1. Summary

Let us summarize a bit what we have done so far and what remains to do.

Fix p a prime, $S \subset S(\mathbb{Q})$ a finite set containing p and ∞, and let $\mathfrak{X}$ be the p-adic character variety of $G_{\mathbb{Q},S}$ of dimension 2. By definition and Procesi-Taylor's theorem, $\mathfrak{X}(\overline{\mathbb{Q}}_p)$ is in bijection with the set of continuous semi-simple representations $G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, say $x \mapsto \rho_x$. We have seen that there is a natural admissible disjoint union

$$\mathfrak{X} = \coprod_{\bar{\rho}} \mathfrak{X}(\bar{\rho})$$

over the semi-simple $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ (up to isom and Frobenius action on image), where $\mathfrak{X}(\bar{\rho})$ is the clopen subspace of ρ such that $\bar{\rho}_x \simeq \bar{\rho}$, and also the analytic generic fiber of the pseudo-deformation ring $R(\mathrm{trace}(\bar{\rho}))$ ($=R(\bar{\rho})$ if $\bar{\rho}$ is irreducible).

We have also seen that for odd regular $\bar{\rho}$, $\mathfrak{X}(\bar{\rho})$ is isomorphic to the open unit ball of dimension 3 over some finite unramified extension of $\mathbb{Q}_p$. We have given an example of such a $\bar{\rho}$ for $p = 5$, $S = \{\infty, 5, 17\}$, namely the representation on $E[5]^\vee$ where E is the elliptic curve $y^2 + xy + y = x^3 - x^2 - x$ over the rationals. We have said that this $\bar{\rho}$ is actually modular : $\bar{\rho} \simeq \bar{\rho}_f$ where f is a generator of the one dimensional space $S_2(17, 1)$. Indeed, the curve E is isogenous to the elliptic curve $X_0(17)$. By the Jacquet-Langlands correspondence, or an explicit argument based involving Theta series similar to the one we have given in discriminant 11, we also have seen that $\bar{\rho} \simeq \bar{\rho}_f$ where f is the unique non-trivial quaternionic eigenform in $S_2^D(1)$, where D is the quaternion algebra of discriminant 17.

Say that a point $x \in \mathfrak{X}$ is modular[1] if ρ_x is[2], a notion that we have defined in lecture 2. Our main aim in the last two lectures will be to show the following :

Theorem 1.1. *(Gouvêa-Mazur, Coleman) Assume that $\bar{\rho}$ is odd regular and modular quaternionic. Then the modular points are Zariski-dense in $\mathfrak{X}(\bar{\rho})$.*

The Zariski-density here simply means that if a power series $F(x, y, z) \in \mathbb{C}_p[[x, y, z]]$ converging on the open unit ball $\mathfrak{X}(\bar{\rho}) = \{(x, y, z), |x|, |y|, |z| < 1\}$ vanishes on the modular points, then it is zero.

In the works of the aforementionned authors, the "quaternionic assumption" is unnecessary, but it is so in the approach that we shall employed. They show more generally that for each modular $\bar{\rho}$ (not necessarily regular), each irreducible component of the Zariski-closure of the modular points in $\mathfrak{X}(\bar{\rho})$ has dimension ≤ 3. The main ingredient of the proof is the construction of the *eigencurve*, which is a rigid analytic curve, whose set-theoretic image $\mathcal{F} \subset \mathfrak{X}$ is the (complete) infinite fern alluded above in the introduction. The construction of this curve is based on the theory of p-adic overconvergent modular forms (Serre, Katz, Hida, Coleman) : the points in $\mathcal{F}$ are the Galois representations attached to certain p-adic modular eigenforms. We shall rather rely on the theory of quaternionic p-adic modular forms which are rather easier to define and study (Buzzard, following ideas of Stevens). As

[1]In the end we may eventually restrict to some modular ρ_x with certain specific level.

[2]For any embedding $k(x) \to \overline{\mathbb{Q}}_p$, if it is so for one it is for all.

we shall see, this theory mostly depends on the most basic properties of the families of locally analytic principal series of a Iwahori subgroup in $\mathrm{GL}_2(\mathbb{Q}_p)$, from which we shall start, after recalling some basic facts of p-adic spectral theory.

References: The articles by Serre and Katz in *Modular functions in one variables* 3 (Antwerp conference), Coleman's papers *Classical and overconvergent modular forms* and *p-adic Banach spaces and families of modular forms* (both in Inv. Math.), Stevens paper *Rigid analytic modular symbols* (unpublished, see his web page), Buzzard's papers *p-adic families of automorphic forms* and *Eigenvarieties* (in L-functions and Galois representations (2007), proceedings of the Durham Conference), Mazur's paper *An infinite fern in the deformation space of Galois representations* ().

2. p-adic spectral theory in families

References : Over a field, see Serre *Endomorphismes complètement continus des espaces de Banach p-adiques* (Publ. Math. IHES 12). For the Banach-module theory, see Coleman *p-adic Banach spaces and families of modular forms*, BGR, and Buzzard *Eigenvarieties.*

2.1. Orthonormalizable Banach modules. Let A is an affinoid $\mathbb{Q}_p$-algebra, viewed as a Banach $\mathbb{Q}_p$-algebra as in lecture 5. We denote by Ban_A the category whose objects are Banach A-modules and whose morphisms are A-linear continuous (=bounded) maps. Recall that a Banach A-module is an A-module M equipped with an ultrametric $\mathbb{Q}_p$-vector space norm $|.|$ for which it is complete and such that there exists a constant $C > 0$ such that $|am| \leq C||a|||m|$ for all $a \in A$ and $m \in M$ (here $||.||$ is a Banach-algebra norm on A, the choice of another norm gives rise to another C).

A standard example is the A-module $A^{(I)}$ of sequences $(x_i)_{i\in I}$ of elements of A converging to 0 with the sup norm $\mathrm{Sup}_i|x_i|$ (convergence of $(x_i)_{i\in I}$ to 0 means that for each real number $C > 0$, then $|x_i| > C$ for at most finitely many $x_i \in A$). If $e_i \in A^{(I)}$ denotes the sequence $(e_i)_j = \delta_{i,j}$, we see that the map $\mathrm{Hom}_{\mathrm{Ban}_A}(A^{(I)}, N) \to NI$, $u \mapsto (u(e_i))_i$ is a bijection onto the subspace of bounded sequences of elements of N. A general Banach A-module is said orthonormalizable, ONable for short, if it is isomorphic to $A^{(I)}$ (in Ban_A). In other words, a Banach A-module M is ONable iff there exists a family $(e_i)_{i\in I}$ of elements of M and a constant $C > 0$ such that :

(ON1) $\forall x \in M$, there is a unique $(x_i) \in A^{(I)}$ such that $x = \sum_{i\in I} x_i e_i$,

(ON2) for all $x \in M$, $|x|/C \leq \mathrm{Sup}_i|x_i| \leq C|x|$.

In this case, (e_i) is called an ON-basis of M. If $A = L$ is a field, it is a classical fact that any L-Banach space is ONable (see e.g. Serre's paper cited above). As another general example, the relative Tate-algebra A-algebra $A\langle t_1, \ldots, t_r\rangle$ is ON-able (with $I = \mathbb{N}$).

There is an obvious notion of direct sum $M \oplus N$ of two Banach A-modules M and N : it is the direct sum as A-module equipped with the norm $|m+n| = \mathrm{Sup}(|m|, |n|)$. We say that a Banach A-module M is *stably orthonormalizable*, or SONable for short, if there exists a Banach A-module N such that $M \oplus N$ is ONable.

2.2. Banach A-modules as families of Banach spaces. If M, N, P are three Banach A-modules, define $\mathrm{Bil}_A(M \times N, P)$ as the set of A-bilinear maps $u : M \times N \to P$ which are continuous, i.e. such that $|u(m,n)| \leq C|m||n|$ for some $C > 0$ and all (m,n) in $M \times N$. Then $P \mapsto \mathrm{Bil}_A(M \times N, P)$ defines an obvious functor $\mathrm{Ban}_A \to \mathrm{Sets}$. A simple but important fact is that

PROPOSITION 2.3. *This functor is representable. We denote by $M\widehat{\otimes}_A N$ the universal object.*

Proof — Indeed, for $x \in M \otimes_A N$ (usual tensor product), define $|x|$ as the infimum over all the possible writings $x = \sum_i m_i \otimes n_i$, of the number $\mathrm{Sup}_i|m_i||n_i|$. This defines a semi-norm on $M \otimes_A N$. Its separated completion, denoted by $M\widehat{\otimes}_A N$, is equipped with the natural A-bilinear map $(m,n) \to \mathrm{Im}(m \otimes n) := m\widehat{\otimes} n$. It has the required universal property : if $u \in \mathrm{Bil}_A(M \times N, P)$ has norm C then u defines an A-linear map $\tilde{u} : M \otimes_A N \to P$ such that $|u(x)| \leq C|x|$ for all $x \in M \otimes_A N$. As P is a Banach A-module, $\tilde{u}$ uniquely extends to a continuous A-linear map $M\widehat{\otimes}_A N \to P$. $\square$

This notion leads to the important notion of *completed scalar extension* : if B is an affinoid A-algebra, that we may view as an object of Ban_A, then $M \mapsto M\widehat{\otimes}_A B$ defines a functor $\mathrm{Ban}_A \to \mathrm{Ban}_B$. If M is a Banach A-module and N is a Banach B-module, one checks at once the adjunction formula

$$\mathrm{Hom}_{\mathrm{Ban}_A}(M, N) = \mathrm{Hom}_{\mathrm{Ban}_B}(M\widehat{\otimes}_A B, N).$$

It follows that $M \mapsto M\widehat{\otimes}_A B$ respects ON-ability, the image of an ON-basis (e_i) being simply $e_i\widehat{\otimes}1$. It also commutes with direct sums, thus respects SON-ability.

If $X = \mathrm{Max}(A)$ and M is a Banach A-module, we denote $M_x = M\widehat{\otimes}_A A/x$ for $x \in X$: *any Banach A-module M can be viewed as a family of p-adic Banach spaces M_x parameterized by X*. SONable M may be interpreted this way as the nicest families of Banach spaces over X.

Exercise: Let M, N be Banach $\mathbb{Q}_p$-vectorspaces. Recall (or show!) that any $\mathbb{Q}_p$-linear continuous surjection $M \to N$ is open ("open mapping theorem"). Deduce that a Banach A-module P is SONable if, and only if, for any surjection $M \to N$ in Ban_A, the map $\mathrm{Hom}_{\mathrm{Ban}_A}(P, M) \to \mathrm{Hom}_{\mathrm{Ban}_A}(P, N)$ is surjective.

2.4. Compact endomorphisms. If M, N are two Banach A-modules, $\mathrm{Hom}_A(M, N)$ is[3] a Banach A-module for the sup norm

$$|u| = \mathrm{sup}_{x \neq 0} \frac{|u(x)|}{|x|},$$

and $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$ is a Banach A-algebra. We say that $u \in \mathrm{Hom}_A(M, N)$ is of finite rank if $u(M)$ is a finitely generated A-module. The closure

$$\mathrm{Comp}_A(M, N) \subset \mathrm{Hom}_A(M, N)$$

(resp. $\mathrm{Com}_A(M) \subset \mathrm{End}_A(M)$) of finite rank operators is the Banach A-module of *compact operators*. It is stable by left and right composition by continuous endomorphisms, as so is the submodule of finite rank operators. In particular $\mathrm{Comp}_A(M)$ of $\mathrm{End}_A(M)$.

[3]From now one we shall write Hom_A for $\mathrm{Hom}_{\mathrm{Ban}_A}$.

Exercise: Let $A = L$ be a finite extension of $\mathbb{Q}_p$, M and N two L-Banach spaces. Show that $u \in \mathrm{Hom}_L(M,N)$ is compact if, and only if, for any bounded subset $B \subset M$, the subset $u(B)$ has a compact closure in N.

Assume that M and N are ONable A-module with respective ON-basis (e_j) and (f_i). Any $u \in \mathrm{Hom}_A(M,N)$ then is uniquely determined by a matrix $(u_{i,j}) \in A^{\mathbb{N}\times\mathbb{N}}$ where $u(e_j) = \sum_i u_{i,j} f_i$. Applying the universal properties of ONable modules to M and N, we see that this matrix can be any matrix with bounded coefficients, each column of which goes to zero. The following propostion is intuitive but its proof (due to Coleman) needs an argument (see also Buzzard's eigenvariety paper).

LEMMA 2.5. *Under the above assumptions, u is compact iff for each $C > 0$, there is at most finitely many $u_{i,j}$ such that $|u_{i,j}| > C$.*

If $u \in \mathrm{Comp}_A(M)$ and if M is ONable, it makes then sense to define[4]

$$\det(1 - Tu) = \sum_{n\geq 0} (-1)^n \mathrm{trace}(\Lambda^n(u)) T^n \in A[[T]]$$

by the obvious formula in terms of the minors of the matrix $(u_{i,j})$ of u in a given ON-basis (e_i) of M. For instance, one set $\mathrm{trace}(u) := \sum_i u_{i,i}$, $\mathrm{trace}(\Lambda^2(u)) := \sum_{i<j}(u_{i,i}u_{j,j} - u_{i,j}u_{j,i})$, etc... It not difficult to check that:

- (F1) This power series is independent on the choice of the ON-basis (e_i) of M. It is called the *characteristic – or Fredholm – power series* of u on M.
- (F2) It belongs to subring $A\{\{T\}\} \subset A[[T]]$ of entire series: $|a_n|R^n$ goes to 0 for any real number $R > 0$. In other words, it defines a rigid analytic function on $X \times \mathbb{A}^1$ (consider the admissible covering by the $X \times B_n$ for $n \geq 1$ where B_n is the closed disc of center 0 and radius p^n) .

Note that if B is an affinoid A-algebra, it follows from the definitions that $u\widehat{\otimes}_A B$ is compact. As $-\widehat{\otimes}_A 1$ sends A-ON basis to B-ON basis as we have seen, the matrix of $u\widehat{\otimes}_A B$ in the $e_i\widehat{\otimes}1$ is its obvious image $(u_{i,j}) \in B^{\mathbb{N}\times\mathbb{N}}$, so $\det(1 - Tu_{|M\widehat{\otimes}_A B})$ is simply the image of $\det(1-Tu)$ in $B\{\{T\}\}$ via the natural map $A\{\{T\}\} \to B\{\{T\}\}$.

As seen by Buzzard, the definition above extends at once to the SONable case: choose some N such that $M \oplus N$ is ONable and apply the definition above to $u \oplus 0$.

In this p-adic world, the elements $u \in \mathrm{Comp}_A(M)$, that we may see as families of compact endomorphisms u_x of the M_x for $x \in X$, have an extremely nice spectral theory or "Fredholm-Riesz" theory. The following result, that we shall admit, is due to Serre when A is a field, to Coleman in the ON-able case, completed by Buzzard in the SONable one. If $G \in A[T]$ is a polynomial, we set $G^*(T) = T^{\deg(G)}G(1/T)$.

THEOREM 2.6. *Let M be a SON-able Banach A-module, $u \in \mathrm{Comp}_A(M)$, and P the characterstic power series of u. Assume that $F = GH \in A\{\{T\}\}$ where $G \in 1 + TA[T]$ is a polynomial whose leading coefficient is a unit in A, and such that $(G,H) = 1$ in $A\{\{T\}\}$.*

[4] Note that we cannot define $\det(T - u)$ in general by any reasonnable formula, even when $u = 0$!

Then there is a unique decomposition $M = P \oplus Q$ into closed, u-stable, A-submodules, such that $G^*(u)$ vanishes on P and is invertible on Q. This decomposition is preserved by any element of $\mathrm{End}(M)$ *commuting with u and has the following properties:*

- *P is projective and finite type over A of rank* $\deg(G)$,
- $\det(1 - Tu_{|P}) = G$ *and* $\det(1 - Tu_{|Q}) = H$.

Let us specify this result when $A = L$ is a field, $M = V$ is an L-Banach space, $u \in \mathrm{Comp}_L(V)$ (this is the case studied by Serre). In this case, the theory of Newton polygon allows to write F – and more generally any entire power series $F \in 1 + TL\{\{T\}\}$ – as an infinite product $\prod_{i\geq 0} F_i$ where each F_i is polynomials in $1 + TL[T]$ with a unique slope $s_i \in \mathbb{Q}$, and $s_i \neq s_j$ for $i \neq j$. In particular $s_i \to \infty$ and the P_i are coprime in $L[T]$ (hence in $L\{\{T\}\}$). It follows that we have a unique topological u-stable decomposition

$$V = V' \oplus \overline{\bigoplus_{i\geq 0} V_{s_i}}$$

where each $V_{s_i} = \mathrm{Ker}(F_i)$ is finite dimensional over L of dimension $\deg(F_i)$, V' is closed, and where $P(u)$ is invertible on V' whenever $P \in 1 + TL[T]$.

Definition 2.7. *The elements of the subspace $\sum_{i\geq 0} V_{s_i} \subset V$ are called the finite slope vectors of u, V_{s_i} being the subspace of elements of slope s_i. Each V_{s_i} is preserved by any $v \in \mathrm{End}_L(V)$ commuting with u. By definition, each eigenvalue of u on $V_{s_i} \otimes_L \overline{L}$ has valuation s_i.*

For later use, let us finally state the following lemma :

Lemma 2.8. *(Parallelogram lemma) Let M and N be SONable Banach A-modules, $u \in \mathrm{Comp}_A(M, N)$ and $v \in \mathrm{Hom}_A(N, M)$. Then $vu \in \mathrm{Com}_A(M)$, $uv \in \mathrm{Comp}_A(N)$ and* $\det(1 - Tuv) = \det(1 - Tvu)$.

See the Serre and Coleman's papers for the proof. Let us simply say that one first reduces to the case where M and N are finite free A-module. Then we reduce to the case $M = N$: consider $M \oplus N$ and the extension of u (resp. v) by 0 on N (resp. M). In this case this is well known !

3. Locally analytic principal series of a Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$.

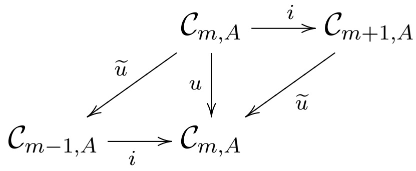
3.1. Locally analytic functions on $\mathbb{Z}_p$. Let A be a $\mathbb{Q}_p$-affinoid algebra. For each integer $m \geq 0$, define the space of m-analytic A-valued functions on $\mathbb{Z}_p$ by

$$\mathcal{C}_{m,A} = \{f : \mathbb{Z}_p \to A \mid \forall x \in \mathbb{Z}_p, f_{x,m}(t) := f(x + p^m t) \in A\langle t\rangle\} \subset A^{\mathbb{Z}_p}.$$

For instance, the 0-analytic (or simply analytic) functions $f : \mathbb{Z}_p \to A$ are the $f(t) \in A\langle t\rangle$. If $|.|$ is a Banach algebra norm on A, then the norm $|f| := \mathrm{Sup}_{x\in\mathbb{Z}_p}|f_{x,m}|_{\mathrm{gauss}}$ is a Banach algebra norm on $\mathcal{C}_{m,A}$. Note that this Sup is really a Sup over the finite set $x \in \{0, \dots, p^m - 1\}$, and that $\mathcal{C}_{m,A}$ is isometric to $(A\langle t\rangle)^{p^m}$. In particular $\mathcal{C}_{m,A}$ is ON-able.

We have (continuous) inclusions $i : \mathcal{C}_{m,A} \to \mathcal{C}_{m+1,A}$. However, $\mathcal{C}_{m,A}$ is not a closed subspace of $\mathcal{C}_{m+1,A}$. In fact, one could check that it is even a dense subspace ! An interesting operator on $A^{\mathbb{Z}_p}$ is the map u defined by $u(f)(t) = f(pt)$.

PROPOSITION 3.2. *("u improves analyticity") For each $m \geq 0$, $u \in \mathrm{Comp}_A(\mathcal{C}_{m,A})$ and $i \in \mathrm{Comp}_A(\mathcal{C}_{m,A}, \mathcal{C}_{m+1,A})$. Moreover, $u(\mathcal{C}_{m+1,A}) \subset \mathcal{C}_{m,A}$ and u induces a continuous map $\widetilde{u} : \mathcal{C}_{m+1,A} \to \mathcal{C}_{m,A}$. For each $m \geq 1$, we have a commutative diagram :*



Proof — For $x \in \mathbb{Z}_p$, we have $(u(f))_{x,m} = f_{px,m+1}$ as functions $\mathbb{Z}_p \to A$. It follows that $u(\mathcal{C}_{m+1,A}) \subset \mathcal{C}_{m,A}$ and that the induced map

$$\widetilde{u} : \mathcal{C}_{m+1,A} \to \mathcal{C}_{m,A}$$

is continuous with norm ≤ 1 ("one forgets the $f_{x,m+1}$ with $x \in \mathbb{Z}_p^\times$"). As the diagramm above is tautologically commutative, it only remains to check that $i \in \mathrm{Com}_A(\mathcal{C}_{m,A}, \mathcal{C}_{m+1,A})$ for each $m \geq 0$. As $f_{x+jp^m,m+1} = f_{x,m}(j+pt)$, the inclusion i is the product of p^m times copies of the map

$$A\langle t\rangle \to A\langle t\rangle^p, f(t) \mapsto (f(i+pt))_{i=0\cdots p-1}.$$

So it is enough to check that $f(t) \mapsto f(i+pt)$ is compact on $A\langle t\rangle$. But it is the composite of the operator $f(t) \mapsto f(i+t)$ by $f(t) \mapsto f(pt)$, the first one being an isometry (whose inverse is $f(t) \mapsto f(-i+t)$) and the second one is typically compact of norm ≤ 1. Indeed, in the ON-basis $1, t, t^2, \ldots$ it is the diagonal matrix

$$(1, p, p^2, \ldots).$$

□

3.3. The weight space. We define *the weight space* $\mathcal{W}$ as the p-adic character variety of $\mathbb{Z}_p^\times$ in dimension 1: for any affinoid $\mathbb{Q}_p$-algebra A, $\mathcal{W}(A)$ is the set of continuous characters $\kappa : \mathbb{Z}_p^\times \to A^\times$ (functorially in A).

From now on we shall assume that p is odd to simplify. Recall that

$$\mathbb{Z}_p^\times \simeq \mu \times (1 + p\mathbb{Z}_p)$$

where $\mu \subset \mathbb{Z}_p^\times$ is the subgroup of $p-1$-th roots of unity. Moreover $x \mapsto (1+p)^x$ is a topological isomorphism $(\mathbb{Z}_p, +) \xrightarrow{\sim} (1 + p\mathbb{Z}_p, \times)$, so $(1 + p\mathbb{Z}_p) = (1+p)^{\mathbb{Z}_p}$.

Let $\mathcal{B}$ be the 1-dim open unit disc $|z - 1| < 1$ over $\mathbb{Q}_p$ (a rigid analytic space) and $\mu^\vee = \mathrm{Hom}_{\mathrm{groups}}(\mu, \mathbb{Q}_p^\times)$ (so $|\mu^\vee| = |\mu|$).

PROPOSITION 3.4. *$\mathcal{W}$ is isomorphic to the disjoint union of $p-1$ copies $\mathcal{B}_\eta$ of $\mathcal{B}$ for $\eta \in \mu^\vee$, the universal character $\kappa^{\mathrm{univ}} : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{W})^\times$ being given on $\mathcal{B}_\eta$ by*

$$\kappa^{\mathrm{univ}}(1+p) = z.$$

Proof — As a warm up for the general case, let us check this statement for the L-points where $L/\mathbb{Q}_p$ is any finite extension. A continuous character $\kappa : \mathbb{Z}_p^\times \to L^\times$ is uniquely determined by its restrictions to μ and to $1+p\mathbb{Z}_p$. As the $(p-1)$-th roots of unity in L already belong to $\mathbb{Q}_p$, it follows that $\kappa(\mu) \subset \mathbb{Q}_p^\times$, so $\kappa_{|\mu} \in \mu^\vee$: this gives $\eta = \kappa_{|\mu}$. Moreover $z := \kappa(1+p) \in L^\times$ has the property that $z^{p^n} \to 1$. It follows that $z \in \mathcal{O}_L^\times$ and that $z \equiv 1 \bmod \pi_L$, so $|z-1| < 1$. Conversely, if $z \in 1 + \pi_L \mathcal{O}_L$, then $z^{p^n} \equiv 1 \bmod \pi^n$, so the character $\kappa : (1+p)^{\mathbb{Z}} \to L^\times$ sending $1+p$ to z extends to a continuous character of $1 + p\mathbb{Z}_p$, what we had to show.

For a similar reason, for each open affinoid $\Omega \subset \mathcal{W}$ then the character

$$(1+p)^{\mathbb{Z}} \to \mathcal{O}(\Omega)^\times$$

sending $1+p$ to $z_{|\Omega}$ extends to a continuous character of $1+p\mathbb{Z}_p$. Indeed, if $|.|$ denotes the sup norm on Ω, then for some integer $M \geq 1$ we have $z_{|\Omega} = 1+u$ with $u \in \mathcal{O}(\Omega)$ and $|u| \leq \frac{1}{p^{1/M}}$ by the "maximum modulus principle". Thus $|(1+u)^{p^m} - 1| \leq \frac{1}{p^{m/N}} \to 0$.

We now prove the general case. A continuous $\kappa : \mathbb{Z}_p^\times \to A^\times$ with say A connected, is uniquely determined by its restriction to μ, which is an element of $\mathrm{Hom}(\mu, \mathbb{Q}_p^\times)$ if A is connected, and by the element $z = \kappa(1+p)$. This latter element is any power bounded element $z \in A$ such that $z^{p^n} \to 1$ and it is an exercise to check that these conditions are equivalent to ask that there is some integer $m \geq 1$ such that $\frac{(z-1)^m}{p}$ is power bounded. As we say in lecture 5, this is also equivalent to ask that for some $m \geq 1$ the element z defines a morphism from $\mathrm{Sp}(A)$ to the closed disc $|z-1|^m \leq 1/p$, and we see that κ is simply the pull-back of κ^{univ} by this morphism.
□

A continuous character $\kappa : \mathbb{Z}_p^\times \to A^\times$, A an affinoid algebra, is said m-analytic if the map $t \mapsto \chi(1+pt)$ is m-analytic on $\mathbb{Z}_p$. By the multiplicativity property, it is equivalent to ask that $t \mapsto \chi(1+p^{m+1}t)$ belongs to $A\langle t \rangle$. This condition defines a subfunctor $\mathcal{W}_m \subset \mathcal{W}$.

PROPOSITION 3.5. *$\mathcal{W}_m$ is representable by an open subspace of $\mathcal{W}$, in the description above it is the open subset defined on each η-component by $|z-1|^{p^m(p-1)} < 1/p$ ("its boundary contains the p^{m+1} roots of unity").*

Proof — Let us give the idea for the points in finite extensions $L/\mathbb{Q}_p$, the general case being similar and left as an exercise to the reader. Let $\kappa : \mathbb{Z}_p^\times \to L^\times$ be a continuous character and $z = \kappa(1+p) \in 1+\pi_L\mathcal{O}_L$. For $t \in \mathbb{Z}_p$, let $\Lambda(t) = \frac{\log_p(1+pt)}{\log_p(1+p)} \in \mathbb{Z}_p$, so that $1+pt = (1+p)^{\Lambda(t)}$. Note that $\Lambda(t) \in \mathbb{Z}_p\langle t \rangle$. If $|z-1| < \frac{1}{p^{1/(p-1)}}$, then

$$\kappa(1+pt) = z^{\Lambda(t)} = \sum_{n \geq 0} \binom{\Lambda(t)}{n} (z-1)^n \in L\langle t \rangle,$$

as the p-adic valuation of $n!$ is less than $\frac{n}{p-1}$, thus κ is 0-analytic. For the same reason, if $|z-1| < \frac{1}{p^{1/p^m(p-1)}}$ then κ is m-analytic. We leave as an exercise using $\exp_p$ and $\log_p$ to check that conversely, all the m-analytic characters have their z in the disc $|z-1| < \frac{1}{p^{1/p^m(p-1)}}$. □

In particular any κ as above is m-analytic for m big enough and we shall denote by $m(\kappa)$ the least such integer m. Explain this with a picture.

3.6. The principal series. Let us introduce some standard subgroups of $G = \mathrm{GL}_2(\mathbb{Q}_p)$:

- B is the subgroup of lower triangular elements in G,
- N is the subgroup of upper triangular unipotent elements in $\mathrm{GL}_2(\mathbb{Z}_p)$, that we shall identify with $\mathbb{Z}_p$ by the map $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$,
- $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$,
- $I \subset \mathrm{GL}_2(\mathbb{Z}_p)$ is the subgroup whose elements are upper triangular mod p,
- $u \in \mathrm{GL}_2(\mathbb{Q}_p)$ the diagonal element $(1,p)$,
- M is the submonoid of $M_2(\mathbb{Z}_p)$ generated by I and u. Note that an element of m has the form[5] $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a,b,c,d \in \mathbb{Z}_p$, $a \in \mathbb{Z}_p^\times$ and $p|c$.

LEMMA 3.7. (i) *The action of I by right translations on $B\backslash G$ has 2 orbits, namely $G = BI \coprod BwI$. The orbit BI has the extra property that it is stable by right translations by M. The multiplication in G induces a bijection $B \times N \overset{\sim}{\to} BI$.*

(ii) *For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$ and $t \in \mathbb{Z}_p$ we have*

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+ct & 0 \\ c & \frac{\det(m)}{a+ct} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{b+td}{a+ct} \\ 0 & 1 \end{pmatrix},$$

and $\frac{b+td}{a+ct} \in \mathbb{Z}_p$ as $a \in \mathbb{Z}_p^\times$ and $p|c$.

(iii) *In particular, the action of M on $B\backslash BI \overset{\sim}{\to} N \overset{\sim}{\to} \mathbb{Z}_p$ is the usual right-action by homographies : $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(t) = \frac{b+td}{a+ct}$. The action of u is thus $u(t) = pt$.*

Proof — The first statement follows for instance from the Iwasawa decomposition $G = B \cdot \mathrm{GL}_2(\mathbb{Z}_p)$ and from the Bruhat decomposition $\mathrm{GL}_2(\mathbb{Z}_p) = I \coprod IwI$. From the Iwahori-decompositon $I = (I \cap B) \times N$ (check that!) one get the decomposition $BI = B \times N$. To check that $BIu \subset BI$ remark that $u^{-1}Nu \subset N$. Assertions (ii) and (iii) are straightforward computations. □

We are now ready to define the principal series representations of I (or rather, of M). We shall view any character χ of $\mathbb{Z}_p^\times$ as a character of the diagonal torus $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ sending any $(p^m x, y)$ with $x \in \mathbb{Z}_p^\times$ to $\chi(x)$, and then as a character of B by inflation. Recall that we identify N with $\mathbb{Z}_p$ as above.

Let L be a finite extension of $\mathbb{Q}_p$ and $\kappa : \mathbb{Z}_p^\times \to L^\times$ a continuous character. For $m \geq m(\kappa)$, set

$$\mathcal{C}_{m,\kappa} = \left\{ \begin{array}{l} f : BI \longrightarrow L,\ f(bx) = \kappa(b)f(x) \quad \forall\, b \in B,\ \forall x \in I, \\ \\ f_{|N} \text{ is } m\text{-analytic.} \end{array} \right\}.$$

[5]The reader can check as an exercise that M is actually the set of all such elements with non-zero determinant.

It follows from the decomposition $BI = B \times N$ that $f \mapsto f_{|N}$ is a bijection $\mathcal{C}_{\kappa,m} \xrightarrow{\sim} \mathcal{C}_{m,L}$. The norm given above on $\mathcal{C}_{m,L}$ gives us then via this identification an L-Banach space norm on $\mathcal{C}_{m,\kappa}$. It follows that as a space, $\mathcal{C}_{m,\kappa}$ does not depend on the choice of κ in $\mathcal{W}_m(L)$.

More interestingly, as BI is stable by right M-translations, it follows that $\mathcal{C}_{m,\kappa}$ is a representation of M for the left action $(m.f)(x) = f(xm)$. Remark that this action preserves m-analyticity. Indeed, this follows from the analyticity of the action by homography for $\kappa = 1$ as

$$\frac{1}{a+ct} = a^{-1} \sum_{i \geq 0} (-1)^i a^{-i} c^i t^i \in \mathbb{Z}_p\langle t \rangle^\times$$

when $a \in \mathbb{Z}_p^\times$ and $c \in p\mathbb{Z}_p$. For general κ, it follows form

$$\kappa(a+ct) = \kappa(a)\kappa(1 + a^{-1}ct) \in \mathcal{C}_{m,L}$$

since $a \in \mathbb{Z}_p^\times$, $c \in p\mathbb{Z}_p$ and $m \geq m(\kappa)$ (see formula (ii) of lemma 3.7). Explicitely the formula for the "κ-action" of M that we obtain on $\mathcal{C}_{m,L}$ is simply :

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot_\kappa f \right)(t) = \kappa(a+ct) f(\frac{b+dt}{a+ct}).$$

What we have done for closed points $\kappa \in \mathcal{W}$ can be done verbatim for any affinoid points $\mathrm{Sp}(A) \to \mathcal{W}$: replace L by A everywhere. It is enough for our purposes to consider the case of open affinoids $\Omega \subset \mathcal{W}$ (e.g. closed balls). If $\Omega \subset \mathcal{W}_m$ is such an affinoid, we define $\mathcal{C}_{m,\Omega}$ by the same formula as above where we replace L by the affinoid algebra $\mathcal{O}(\Omega)$ and κ by the restriction to Ω of the universal character κ^{univ}. Again, this is an $\mathcal{O}(\Omega)$-module isometric to $\mathcal{C}_{m,\mathcal{O}(\Omega)}$ via $f \mapsto f_{|N}$, as well as an $\mathcal{O}(\Omega)$-linear representation of M. The collection of M-representations $\{\mathcal{C}_{m,\Omega}, \Omega \subset \mathcal{W}, m \geq m(\Omega)\}$ is called the *p-adic family of locally analytic principal series of I* (or of M).

LEMMA 3.8. (a) *The formation of the $\mathcal{O}(\Omega)[M]$-module $\mathcal{C}_{m,\Omega}$ commutes with completed scalar extension to any $\Omega' \subset \Omega$ or $\kappa \in \Omega(L)$: the natural maps $\mathcal{C}_{m,\Omega}\widehat{\otimes}_{\mathcal{O}(\Omega)}\mathcal{O}(\Omega') \to \mathcal{C}_{m,\Omega'}$ and $\mathcal{C}_{m,\Omega}\widehat{\otimes}_\kappa L \to \mathcal{C}_{m,\kappa}$ are isomorphisms and M-equivariant.*

(b) *The elements of M have norm ≤ 1 on each of these spaces, the elements in MuM act compactly by improving analyticity.*

Proof — Indeed, (a) follow via the isometry $f \mapsto f_{|N}$ from a similar statement for the m-analytic functions on $\mathbb{Z}_p$ with values in $\mathcal{O}(\Omega)$ or L, which follows in turn from the isomorphism $A\langle t\rangle\widehat{\otimes}_A B = B\langle t \rangle$. Note that the given maps are M-equivariant by definition. Part (b) follows for instance from the formulas given in lemma 3.7 (ii) for the elements in M, and from the prop. "u improves analyticity" for the element u : note that the u here acts as the u there via the identification of $\mathcal{C}_{m,*}$ with analytic functions. The case of a general element in MuM follows form these two ones. □

3.9. The finite dimensional representation. We embed $\mathbb{Z}$ in $\mathcal{W}(\mathbb{Q}_p)$ by the map $k \mapsto (x \mapsto x^{k-2})$. Note that $\mathbb{Z} \subset \mathcal{W}_0(\mathbb{Q}_p)$. The presence of the shift by 2 is a convention explained by the relation to modular forms. Fix $k \geq 2$ an integer. Remark that the (closed) subspace

$$W_k \subset \mathcal{C}_{0,k} \xrightarrow{\sim} \mathcal{C}_0 = \mathbb{Q}_p\langle t\rangle$$

of polynomials of degree $\leq k-2$ in t is stable under the action of M on $\mathcal{C}_{0,k}$.

PROPOSITION 3.10. *("Small slope function is polynomial") As a representation of M, we have $W_k \simeq \mathrm{Sym}^{k-2}(\mathbb{Q}_p^2)$. Furthermore, if $m \in MuM$ then the norm of m acting on $\mathcal{C}_{0,k}/W_k$ is less than $1/p^{k-1}$.*

Proof — The first statement is straightforward. For the second, it is enough to prove it for $m = u$ as any $m \in M$ has norm ≤ 1 on $\mathcal{C}_{0,k}$, hence on $\mathcal{C}_{0,k}/W_k$. But for u this is obvious as an ON-basis for this quotient is $t^{k-1}, t^k, \ldots$ on which u acts diagonally by $(p^{k-1}, p^k, \ldots)$. □

Give a picture summarizing the constructions in this section.

4. Some spaces of p-adic quaternionic modular forms

We fix a definite quaternion algebra D of discriminant d, N a integer prime to d, and p a prime number prime to Nd.

Consider the compact open subgroup $K_1(N,p) \subset K_1(N)$ whose elements (x_ℓ) have the extra property that $x_p \in I$ (recall that $K_1(N)_p = \mathrm{GL}_2(\mathbb{Z}_p)$). For each $\mathbb{Q}_p[M]$-module V, it makes sense to consider the vector space

$$F(V) := \{f : D_f^\times \to V,\ f(\gamma x y) = y_p^{-1} f(x),\ \forall (\gamma, x, y) \in D^\times \times D_f^\times \times K\}.$$

This space is naturally equipped with commuting actions of the Hecke-algebras of (M, I) and of the $(D_\ell^\times, K_1(N,p)_\ell)$ for $\ell \neq p$, of which we shall only remember

$$\mathcal{H} := \mathbb{Z}[U_p, \{T_\ell, (\ell, Nd) = 1\}]$$

where U_p is the Hecke operator at p given by the double class of the diagonal element $u = (1,p) \in M$. This action of Hecke operators exists for the same reason as for the $S_k(K)$: the space $F(V)$ is actually the $K_1(N,p)$-invariant subspace of the $M \times \prod_{\ell \neq p} D_\ell^\times$-module of functions $D^\times \backslash D_f^\times \to V$ for the action $(g.f)(x) = g_p f(x g_p)$. For instance, as

$$IuI = \bigcup_{i=0}^{p-1} u_i I$$

where $u_i = \begin{pmatrix} 1 & 0 \\ pi & p \end{pmatrix}$, and if we view each such u_i as an element of $D_f^\times$ trivial outside p, then for $f \in S_k(K)$ we have

$$U_p(f)(x) = \sum_{i=0}^{p-1} u_i f(x u_i).$$

The formula for the T_ℓ for ℓ prime to Npd is actually given exactly by the same formula as in the classical quaternionic case.

As a consequence, we may view F as a functor

$$F : \mathrm{Mod}(\mathbb{Q}_p[M]) \to \mathrm{Mod}(\mathcal{H}).$$

As before, if we write $D_f^\times = \coprod_{i=1}^s D^\times x_i K_1(N,p)$ and set $\Gamma_i = K_1(N,p) \cap x_i^{-1} D^\times x_i$ (a finite group – often even trivial), then $f \mapsto (f(x_i))$ induces a $\mathbb{Q}_p$-linear bijection

$$F(V) \xrightarrow{\sim} \prod_i V^{\Gamma_i}.$$

It follows the functor F is extremely well behaved, and that $F(V)$ inherits most of the extra properties that V might have :

(i) First of all, F is exact (i.e. transforms exact sequences into exact sequences).
(ii) If V is a Banach A-module with a norm $|.|$, then $|f| = \mathrm{Sup}_i |f(x_i)|$ defines a Banach A-module norm on $F(V)$. It is SON-able if V is and it commutes with scalar extensions : $F(V)\widehat{\otimes}_A B \xrightarrow{\sim} F(V\widehat{\otimes}_A B)$.
(iii) (continuation) If the elements of M act continuously with norm ≤ 1 on V, then so do the $h \in \mathcal{H}$ on $F(V)$. If furthermore MuM acts by compact operators on V, then U_p is compact on $F(V)$.

Definition 4.1. *The space of m-analytic p-adic modular forms of level N and weight $\kappa \in \mathcal{W}_m(L)$ is the L-Banach space $S_{m,\kappa}(N) := F(\mathcal{C}_{m,\kappa})$.*

The space of families of m-analytic p-adic modular forms of level N and weight in $\Omega \subset \mathcal{W}_m$ is the Banach $\mathcal{O}(\Omega)$-module $S_{m,\Omega}(N) := F(\mathcal{C}_{m,\Omega})$.

From the remarks (i), (ii) and (iii) above and the study of the $\mathcal{C}_{m,A}$ we deduce the following facts: for $* \in \{\kappa, \Omega\}$ as above

(a) $S_{m,*}(N)$ is equipped with a linear action of $\mathcal{H}$, each element being continuous of norm ≤ 1, and U_p being compact.
(b) We have a natural compact inclusion $i : F(\mathcal{C}_{m,*}) \subset F(\mathcal{C}_{m,*})$ and U_p actually improves analyticity : for $m-1 \geq m(*)$ we have a factorization/diagramm

$$\begin{array}{ccccc} & & S_{m,*}(N) & \xrightarrow{\;i\;} & S_{m+1,*}(N) \\ & \swarrow^{\widetilde{U_p}} & \downarrow{\scriptstyle U_p} & \swarrow_{\widetilde{U_p}} & \\ S_{m-1,*}(N) & \xrightarrow[i]{} & S_{m,*}(N) & & \end{array}$$

(c) The $S_{m,*}(N)$ are SONable and their formation when $*$ vary commutes with completed tensor products.

It follows from (a) and (c) that the Fredholm series $\det(1-TU_p|S_{m,*}(N))$ is well-defined, and by (b) and the parallelogramm lemma it is independent on $m \geq m(*)$. The compatibility of the formation of the Fredholm series with scalar extension and (c) :

Corollary 4.2. *There is a unique power series* $\mathrm{Fred}_{U_p} \in 1 + T\mathcal{O}(\mathcal{W})[[T]]$ *such that for each $*$ as above, the image of* $\mathrm{Fred}_{U_p} \in 1 + \mathcal{O}(*)[[T]]$ *is*

$$\det(1 - TU_p|S_{m,*}(N))$$

for any $m \geq m(*)$. This power series defines a rigid analytic function on $\mathcal{W} \times \mathbb{A}^1$.

This corollary will be combined with Riesz-Coleman theory in the next lecture to prove the existence of families of eigenforms.

DEFINITION 4.3. *Let $\kappa \in \mathcal{W}_m(L)$. A p-adic modular form $f \in S_{m,\kappa}(N)$ is of finite slope $s \in \mathbb{Q}$ it belongs to the finite dimensional subspace $S_{m,\kappa}(N)_s$ of slope s for the action of the U_p-operator. This latter space is an $\mathcal{H}$-submodule of $S_{m,\kappa}(N)$. It does not depend on $m \geq m(\kappa)$.*

We now compare these spaces of p-adic modular forms with the space of complex quaternionic modular forms. For this we fix a pair of embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

PROPOSITION 4.4. *Assume $k \geq 2$ is an integer.*

(i) *Via these embeddings, $S_k^D(N,p) := F(\mathrm{Sym}^{k-2}(\mathbb{Q}_p^2))$ is a $\mathbb{Q}_p$-model of the $\mathcal{H}$-module $S_k(K_1(N,p))$ for $k \geq 2$.*
(ii) *There is an $\mathcal{H}$-equivariant embedding $S_k^D(N,p) \subset S_{0,k}(N)$.*
(iii) *If $s < k-1$, then $S_{0,k}(N)_s \subset S_k^D(N,p)$ ("small slope forms are classical").*

Proof — To check (ii), apply the exact functor F to the M-equivariant inclusion $W_k = \mathrm{Symm}^{k-2}(\mathbb{Q}_p^2) \to \mathcal{C}_{0,k}$. We even have a U_p-equivariant exact sequence

$$0 \to F(W_k) \to F(\mathcal{C}_{0,k}) \to F(\mathcal{C}_{0,k}/W_k) \to 0.$$

This proves (iii) as we have seen that MuM acts by endomorphisms of norm $\leq \frac{1}{p^{k-1}}$ on $\mathcal{C}_{0,k}/W_k$, hence so does U_p on $F(\mathcal{C}_{0,k}/W_k)$.

We finally check (i). As in the previous lecture, and thanks to the embedding ι_∞, one can define a $\overline{\mathbb{Q}}$-model of $S_k(K_1(N,p))$ by considering its $\overline{\mathbb{Q}}$-subvectorspace of functions

$$D_f^\times \to \mathrm{Symm}^{k-2}(\overline{\mathbb{Q}}).$$

Using ι_p one can extend scalars from this space to $\overline{\mathbb{Q}}_p$, and one simply gets the space of functions with values in $\mathrm{Symm}^{k-2}(\overline{\mathbb{Q}}_p)$ such that $f(\gamma x y) = \gamma f(x)$ where $\gamma \in D^\times$, $x \in D_f^\times$ and $y \in K_1(N,p)$. But this last space can then be identified with $F(\mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2))$ by the map

$$f \mapsto f',\ f'(x) := x_p^{-1} f(x).$$

We conclude the proof as the space $F(\mathrm{Sym}^{k-2}(\mathbb{Q}_p^2))$ is a $\mathbb{Q}_p$-model of $F(\mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2))$. □

The previous proposition shows that "small slope eigenforms are classical". On the opposite, it is a classical fact that the Hecke-operator $[IuI]$ is invertible in the Hecke-algebra of $(\mathrm{GL}_2(\mathbb{Q}_p), I)$ (see Henniart's lecture), from which it follows that any classical p-adic eigenform is actually of finite slope. We will prove a more precise statement in the next lecture.