

with
(Laurent Clozel)

Number fields with given ramification and self-dual automorphic forms for $GL(2m)$

I Arithmetic problem

S finite set of primes, $\mathbb{Q}_S \subset \bar{\mathbb{Q}}$ max. alg. unramified outside S
 $\overset{\text{Fix}}{P} \in S, \bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_P \Rightarrow \text{Gal}(\bar{\mathbb{Q}}_P/\mathbb{Q}_P) \longrightarrow \text{Gal}(\mathbb{Q}_S/\mathbb{Q}) = G_S$

Theorem A if $|S| \geq 2$, this map is injective.

there are so many n. fields un. outside S that they generate $\bar{\mathbb{Q}}_P$

Cor: $|G_S| = |\hat{\mathbb{Z}}|$

Idea to attack them Use numb. fields coming from Galois action on l -adic cohom. of unitary Shimura varieties that we control well by work of Kottwitz, Clozel, Harris-Taylor. We are reduced to construct coh. classes on aut. forms which have prescribed prop. (analytic pt.)

- Remark:
- * Ch 04', theorem holds if $S \ni l \neq p, l \geq 3$ (4) and $\frac{l-1}{r} = 1$
 - * it suffices to consider $S = \{l, p\}$.
 - * mention Greenberg, elementary approach?

II Aut. forms

Fix ω supercuspidal rep. of $GL_m(\mathbb{Q}_p)$,
 Fix $l \neq p$. Denote $\rho(\omega) =$ Beuzdek comp. of $GL_{2m}(\mathbb{Q}_p)$ of $\omega \times \omega^v$

$\omega^v \neq \omega \chi \quad \forall \chi$ unram.
 $\{ \pi(\chi) = \text{Ind}_{(m,m)}^{GL_m} \omega \chi \times \omega^v \chi^{-1} \}$

Theorem B $\exists \pi$ unrep aut form on $GL_{2m}(\mathbb{A}_S)$,

- i) $\pi^v \cong \pi$
- ii) π_p alg. regular
- iii) π_e Steinberg
- iv) $\pi_p \in \rho(\omega)$
- v) π unram. outside $\{l, p, \infty\}$

Rmk (i) Rigidity: $\pi^V \supset \pi$ and control at every place.

⚠ we can't prescribe anything, e.g. we can't produce antidead gold with π_q supersing. rkh. and $\pi_q^!$ supersing. symplectic!

(ii) thm B \Rightarrow thm A using HT theorem (l -adic real.) and some elementary steps.

(iii) In my previous work, I used unitary aut. forms, here $\Rightarrow GL_n/\mathbb{Q}$.

Our proof will use Arthur's trace form. for $G = GL_n/\mathbb{Q}$ twisted by θ .

$\theta(g) = J g^{-1} J^{-1}$, $\theta^2 = 1$, θ permutes T, B, epimorph.

$\text{tr}(\text{I}_\theta R(f))$, $R = G \times_{\text{anp}}^2 (\mathbb{R}^* \times G(\mathbb{Q})) \setminus G(\mathbb{A})$

(\rightarrow Need some results that might be contained in Arthur's book at the end)

III Test functions f_x

We are interested in following rep. of $GL_n(\mathbb{R})$

$\left\{ \begin{aligned} \phi: W_{\mathbb{R}} &\rightarrow GL_n(\mathbb{C}) \\ &= \bigoplus_{i=1}^n \begin{pmatrix} (\frac{3}{2})^{p_i} \\ (\frac{3}{2})^{-p_i} \end{pmatrix} \otimes W_{\mathbb{R}} \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} \phi: W_{\mathbb{R}} &\rightarrow \mathcal{F}_{2n}(\mathbb{C}) \\ &\text{discrete} \end{aligned} \right\} \hookrightarrow \left\{ \begin{aligned} L\text{-packet of discrete} \\ &\text{seis on } SO_{2n+1}^*(\mathbb{R}) \end{aligned} \right\}$

$p_i \in \frac{1}{2} + \mathbb{Z}$, $2 \text{ by } 2 \neq$

\hookrightarrow isom., θ -discrete.

$\{\pi_{\lambda, \lambda}\}$

$\{\nu_{\lambda, \lambda}\}$ ind. rep. of $SO_{2n+1}^*(\mathbb{R})$

M_{30} $\exists f_{\lambda}$ twisted pseudo-coeff for π_{λ} : $\left| \begin{aligned} \text{trace}(A_{\theta} R(f)) &= 0 \text{ if } f \neq \pi_{\lambda} \\ &= 1 \text{ if } f = \pi_{\lambda} \end{aligned} \right.$

choice of A_{θ} : if $\pi^V \supset \pi \Leftrightarrow \pi = \pi \circ \theta$, $\exists A_{\theta} \cdot \pi \rightarrow \pi \circ \theta$, $A_{\theta}^2 = 1$
determined up to a sign. if π generic, I choose the one which acts by +1 on $W_{\mathbb{R}}$.

Norm map
in this context
(Waldspurger)

$$\gamma \longrightarrow N\gamma \quad (\sim r^\theta \gamma)$$

$$\theta\text{-ell} \rightsquigarrow N\gamma \text{ conj. class in } SO(2n+1)(\mathbb{R})$$

Thm C γ θ -simple, $TO_\gamma(f_\lambda) = \begin{cases} \epsilon(\gamma) \text{ tr}(N\gamma | V_\lambda) & \text{if } \gamma \text{ ellipt.} \\ 0 & \text{else.} \end{cases}$

(for some choice of > 0 Haar measure).

Pf: Use results of Bouaziz, Labesse.

Ⓧ test function at finite primes

• unit of spherical Hecke algebra outside l, p .

• At l , $f_l =$ pseudo coeff. of St_l (via Kottwitz method, EP functions)

• At p , Recall $\zeta_p(w)$

positive pseudo coefficients $f_p: \begin{cases} \text{tr}(A_\theta \rho(f_p)) = 0 & p \notin \zeta_p(w) \\ \text{tr}(A_\theta I(x)(f_p)) > 0 & I(x) \text{ unitary} \end{cases}$

Thm D: Fix f_p as above (it exists)
 (i) $\gamma \rightarrow STO_\gamma(f_p)$ (on sh conj. cl's)
 is not $\equiv 0$

(ii) $TO_{\gamma_0}(f_p) = STO_{\gamma_0}(f_p) > 0$

$$\left. \begin{array}{l} \gamma_0 \text{ is the unique } \theta\text{-conj. class with Norm } 1. \\ \text{"} \\ \text{"} \\ 1 \end{array} \right\} \begin{array}{l} \text{Its centraliser is } Sp_{2n}, \theta\text{-conj } \gamma_0 = \text{st } \theta\text{-conj } \gamma_0 \\ H^1(\mathbb{Q}_p, Sp_{2n}) = 0. \end{array}$$

Remark: In agreement with results of Shahidi when f_p is
 a coeff. of a selfdual supercuspidal π_p . ($f_p(1) \neq 1$)
 $TO_{\gamma_0}(f_p) \neq 0$ related to symplectic/orthogonal
 alternative.

V Pf of Thm B

ATF, $f = f_\lambda \otimes f_\rho \otimes f_p \otimes f^{\omega, l, p}$ just defined, $\lambda \xrightarrow{\text{off the walls}} \infty$

$$\text{tr}(I_{\mathbb{R}} R(f)) = \sum_{\gamma \in G(\mathbb{C})_{\text{ell}}} \nu_\gamma \text{TO}_\gamma(f), \quad \left. \begin{array}{l} \text{the sum is} \\ \text{finite indep.} \\ \text{of } \lambda \end{array} \right\}$$

" (Kun C)
 $\text{tr}(N\sigma(V_\lambda)) \text{TO}_\gamma(f)$

Prop: G is a compact connected Lie group $(T, X^*(T), \alpha; V_\alpha)$

$\gamma \in G$ non central $\Rightarrow \frac{\text{tr}(\gamma | V_\lambda)}{\dim V_\lambda} \xrightarrow{\lambda \rightarrow \infty} 0$

\oplus properties of γ_0 (unique class $N\sigma=1$) $\Rightarrow \frac{\text{tr}(I_{\mathbb{R}} R(f))}{\dim V_\lambda} \rightarrow \nu_{\gamma_0} \text{TO}_{\gamma_0}(f)$

Remark: Same G_2 , RR formula. $\neq 0$ by Thm D (and some work at ℓ)

VI Pf Thm D (i) $(\text{TO}_{\gamma_0}(f_p) \neq 0)$ local statement but global #

Step 1 fix λ_0 (e.g. 0) Using S.T.F (KS) show that some π_0 exists as in Thm B but maybe ramified outside $\{\infty, l, p\}$. (uses Thm 0 (i)) \wedge at say S'

Step 2 positivity argument, with mammalshion.

define > 0 pseudo coeff. fns $\pi_{0, S'}$ say $f_{S'}$, and apply

ATF: $\sum_{\gamma \in G(\mathbb{C})_{\text{ell}}} \nu_\gamma \text{TO}_\gamma(f) \geq 0 \quad \forall \lambda$
 $> 0 \quad , \quad \lambda = \lambda_0$

Prop: G compact Lie group, $\gamma_1, \gamma_2, \dots, \gamma_n$ 2 by 2 + cong. classes in G
 $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ complex numbers.

if $\left| \sum_{i=1}^n \lambda_i \text{tr}(\rho(\gamma_i)) \right| \geq 0 \quad \forall \rho \in \hat{G}, > 0 \quad \rho = \rho_0 \Rightarrow \lambda_1 > 0$