Kneser neighbours and orthogonal Galois representations in dimensions 16 and 24

Gaëtan Chenevier

(joint work with Jean Lannes)

Let $n \geq 1$ be an integer. Recall that an even unimodular lattice in the standard euclidean space $\mathbb{R}^n$ is a lattice $L \subset \mathbb{R}^n$ of covolume 1 with $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$. Let $X_n$ denote the set of isometry classes of even unimodular lattices in $\mathbb{R}^n$. As is well-known, $X_n$ is a finite set which is non-empty if and only if $n \equiv 0 \mod 8$. For example, the lattice $E_n = D_n + \mathbb{Z} e_1 + \ldots + e_n$, $\{e_1, \ldots, e_n\}$ denoting the canonical basis of $\mathbb{R}^n$ and $D_n$ the sublattice of index 2 in $\mathbb{Z}^n$ whose elements $(x_i)$ satisfy $\sum_i x_i \equiv 0 \mod 2$, is even unimodular for $n \equiv 0 \mod 8$.

The set $X_n$ has been determined in only three cases. One has $X_8 = \{E_8\}$ (Mordell), $X_{16} = \{E_8 \oplus E_8, E_{16}\}$ (Witt) and Niemeier showed that $X_{24}$ has 24 explicit elements (see [V]). The number of numerical coincidences related to Niemeier’s lists is quite extraordinary and makes that list still mysterious. For the other values of $n$ the Minkowski-Siegel-Smith mass formula shows that $X_n$ is huge, perhaps impossible to describe. For instance, $X_{32}$ already has more than $8.10^6$ elements ([S]).

Let $L \subset \mathbb{R}^n$ be an even unimodular lattice, and let $p$ be a prime; Kneser defines a $p$-neighbour of $L$ as an even unimodular lattice $M \subset \mathbb{R}^n$ such that $M \cap L$ has index $p$ in $L$ (hence in $M$). The relation of being $p$-neighbours turns $X_n$ into a graph which was shown to be connected by Kneser, providing a theoretical way to compute $X_n$ from the single lattice $E_n$. This is actually the way Kneser and Niemeier computed $X_n$ for $n \leq 24$, using the prime $p = 2$ and the huge number of symmetries present in those cases.

In this paper, we are interested in giving an explicit formula for the number $N_p(L, M)$ of $p$-neighbours of $L$ which are isometric to $M$. Equivalently, it amounts to determining the $\mathbb{Z}$-linear operator $T_p : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ defined by $T_p[L] = \sum[N]$, $[-]$ denoting the isometry class of a lattice, the summation being over all the $p$-neighbours of $L$.

Before stating our main results, let us mention that the $p$-neighbours of a given even unimodular lattice $L$ are in canonical bijection with the $\mathbb{F}_p$-points of the projective quadric $C_L$ over $\mathbb{Z}$ defined by the quadratic form $x \mapsto x^2$ on $L$. The quadric $C_L$ is hyperbolic over $\mathbb{F}_p$ for each prime $p$, thus $L$ has exactly

$$c_n(p) = |C_L(\mathbb{F}_p)| = 1 + p + p^2 + \ldots + p^{n-2} + p^{n/2-1}$$

$p$-neighbours, where $n = \text{rk}_\mathbb{Z} L$. We are thus interested in the partition of the quadric $C_L(\mathbb{F}_p)$ into $|X_n|$ parts (some of them being possibly empty) given by the isometry classes. Of course, $N_p(E_8, E_8) = c_8(p)$ as $X_8 = \{E_8\}$, thus the first interesting case (perhaps known to specialists!) is $n = 16$. 

1
**Theorem A**: Let \( n = 16 \). In the basis \( E_8 \oplus E_8, E_{16} \) the matrix of \( T_p \) is
\[
c_{16}(p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 + p + p^2 + p^3) \begin{bmatrix} 1 + p^{11} - \tau(p) \\ 691 \end{bmatrix} \begin{bmatrix} -405 & 286 \\ 405 & -286 \end{bmatrix}
\]
\[
\sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24} \text{ denoting Ramanujan’s } \Delta \text{ function.}
\]

One first makes two well-known observations. First, the operators \( T_p \) commute with each others. Second, they are self-adjoint for the scalar product defined by \( \langle [L], [M] \rangle = \delta_{[L],[M]} |O(L)| \), which amounts to saying that for all \( L, M \in X_n \) we have
\[
(1) \quad N_p(L, M) |O(M)| = N_p(M, L) |O(L)|.
\]
Our main question (for any \( n \)) is thus equivalent to first finding a basis of \( \mathbb{R}[X_n] \) made of eigenvectors common to all of the \( T_p \) operators, and then to describing the system of eigenvalues \( (\lambda_p) \) of the \( (T_p) \) on each of these eigenvectors. If \( n = 16 \) it is not difficult to compute \( T_2 \), and this was essentially done by Borcherds for \( n = 24 \) (see [N-V]). In both cases, the eigenvalues of \( T_2 \) are distinct integers (this was noticed by Nebe and Venkov [N-V] for \( n = 24 \)). Let us mention the important presence of \( (c_n(p)) \) as “trivial” system of eigenvalues : formula (1) shows that \( \sum_{L \in X_n} [L] |O(L)|^{-1} \in \mathbb{Q}[X_n] \) is an eigenvector for \( T_p \) with eigenvalue \( c_n(p) \).

Assume now \( n = 16 \). The non-trivial system of eigenvalues is related to Ramanujan’s \( \Delta \)-function in a non-trivial way. Our proof relies on Siegel theta series
\[
\vartheta_g : \mathbb{Z}[X_n] \rightarrow \text{M}_2(\text{Sp}_{2g}(\mathbb{Z})),
\]
the latter space being the space of classical Siegel modular forms of weight \( n/2 \) and genus \( g \). The generalized Eichler commutation relation ([R], [W]) asserts that \( \vartheta_g \) intertwines \( T_p \) with some explicit Hecke operator on the space of Siegel modular forms. By a classical result of Witt, Kneser and Igusa (see [K]),
\[
\vartheta_g(E_8 \oplus E_8) = \vartheta_g(E_{16}) \text{ if } g \leq 3,
\]
whereas \( F = \vartheta_4(E_8 \oplus E_8) - \vartheta_4(E_{16}) \) does not vanish, thus the \( T_p \)-eigenvale we are looking for is related to the Hecke eigenvalues of \( F \in S_8(\text{Sp}_{2g}(\mathbb{Z})) \). A result by Poor and Yuen [P-Y] asserts that the latter space is 1-dimensional (generated by the famous Schottky form). But another non-trivial member of this space is Ikeda’s lift of Ramanujan’s \( \Delta \) function (see [I]), whose Hecke eigenvalues are explicitly given in terms of \( \Delta \). By unravelling the precise formulæ we obtain Theorem A. Actually, we found a direct proof of the existence of Ikeda’s lift of \( \Delta \) that relies on the triality for the reductive group \( \text{PGO}_{E_8}^+ \) over \( \mathbb{Z} \) and two theta series constructions.

**Pre-Theorem** B : There is an explicit formula as well for \( T_p \) if \( n = 24 \).

In this case it is more difficult to find the non trivial systems of eigenvalues on \( \mathbb{Q}[X_{24}] \). Five of them were actually identified as Ikeda lifts in the work of Nebe and Venkov [N-V], with a particular one due to Borcherds-Freitag-Weissauer. We rather rely on Chapter 9 of the book [A] by Arthur . . . which is still unpublished at
the moment; hence the * in the statement above. The relation with automorphic forms comes from the canonical identification $X_n = G(\mathbb{Q}) \backslash G(A_f) / G(\mathbb{Z})$ where $G$ is the $\mathbb{Z}$-orthogonal group of $E_n$, so that $\mathbb{C}[X_n]$ is canonically the dual of the space of automorphic forms of $G$ of level 1 and trivial coefficients. The quickest way (although perhaps inappropriate!) to state our results is in terms of Galois representations.

Fix a prime $\ell$. Thanks to the works of many authors (including [A]), for any system of eigenvalues $\pi = (\lambda_p)$ of $(T_p)$ on $\overline{\mathbb{Q}}[X_n]$, there exists a unique continuous, semi-simple representation

$$\rho_{\pi, \ell} : \text{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \longrightarrow \text{GO}(n, \overline{\mathbb{Q}}_\ell)$$

which is unramified outside $\ell$ and such that $\text{Trace}(\rho_{\pi, \ell}(\text{Frob}_p)) = \lambda_p$ for each prime $p \neq \ell$. This Galois representation is furthermore crystalline at $\ell$ with Hodge-Tate numbers $0, 1, \ldots, n - 2$, the number $n/2 - 1$ occurring twice.

For example, if $\pi = (c_n(p))$ then $\rho_{\pi, \ell} = (\oplus_{i=0}^{n-2} \omega^i) \oplus \omega^{n/2 - 1}$, where $\omega$ is the $\ell$-adic cyclotomic character. Moreover, Theorem A implies that if $\pi$ is the non-trivial system of eigenvalues when $n = 16$, then $\rho_{\pi, \ell} = \rho_{\Delta, \ell} \otimes (\oplus_{i=0}^{8} \omega^i) \oplus (\oplus_{i=0}^{8} \omega^i) \otimes \omega^4$, where $\rho_{\Delta, \ell}$ is Deligne’s Galois representation attached to $\Delta$.

Consider now the following collection of orthogonal $\ell$-adic Galois representations of $\text{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ with similitude factor $\omega^{22}$ and distinct Hodge-Tate numbers in \{0, 2, \ldots, 22\}. For $r$ odd and $1 \leq r \leq 23$, set

$$[r] = (\oplus_{i=0}^{-r} \omega^i) \otimes \omega^{23-r}.$$

For $r$ even, $k \in \{12, 16, 18, 20, 22\}$, $k + r \leq 24$, set

$$\Delta_k[r] = \rho_{\Delta_k, \ell} \otimes (\oplus_{i=0}^{-r} \omega^i) \otimes \omega^{24-(k+r)},$$

where $\Delta_k$ is a generator of $S_k(\text{SL}_2(\mathbb{Z}))$. Set

$$\text{Sym}^2 \Delta = \text{Sym}^2 \rho_{\Delta, \ell}.$$

For $(j, k) \in \{(6, 8), (4, 10), (8, 8), (12, 6)\}$ then the space of vector-valued Siegel modular cusp forms of genus 2 (for $\text{Sp}_4(\mathbb{Z})$) and coefficient $\text{Sym}^2 \otimes \det^k$ has dimension one (Tsushima, see [VdG]). Let $\Delta_{j,k}$ be a generator and let $\rho_{\Delta_{j,k}, \ell}$ be its associated 4-dimensional $\ell$-adic Galois representation (Weissauer); it is symplectic with Hodge-Tate numbers $0, k - 2, j + k - 1$, and $j + 2k - 3$. Set

$$\Delta_{j,k}[2] = \rho_{\Delta_{j,k}, \ell} \otimes (1 \oplus \omega) \otimes \omega^{24-2k-r}.$$

**Fact:** There are exactly 24 representations of dimension 24 which are direct sums of representations in the list (2)–(5) above and whose Hodge-Tate numbers are 0, 1, \ldots, 22 with 11 occurring twice.

For instance, $\text{Sym}^2 \Delta \oplus \Delta_{20}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [9], \Delta_{4,10}[2] \oplus \Delta_{18}[2] \oplus \Delta[4] \oplus [3] \oplus [1]$, and $\text{Sym}^2 \Delta \oplus \Delta_{4,8}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [5]$ are three of them. A more precise form of pre-Theorem B (still conditional to Arthur’s results) is then the following.
Theorem* B: The Galois representations attached to the 24 systems of eigenvalues $\pi$ occurring in $\mathbb{Q}[X_{24}]$ are exactly the Galois representations above.

Thanks to this list, the only unknown to determine $T_p$ on $\mathbb{Z}[X_{24}]$ are the Hecke eigenvalues of the four Siegel cusp forms $\Delta_{j,k}$. They have actually been computed by Van der Geer and Faber for all $p \leq 11$, and even for some of them up to $p = 37$; see [VdG]. We checked that their results fit with Borcherds’s computation of $T_2$. Better yet, we directly proved that for any even unimodular root lattice $L \subset \mathbb{R}^{24}$ with Coxeter number $h(L)$ we have

$$N_p(\text{Leech}, L) = 0 \text{ if } p < h(L).$$

In particular, if $L \in \{E_{24}, E_8 \oplus E_8, A_{24}, E_{38} \}$ and $p \leq 23$ then $N_p(\text{Leech}, L) = 0$. This simple fact allowed us to confirm all the eigenvalues given in the table of Van der Geer and Faber for $p \leq 23$ and to compute $^1 T_p$ for those $p$. Using the Ramanujan estimates for the $\Delta_k$ and $\Delta_{j,k}$, and with the convention $h(\text{Leech}) = 1$, we obtain for instance:

Corollary* : Assume $p \geq 11$. If $L$ and $M$ are two even unimodular lattices in $\mathbb{R}^{24}$ with Coxeter numbers $h(L) \geq h(M)$, then $N_p(L, M) \geq 1$ if, and only if, $p \geq h(L)/h(M)$.

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References


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