Kneser neighbours and orthogonal Galois representations in dimensions 16 and 24

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(joint work with Jean Lannes)

Let $n \geq 1$ be an integer. Recall that an even unimodular lattice in the standard euclidean space $\mathbb{R}^n$ is a lattice $L \subset \mathbb{R}^n$ of covolume 1 with $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$. Let X_n denote the set of isometry classes of even unimodular lattices in $\mathbb{R}^n$. As is well-known, X_n is a finite set which is non-empty if and only if $n \equiv 0 \bmod 8$. For example, the lattice

$$\mathrm{E}_n = \mathrm{D}_n + \mathbb{Z}\,\frac{e_1 + \ldots + e_n}{2},$$

$\{e_1, \ldots, e_n\}$ denoting the canonical basis of $\mathbb{R}^n$ and D_n the sublattice of index 2 in $\mathbb{Z}^n$ whose elements (x_i) satisfy $\sum_i x_i \equiv 0 \bmod 2$, is even unimodular for $n \equiv 0 \bmod 8$.

The set X_n has been determined in only three cases. One has $\mathrm{X}_8 = \{\mathrm{E}_8\}$ (Mordell), $\mathrm{X}_{16} = \{\mathrm{E}_8 \oplus \mathrm{E}_8, \mathrm{E}_{16}\}$ (Witt) and Niemeier showed that X_{24} has 24 explicit elements (see [V]). The number of numerical coincidences related to Niemeier's lists is quite extraordinary and makes that list still mysterious. For the other values of n the Minkowski-Siegel-Smith mass formula shows that X_n is huge, perhaps impossible to describe. For instance, X_{32} already has more than 80.10^6 elements ([S]).

Let $L \subset \mathbb{R}^n$ be an even unimodular lattice, and let p be a prime; Kneser defines a p-neighbour of L as an even unimodular lattice $M \subset \mathbb{R}^n$ such that $M \cap L$ has index p in L (hence in M). The relation of being p-neighbours turns X_n into a graph which was shown to be connected by Kneser, providing a theoretical way to compute X_n from the single lattice E_n. This is actually the way Kneser and Niemeier computed X_n for $n \leq 24$, using the prime $p = 2$ and the huge number of symmetries present in those cases.

In this paper, we are interested in giving an explicit formula for the number $\mathrm{N}_p(L, M)$ of p-neighbours of L which are isometric to M. Equivalently, it amounts to determining the $\mathbb{Z}$-linear operator $\mathrm{T}_p : \mathbb{Z}[\mathrm{X}_n] \to \mathbb{Z}[\mathrm{X}_n]$ defined by $\mathrm{T}_p[L] = \sum [N]$, $[-]$ denoting the isometry class of a lattice, the summation being over all the p-neighbours of L.

Before stating our main results, let us mention that the p-neighbours of a given even unimodular lattice L are in canonical bijection with the $\mathbb{F}_p$-points of the projective quadric C_L over $\mathbb{Z}$ defined by the quadratic form $x \mapsto \frac{x \cdot x}{2}$ on L. The quadric C_L is hyperbolic over $\mathbb{F}_p$ for each prime p, thus L has exactly

$$\mathrm{c}_n(p) = |\mathrm{C}_L(\mathbb{F}_p)| = 1 + p + p^2 + \ldots + p^{n-2} + p^{n/2-1}$$

p-neighbours, where $n = \mathrm{rk}_{\mathbb{Z}} L$. We are thus interested in the partition of the quadric $\mathrm{C}_L(\mathbb{F}_p)$ into $|\mathrm{X}_n|$ parts (some of them being possibly empty) given by the isometry classes. Of course, $\mathrm{N}_p(\mathrm{E}_8, \mathrm{E}_8) = \mathrm{c}_8(p)$ as $\mathrm{X}_8 = \{\mathrm{E}_8\}$, thus the first interesting case (perhaps known to specialists!) is $n = 16$.

Theorem A : *Let $n = 16$. In the basis $\mathrm{E}_8 \oplus \mathrm{E}_8, \mathrm{E}_{16}$ the matrix of T_p is*

$$\mathrm{c}_{16}(p)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + (1+p+p^2+p^3)\,\frac{1+p^{11}-\tau(p)}{691}\begin{bmatrix}-405 & 286\\ 405 & -286\end{bmatrix},$$

$\sum_{n\geq 1}\tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24}$ denoting Ramanujan's Δ function.

One first makes two well-known observations. First, the operators T_p commute with each others. Second, they are self-adjoint for the scalar product defined by $\langle [L],[M]\rangle = \delta_{[L],[M]}|\mathrm{O}(L)|$, which amounts to saying that for all $L, M \in \mathrm{X}_n$ we have

$$\mathrm{N}_p(L,M)|\mathrm{O}(M)| = \mathrm{N}_p(M,L)|\mathrm{O}(L)|. \tag{1}$$

Our main question (for any n) is thus equivalent to first finding a basis of $\mathbb{R}[\mathrm{X}_n]$ made of eigenvectors common to all of the T_p operators, and then to describing the system of eigenvalues (λ_p) of the (T_p) on each of these eigenvectors. If $n = 16$ it is not difficult to compute T_2, and this was essentially done by Borcherds for $n = 24$ (see [N-V]). In both cases, the eigenvalues of T_2 are distinct integers (this was noticed by Nebe and Venkov [N-V] for $n = 24$). Let us mention the important presence of $(\mathrm{c}_n(p))$ as "trivial" system of eigenvalues : formula (1) shows that $\sum_{L\in X_n}[L]|O(L)|^{-1} \in \mathbb{Q}[X_n]$ is an eigenvector for T_p with eigenvalue $\mathrm{c}_n(p)$.

Assume now $n = 16$. The non-trivial system of eigenvalues is related to Ramanujan's Δ-function in a non-trivial way. Our proof relies on Siegel theta series

$$\vartheta_g : \mathbb{Z}[X_n] \to \mathrm{M}_{\frac{n}{2}}(\mathrm{Sp}_{2g}(\mathbb{Z})),$$

the latter space being the space of classical Siegel modular forms of weight $n/2$ and genus g. The generalized Eichler commutation relation ([R], [W]) asserts that ϑ_g intertwines T_p with some explicit Hecke operator on the space of Siegel modular forms. By a classical result of Witt, Kneser and Igusa (see [K]),

$$\vartheta_g(\mathrm{E}_8 \oplus \mathrm{E}_8) = \vartheta_g(\mathrm{E}_{16}) \text{ if } g \leq 3,$$

whereas $\mathrm{F} = \vartheta_4(\mathrm{E}_8 \oplus \mathrm{E}_8) - \vartheta_4(\mathrm{E}_{16})$ does not vanish, thus the T_p-eigenvalue we are looking for is related to the Hecke eigenvalues of $F \in \mathrm{S}_8(\mathrm{Sp}_8(\mathbb{Z}))$. A result by Poor and Yuen [P-Y] asserts that the latter space is 1-dimensional (generated by the famous Schottky form!). But another non-trivial member of this space is Ikeda's lift of Ramanujan's Δ function (see [I]), whose Hecke eigenvalues are explicitly given in terms of Δ. By unravelling the precise formulae we obtain Theorem A. Actually, we found a direct proof of the existence of Ikeda's lift of Δ that relies on the triality for the reductive group $\mathrm{PGO}^+_{E_8}$ over $\mathbb{Z}$ and two theta series constructions.

Pre-Theorem* B : *There is an explicit formula as well for T_p if $n = 24$.*

In this case it is more difficult to find the non trivial systems of eigenvalues on $\mathbb{Q}[\mathrm{X}_{24}]$. Five of them were actually identified as Ikeda lifts in the work of Nebe and Venkov [N-V], with a particular one due to Borcherds-Freitag-Weissauer. We rather rely on Chapter 9 of the book [A] by Arthur . . . which is still unpublished at the moment; hence the * in the statement above. The relation with automorphic forms comes from the canonical identification $X_n = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}})$ where G is the $\mathbb{Z}$-orhogonal group of E_n, so that $\mathbb{C}[X_n]$ is canonically the dual of the space of automorphic forms of G of level 1 and trivial coefficients. The quickest way (although perhaps inappropriate!) to state our results is in terms of Galois representations.

Fix a prime ℓ. Thanks to the works of many authors (including [A]), for any system of eigenvalues $\pi = (\lambda_p)$ of (T_p) on $\overline{\mathbb{Q}}_\ell[X_n]$, there exists a unique continuous, semi-simple representation

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GO}(n, \overline{\mathbb{Q}_\ell})$$

which is unramified outside ℓ and such that $\mathrm{Trace}(\rho_{\pi,\ell}(\mathrm{Frob}_p)) = \lambda_p$ for each prime $p \neq \ell$. This Galois representation is furthermore crystalline at ℓ with Hodge-Tate numbers $0, 1, \ldots, n-2$, the number $n/2 - 1$ occuring twice.

For example, if $\pi = (c_n(p))$ then $\rho_{\pi,\ell} = (\oplus_{i=0}^{n-2} \omega^i) \oplus \omega^{n/2-1}$, where ω is the ℓ-adic cyclotomic character. Moreover, Theorem *A* implies that if π is the non-trivial system of eigenvalues when $n = 16$, then $\rho_{\pi,\ell} = \rho_{\Delta,\ell} \otimes (\oplus_{i=0}^{3} \omega^i) \oplus \omega^7 \oplus (\oplus_{i=0}^{6} \omega^i) \otimes \omega^4$, where $\rho_{\Delta,\ell}$ is Deligne's Galois representation attached to Δ.

Consider now the following collection of orthogonal ℓ-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with similitude factor ω^{22} and distinct Hodge-Tate numbers in $\{0, 2, \ldots, 22\}$. For r odd and $1 \leq r \leq 23$, set

$$[r] = (\oplus_{i=0}^{r-1} \omega^i) \otimes \omega^{\frac{23-r}{2}}. \qquad (2)$$

For r even, $k \in \{12, 16, 18, 20, 22\}$, $k + r \leq 24$, set

$$\Delta_k[r] = \rho_{\Delta_k,\ell} \otimes (\oplus_{i=0}^{r-1} \omega^i) \otimes \omega^{\frac{24-(k+r)}{2}}, \qquad (3)$$

where Δ_k is a generator of $S_k(SL_2(\mathbb{Z}))$. Set

$$\mathrm{Sym}^2\Delta = \mathrm{Sym}^2\rho_{\Delta,\ell}. \qquad (4)$$

For $(j, k) \in \{(6, 8), (4, 10), (8, 8), (12, 6)\}$ then the space of vector-valued Siegel modular cusp forms of genus 2 (for $\mathrm{Sp}_4(\mathbb{Z})$) and coefficient $\mathrm{Sym}^j \otimes \det^k$ has dimension one (Tsushima, see [VdG]). Let $\Delta_{j,k}$ be a generator and let $\rho_{\Delta_{j,k},\ell}$ be its associated 4-dimensional ℓ-adic Galois representation (Weissauer) ; it is symplectic with Hodge-Tate numbers $0, k-2, j+k-1$, and $j+2k-3$. Set

$$\Delta_{j,k}[2] = \rho_{\Delta_{j,k},\ell} \otimes (1 \oplus \omega) \otimes \omega^{\frac{24-2k-j}{2}}. \qquad (5)$$

Fact: *There are exactly* 24 *representations of dimension* 24 *which are direct sums of representations in the list* (2)–(5) *above and whose Hodge-Tate numbers are* $0, 1, \ldots, 22$ *with* 11 *occuring twice.*

For instance, $\mathrm{Sym}^2\Delta \oplus \Delta_{20}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [9]$, $\Delta_{4,10}[2] \oplus \Delta_{18}[2] \oplus \Delta[4] \oplus [3] \oplus [1]$, and $\mathrm{Sym}^2\Delta \oplus \Delta_{6,8}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [5]$ are three of them. A more precise form of pre-Theorem B (still conditionnal to Arthur's results) is then the following.

Theorem* **B** : *The Galois representations attached to the* 24 *systems of eigenvalues* π *occuring in* $\mathbb{Q}[\mathrm{X}_{24}]$ *are exactly the Galois representations above.*

Thanks to this list, the only unknown to determine T_p on $\mathbb{Z}[\mathrm{X}_{24}]$ are the Hecke eigenvalues of the four Siegel cusp forms $\Delta_{j,k}$. They have actually been computed by Van der Geer and Faber for all $p \leq 11$, and even for some of them up to $p = 37$: see [VdG]. We checked that their results fit with Borcherds's computation of T_2. Better yet, we directly proved that for any even unimodular root lattice $L \subset \mathbb{R}^{24}$ with Coxeter number $\mathrm{h}(L)$ we have

$$\mathrm{N}_p(\mathrm{Leech}, L) = 0 \quad \text{if } p < \mathrm{h}(L).$$

In particular, if $L \in \{\mathrm{E}_{24}, \mathrm{E}_8^3, \mathrm{E}_8 \oplus \mathrm{E}_{16}, \mathrm{A}_{24}^+\}$ and $p \leq 23$ then $\mathrm{N}_p(\mathrm{Leech}, L) = 0$. This simple fact allowed us to confirm all the eigenvalues given in the table of Van der Geer and Faber for $p \leq 23$ and to compute[1] T_p for those p. Using the Ramanujan estimates for the Δ_k and $\Delta_{j,k}$, and with the convention h(Leech) = 1, we obtain for instance :

Corollary* **:** *Assume* $p \geq 11$. *If* L *and* M *are two even unimodular lattices in* $\mathbb{R}^{24}$ *with Coxeter numbers* $\mathrm{h}(L) \geq \mathrm{h}(M)$, *then* $\mathrm{N}_p(L, M) \geq 1$ *if, and only if,* $p \geq \mathrm{h}(L)/\mathrm{h}(M)$.

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[1]See `http://www.math.polytechnique.fr/~chenevier/niemeier/niemeier.html` for tables.