AN EXPLICIT FORMULA FOR $N_p(L,M)$

G. CHENEVIER & J. LANNES

For $k \in \{12, 16, 18, 20, 22\}$, let $\Delta_k = q + \sum_{n\geq 2} \tau_k(n) q^n$ be the normalized generator of the space of cuspforms of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$. In particular, the modular form $\Delta_{12} = q\prod_{n\geq 1}(1-q^n)^{24}$ is Jacobi's Δ function. Moreover, we have

$$\Delta_{16} = \mathrm{E}_4\Delta,\ \ \Delta_{18} = \mathrm{E}_6\Delta,\ \ \Delta_{20} = \mathrm{E}_4^2\Delta,\ \ \Delta_{22} = \mathrm{E}_4\mathrm{E}_6\Delta.$$

Here, we have set $\mathrm{E}_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$, $\mathrm{E}_6 = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$ and $\sigma_k(n) = \sum_{d|n} d^{k-1}$.

For $(j,k) \in \{(6,8),(4,10),(8,8),(12,6)\}$ let $\Delta_{j,k}$ be a generator of the space $\mathrm{S}_{j,k}(\mathrm{Sp}_4(\mathbb{Z}))$ of Siegel cusp forms of genus 2 and coefficient $\mathrm{Sym}^j \otimes \det^k$, and let $\tau_{j,k}(p)$ be the eigenvalue of the standard Hecke operator [1] T_p on $\Delta_{j,k}$. As far as we know, there is no simple explicit formula for $\tau_{j,k}(p)$.

Following [C-L], for each even unimodular lattices L and M in $\mathbb{R}^{24}$, there are unique polynomials with rational coefficients P, P_k for $k \in \{12, 16, 18, 20, 22\}$, $P_{j,k}$ for $(j,k) \in \{(6,8),(4,10),(8,8),(12,6)\}$, and a constant $c \in \mathbb{Q}$, such that for all primes p we have :

$$\mathrm{N}_p(L,M) = P(p) + \sum_k \tau_k(p)P_k(p) + \sum_{j,k}\tau_{j,k}(p)P_{j,k}(p) + c\,(\tau_{12}(p)^2 - p^{11}),$$

where $\mathrm{N}_p(L,M)$ denotes the number of p-neighbors of L which are isometric to M. The polynomial P has degree 22 and is the dominant term in this formula.

Let us write $(Q_1,\ldots,Q_{11}) = (P_{12}, P_{16}, P_{18}, P_{20}, P_{22}, c, P_{6,8}, P_{8,8}, P_{4,10}, P_{12,6}, P)$. Each Q_s depends on (the isometry class of) L and M, hence should really be denoted by $Q_s(L,M)$. Let us order the Niemeier lattices as before. The list of all tuples $[s,i,j,Q_s(L(i),L(j))]$ is here :

`http://gaetan.chenevier.perso.math.cnrs.fr/niemeier/matrice`.

References

[C-L] G. Chenevier & J. Lannes, *Kneser neighbors and orthogonal galois representations in dimensions* 16 *and* 24, Oberwolfach Report (2011), article plus complet en cours de rédaction.

[1] See `http://gaetan.chenevier.perso.math.cnrs.fr/niemeier/listes_vp.pdf`