actually have $\mathrm{L}(s, \pi) = \mathrm{L}(s, r_\pi)$ (Godement-Jacquet, Harris-Taylor–Henniart).

Theorem. (Generalized Artin conjecture) *Let $r : \mathrm{L}_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$ be irreducible. Then $\mathrm{L}(s, r)$ has meromorphic continuation to $\mathbb{C}$, and is holomorphic if $r \neq 1$.*

Proof: write $r = r_\pi$ for some $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and apply Godement-Jacquet.

Remark: $\mathrm{L}(s, 1)$ is the complete Riemann zeta function, which has (simple) poles at $s = 0, 1$.

The usual Artin conjecture

Recall Axiom 3 (not used so far):

$$\pi_0(\mathrm{L}_{\mathbb{Q}}) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ coincide with finite image representations of $\mathrm{L}_{\mathbb{Q}}$.

The generalized Artin conjecture implies thus the usual Artin conjecture:

Theorem: *Artin L-functions of non-trivial irreducible complex representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *are entire.*

This is wide open.

Langlands' Euler products

For $\pi \in \Pi_{\text{cusp}}(\mathrm{GL}_n)$ and any $\rho : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_m(\mathbb{C})$, Langlands' L-function $\mathrm{L}(s, \pi, \rho)$ satisfies

$$\mathrm{L}(s, \pi, \rho) = \mathrm{L}(s, \rho \circ r_\pi)$$

(at least at unramified places, this is really by definition; this is the right definition at all places).

Theorem: $\mathrm{L}(s, \pi, \rho)$ *has a meromorphic continuation to* $\mathbb{C}$*, with possible poles at* $s = 0, 1$ *only. Moreover, the order of the pole at* $s = 0, 1$ $=$ *number of times* 1 *occurs in* $\rho \circ r_\pi$*.*

Proof : write $\rho \circ r_\pi = \oplus_{i=1}^k r_{\pi_i}$, then we have $\mathrm{L}(s, \pi, \rho) = \prod_{i=1}^k \mathrm{L}(s, \pi_i)$.
We conclude using $\mathrm{L}(1, \pi)$ for $\pi \neq 1$ (JS). $\square$

This is much stronger than the conjecture stated in the end of the last lecture.

Rankin–Selberg and Schur's lemma

For $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and $\pi' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_m)$, set

$$L(s, \pi' \times \pi^\vee) := L(s, r_{\pi'} \otimes r_\pi^\vee).$$

Corollary: (JPSS) $\mathrm{L}(s, \pi' \times \pi^\vee)$ *has a meromorphic continuation to* $\mathbb{C}$, *with a pole at* $s = 1$ *iff* $\pi' \simeq \pi$.

Proof : The order of the pole at $s = 1$ is

$$\dim(r_{\pi'} \otimes r_\pi^\vee)^{\mathrm{L}_\mathbb{Q}}.$$

By Schur's lemma, this is 1 iff $r_{\pi'} \simeq r_\pi$, and 0 otherwise. $\square$

Tensor products

For $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and $\pi' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_m)$, write

$$r_\pi \otimes r_{\pi'} = r_{\pi_1} \oplus \cdots \oplus r_{\pi_r}.$$

This predicts cuspidal automorphic representations $\pi_1, \ldots, \pi_r$ such that

$$c(\pi_p) \otimes c(\pi'_p) = c((\pi_1)_p) \oplus \cdots \oplus c((\pi_r)_p)$$

for almost all p.

This is largely unknown on the automorphic side!

One reason: seems difficult to guess the π_i's. One would need to understand triple product L-functions

$$\mathrm{L}(s, \pi \times \pi' \times \pi'') = \mathrm{L}(s, r_\pi \otimes r_{\pi'} \otimes r_{\pi''})$$

for all $\pi'' \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_\ell)$.

(Universal) Sato–Tate for $\mathrm{L}_{\mathbb{Q}}$

Theorem: *For each finite set S, the conjugacy classes Frob_p, $p \notin S$, are equidistributed in the space of conjugacy classes of the compact group $\mathrm{L}_{\mathbb{Q},S}$: we have*

$$\lim_{x \to \infty} \frac{1}{\#\{p \leq x,\ p \notin S\}} \sum_{p \leq x,\ p \notin S} f(\mathrm{Frob}_p) = \int_{\mathrm{L}_{\mathbb{Q},S}} f(g)\, dg,$$

for all central continuous functions $f : \mathrm{L}_{\mathbb{Q},S} \longrightarrow \mathbb{C}$, with dg = proba Haar measure on $\mathrm{L}_{\mathbb{Q},S}$.

Proof ideas (close to Dirichlet and Hecke):

- reduce to $f = \mathrm{trace}\, r$ with r any irreducible representation of $\mathrm{L}_{\mathbb{Q},S}$ (Peter-Weyl),
- Using $\log \mathrm{L}(s, r)$ and a Tauberian theorem, reduce to $\mathrm{L}(1, r) \neq 0$ for $r \neq 1$,
- by Axiom 4, may assume $r = r_\pi$ for some cuspidal π,
- the required non-vanishing $\mathrm{L}(1, \pi) \neq 0$ is a theorem of JS.

(See e.g. some appendix in Serre's abelian ℓ-adic representation.)

The argument actually shows that Axioms 1 & 4 imply Axiom 2.

The Sato–Tate group of a $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$

For a fixed cuspidal π, its Sato–Tate group is defined as

$$\mathrm{ST}(\pi) = r_\pi(\mathrm{L}_{\mathbb{Q}}) \subset \mathrm{GL}_n(\mathbb{C})$$

(an irreducible compact subgroup of $\mathrm{GL}_n(\mathbb{C})$ well defined up to conjugacy).

The Sato–Tate conjecture for π is the equidistribution of the conjugacy classes

$$r_\pi(\mathrm{Frob}_p) \subset \mathrm{ST}(\pi),\ p \notin S,$$

with respect to Haar measure. It follows from the Theorem by a simple pushforward!

Example: if π is unramified at all primes, then $\mathrm{ST}(\pi)$ is connected (as $\mathrm{L}_{\mathbb{Z}}$ is) and in $\mathrm{SL}_n(\mathbb{C})$ (why?).
For $n = 2$, unique possibility is $\mathrm{ST}(\pi) \simeq \mathrm{SU}(2)$: usual Sato-Tate conjecture for $\pi = \pi_\Delta$.

And $\Pi_{\mathrm{disc}}(\mathrm{GL}_n)$?

The discrete spectrum is bigger than the cuspidal spectrum:

$$A_{\mathrm{disc}}(\mathrm{GL}_n) \supsetneq A_{\mathrm{cusp}}(\mathrm{GL}_n).$$

Ramanujan defect : the trivial representation already violates Ramanujan in the naive sense.

Arthur's idea (inspired by Lefschetz) is to introduce irreducible morphisms

$$\mathrm{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

(polynomial on $\mathrm{SL}_2(\mathbb{C})$)

For GL_n, these are exactly of the form

$$r_\pi \otimes \mathrm{Sym}^{d-1}\mathbb{C}^2, \qquad d \mid n, \quad \pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_{n/d}).$$

This fits the description $\Pi_{\mathrm{disc}}(\mathrm{GL}_n)$ proved by Moeglin–Waldspurger.

Beyond GL_n

For (say) a split reductive group $G/\mathbb{Q}$, with complex dual group $\widehat{G}(\mathbb{C})$, Langlands parameters should be continuous morphisms

$$\mathrm{L}_{\mathbb{Q}} \longrightarrow \widehat{G}(\mathbb{C}).$$

WARNING: for general G, cusp forms do not seem to play the central role anymore. One rather expects a full description of $\Pi_{\mathrm{disc}}(G)$ in terms of *discrete* Arthur parameters

$$\psi : \mathrm{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}(\mathbb{C}).$$

discrete = the centralizer of $\mathrm{Im}\,\psi$ in $\widehat{G}(\mathbb{C})$ contains the center of $\widehat{G}$ with finite index.

Several new ingredients to describe $\Pi_{\mathrm{disc}}(G)$: local and global component groups of the centralizer of ψ, local and global $L-$ and A-packets, Arthur-Langlands multiplicity formula ... for another time!

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