1. Affinoid algebras and Tate's p-adic analytic spaces : a brief survey

1.1. Introduction. (rigid) p-adic analytic geometry has been discovered by Tate after his study of elliptic curves over $\mathbb{Q}_p$ with multiplicative reduction. This theory is a p-adic analogue of complex analytic geometry. There is however an important difference between the complex and p-adic theory of analytic functions coming from the presence of too many locally constant functions on $\mathbb{Z}_p^m$, which are locally analytic in any reasonable sense. A locally ringed topological space which is locally isomorphic to ($\mathbb{Z}_p^n$, sheaf of locally analytic function) is called a $\mathbb{Q}_p$-analytic manifold, and it is NOT a rigid analytic space in the sense of Tate. It is for instance always totally disconnected as a topological space, which already prevents from doing interesting geometry Tate actually has a related notion of "woobly analytic space" (espace analytique bancal en francais) that he opposes to its "rigid analytic space". We strongly recommend to read Tate's inventiones paper "Rigid analytic spaces" and to glance through Bosch-Guntzer-Remmert "Non archimedean analysis" for more details (especially for G-topologies). For a nice introduction to this and other approaches, we recommend Conrad's notes on rigid analytic geometry on his web page.

For the purposes of this course, we shall use the language of rigid analytic geometry when studying families of modular forms and when studying the generic fiber of deformation rings (see below). The eigencurve for instance will be a rigid analytic curve. Our aim here is to give the general ideas of Tate's definition of a rigid analytic space. Just as algebraic geometry is built from affine varieties and finitely generated k-algebras, rigid geometry is built from affinoids and the so-called affinoid algebras. The maximal spectrum $\mathrm{Max}(A)$ of an affinoid algebra A may then be endowed with a G-topology ("admissible open and admissible coverings"), a special kind of Grothendieck topology, and a sheaf of $\mathbb{Q}_p$-algebras for this G-topology whose global sections coincide with A. The resulting object is called an affinoid. A key idea of Tate is to restrict the sheaf property to collections of open subsets that "overlap enough", such as "finite coverings by affinoid subdomains". A general rigid analytic space is then obtained by gluing affinoids in an appropriate geometric category.

1.2. Affinoid algebras. The *Tate algebra* in m variables over $\mathbb{Q}_p$ is the subalgebra

$$\mathbb{Q}_p\langle t_1, \dots, t_m\rangle \subset \mathbb{Q}_p[[t_1, \dots, t_m]]$$

of functions $f = \sum_\alpha a_\alpha t^\alpha$ such that $a_\alpha \to 0$ in $\mathbb{Q}_p$ when $|\alpha| \to \infty$ (check that it is stable by product!). The convergence property of such a power series exactly ensures that it can be evaluated on $\mathbb{Z}_p^m$. The elements of the Tate algebra are called a rigid analytic function on $\mathbb{Z}_p^m$. The Tate algebra is a domain. Despite its analytic flavour it actually shares many properties with the polynomial algebra :

Proposition 1.3. *(Tate) $\mathbb{Q}_p\langle t_1, \dots, t_m\rangle$ is noetherian, Jacobson, factorial, regular of equidimension n, and the nullstellensatz holds : for each maximal ideal M, $\mathbb{Q}_p\langle t_1, \dots, t_m\rangle/M$ is a finite extension of $\mathbb{Q}_p$.*

(An important tool in its study is the so-called Weierstrass preparation theorem. Note that $\mathbb{Q}_p\langle t\rangle$ is in particular a principal ring.)

Definition 1.4. *An affinoid algebra is a $\mathbb{Q}_p$-algebra isomorphic to $\mathbb{Q}_p\langle t_1, \dots, t_m\rangle/I$ for some n and some ideal I of $\mathbb{Q}_p\langle t_1, \dots, t_m\rangle$.*

In particular, an affinoid algebra is Noetherian, Jasobson and the nullstellensatz holds as well.

1.5. Maximal spectrum. If A is an affinoid algebra, we denote by $\mathrm{Max}(A)$ its maximal spectrum, i.e. the set of maximal ideals. It is equipped with the Zariski topology, as well as a totally discontinous topology as we could see, but none of these two topology is really pertinent for rigid geometry.

Thanks to the nullstellensatz, any $\mathbb{Q}_p$-algebra homomorphism $f : A \to B$ between two affinoid algebras induces a map $f : \mathrm{Max}(B) \to \mathrm{Max}(A)$, so we are in a situation similar to the old-fashioned affine algebraic geometry.

We need two other definitions. Let A be an affinoid $\mathbb{Q}_p$-algebra and $X = \mathrm{Max}(A)$. If $L/\mathbb{Q}_p$ is a finite extension, we denote by $A(L)$ or by $X(L)$ the set of $\mathbb{Q}_p$-algebra homomorphisms $A \to L$, and we set

$$X(\overline{\mathbb{Q}}_p) = \bigcup_{L\subset\overline{\mathbb{Q}}_p} X(L).$$

The natural map $X(\overline{\mathbb{Q}}_p) \to X$ is surjective by the nullstellensatz, and $G_{\mathbb{Q}_p}$ acts transitively in each fiber of it, so we get a bijection $X(\overline{\mathbb{Q}}_p)/G_{\mathbb{Q}_p} = X$. Knowing X is essentially equivalent to knowing $X(L)$ for all L.

If $a \in A$ and $x \in X$, it makes sense to define $|a(x)|$ as A/M_x is a finite extension of $\mathbb{Q}_p$, thus it has a unique norm extending the one of $\mathbb{Q}_p$. We define $|a|_{\sup}$ as $\mathrm{Sup}_{x\in X}|a(x)|$. It will follow from the next result that it is finite (it is even attained : "maximum modulus principle").

1.6. Banach algebra topology. Before computing X in some examples let us make some topology considerations. If $f = \sum_\alpha a_\alpha t^\alpha \in \mathbb{Q}_p\langle t_1, \dots, t_n\rangle$ we define its Gauss norm $|f|$ as $\sup_\alpha |a_\alpha|$ where $|.|$ is the usual norm on $\mathbb{Q}_p$ say. Then $|.|$ is a $\mathbb{Q}_p$-algebra norm[1], which even satisfies $|fg| = |f||g|$ for all f, g, and the Tate algebra is complete for this norm : it is a Banach $\mathbb{Q}_p$-algebra. One can show that *any ideal of the Tate algebra is closed* for this norm, so any affinoid algebra is a Banach algebra as well once we choose a presentation as a quotient by a Tate-algebra. This structure is even independent of the presentation as a quotient of a Tate algebra. Indeed :

Proposition 1.7. *(Tate) Any two Banach-algebra norms on an affinoid algebra are equivalent. Furthermore, any $\mathbb{Q}_p$-algebra morphism between two affinoid algebras is continuous. If A is reduced, $|.|_{\sup}$ is a Banach $\mathbb{Q}_p$-algebra norm on A.*

The sup-norm and the Gauss norm turn out to coincide on $\mathbb{Q}_p\langle t_1, \dots, t_m\rangle$. Let us mention that for an general A, the existence of multiplicative (rather than sub-multilpicative) a subtle question (it does not always exist, even if A is a domain).

[1]If A is a $\mathbb{Q}_p$-algebra, a $\mathbb{Q}_p$-algebra norm is on A is an ultrametric norm $|.| : A \to \mathbb{R}_+$ such that $|\lambda x| = |\lambda||x|$ for $\lambda \in \mathbb{Q}_p$ and $x \in A$ and such that $|xy| \le |x||y|$ for $x, y \in A$. If A is complete for $|.|$ we say that $|.|$ is a Banach-algebra norm, and that A is a Banach algebra.

Example: (Power-bounded elements) One consequence of this is that there is a well-defined notion of power-bounded element in an affinoid algebra, i.e. the elements a such that $\{a^n, n \in \mathbb{N}\}$ is bounded in A. By the proposition, this notion is also preserved under $\mathbb{Q}_p$-algebra homomorphisms. For instance, t is power-bounded in $\mathbb{Q}_p\langle t\rangle$. It follows that an element $a \in A$ is power-bounded iff there is a $\mathbb{Q}_p$-algebra morphism $\mathbb{Q}_p\langle t\rangle \to A$ sending t to a (universal property of Tate algebras). Tate has shown that $a \in A$ is power-bounded iff $|a|_{\sup} \leq 1$ (this is non-trivial).

1.8. Examples of affinoid algebras and maximal spectra. Here are some examples.

(o) If $F/\mathbb{Q}_p$ is a finite extension then F is an affinoid algebra. Indeed, choose $\alpha \in \mathcal{O}_F$ such that $\mathbb{Q}_p(\alpha) = F$ (there is always such an element), then the $\mathbb{Q}_p$-algebra morphism $\mathbb{Q}_p\langle t\rangle \to F$ sending t to α is surjective. More generally, any finite dimensional $\mathbb{Q}_p$-algebra is an affinoid algebra (use that it is complete for some $\mathbb{Q}_p$-algebra norm).

(i) (closed unit disc in m-variables) If $A = \mathbb{Q}_p\langle t_1, \dots, t_m\rangle$ then $\varphi \mapsto (\varphi(t_i))$ induces a bijection $X(L) = \{(t_1, \dots, t_m) \in L, |t_i| \leq 1\} = \mathcal{O}_L^m$.

(ii) (general case) If $A = \mathbb{Q}_p\langle t_1, \dots, t_m\rangle/I$ then $\varphi \mapsto (\varphi(t_i))$ induces a bijection between $X(L)$ and the zero locus of (a system of generators of) I inside $\mathcal{O}_L^m$.

(iii) (closed disc of any radius) The affinoid closed disc with radius p^r, $r \in \mathbb{Z}$, is $\mathbb{Q}_p\langle p^r t_1, \dots, p^r t_m\rangle$ (they are all isomorphic if m is fixed). We also have discs of any radius in $p^{\mathbb{Q}}$. For instance, if $A = \mathbb{Q}_p\langle t, u\rangle/(pu - t^r)$ and $r \in \mathbb{N}$, then $X(L)$ is the set of $t \in L$ such that $|t| \leq 1/p^{1/r}$.

(iv) (annuli) If $A = \mathbb{Q}_p\langle x, y\rangle/(xy - p)$ then $X(L)$ is the annulus $1/p \leq |x| \leq 1$ in L.

(v) (relative Tate algebra) If A is an affinoid algebra, we may define as well $A\langle t_1, \dots, t_n\rangle \subset A[[t_1, \dots, t_n]]$ as the power series whose coefficients goes to 0 in A. It is still an affinoid algebra. What are its L-points ?

(vi) (sums) The category of affinoid algebras and $\mathbb{Q}_p$-algebra morphisms admit finite (amalgamated) sums. Indeed, if we have $A \to B$ and $A \to C$ then their sum is the completed tensor product $B\widehat{\otimes}_A C$. Concretely, if $B = A\langle t_1, \dots, t_m\rangle/I$ and $C = A\langle u_1, \dots, t_n\rangle/J$ then

$$B\widehat{\otimes}_A C = A\langle t_1, \dots, t_m, u_1, \dots, u_n\rangle/(I, J)$$

(it is clearly an affinoid algebra and a sum).

(vii) If A is an affinoid algebra and B is an A-algebra of finite type as A-module, then B is an affinoid algebra (this is not obvious).

1.9. Tate's acyclicity theorem. Following Tate, we now define a collection of particular subsets of $\mathrm{Max}(A)$ usually called affinoid subdomains (and "affine subsets" in Tate's original paper). Let An be the category of affinoid algebras over $\mathbb{Q}_p$ with $\mathbb{Q}_p$-algebra morphisms.

DEFINITION 1.10. *(Tate) Let A be an affinoid algebra and $X = \mathrm{Max}(A)$. A subset $U \subset X$ is called an affinoid subdomain, if the covariant functor* $\mathrm{An} \to \mathrm{Sets}$, *associating to B the set $\{f \in \mathrm{Hom}_{\mathrm{An}}(A, B), f(\mathrm{Max}(B)) \subset U\}$, is representable.*

In other words, if there exists a morphism $g : A \to A'$ such that for any $f : A \to B$, $f(\mathrm{Max}(B)) \subset U$ iff $f = h \circ g$ for some $h : A' \to B$. In this case, we set $\mathcal{O}(U) = A'$.

In particular X (and $\emptyset$) itself is an affinoid subdomain and $\mathcal{O}(X) = A$. If U is an affinoid subdomain there is a natural $\mathbb{Q}_p$-algebra homomorphism $\mathcal{O}(X) \to \mathcal{O}(U)$.

This definition unfortunately a little difficult to grasp. Nevertheless, it is not too difficult to show that:

- if U is an affinoid subdomain as in the statement, then $g_{|\mathrm{Max}(A')}$ is injective with image exactly U (for instance for this last property, apply the definition to the closed points),

- being an affinoid subdomain is a transitive property (clear),

- finite intersections of affinoid subdomains are still affinoid subdomains (this follows at once from the existence of amalgamated sums in An).

- if $f : A \to B$ is a $\mathbb{Q}_p$-algebra morphism then f^{-1} sends affinoid subdomains of $\mathrm{Max}(A)$ to affinoid subdomains of $\mathrm{Max}(B)$. (Idem : if $A \to A'$ represents $U \subset \mathrm{Max}(A)$ then $A'\widehat{\otimes}_A B$ represents $f^{-1}(U) \subset \mathrm{Max}(B)$.

Here is are the most important examples of affinoid subdomains. If $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are in A, the locus of $X = \mathrm{Max}(A)$ defined by $|a_i(x)| \geq 1$ and $|b_j(x)| \leq 1$ for all i, j is an affinoid subdomain ("Laurent/Weierstrass domain"). One can show that it is represented by the affinoid A-algebra

$$A\langle u_i, v_j\rangle/(1 - a_i u_i, b_j - v_j).$$

Let $f_1, \ldots, f_r, g \in A$ having no common zeros in X (i.e. generating the ideal A) then the subset of $X = \mathrm{Max}(A)$ defined by $|f_i(x)| \leq |g(x)|$ for all i is an affinoid subdomain as well ("rational domain"). Indeed,[2] one can show that it is represented by the affinoid A-algebra

$$A' = A\langle t_1, \ldots, t_r\rangle/(gt_i - f_i).$$

Example. For $n \in \mathbb{Z}$, the natural map $\mathbb{Q}_p\langle (p^{n+1}t)\rangle \to \mathbb{Q}_p\langle (p^n t)\rangle$, sending $p^{n+1}t$ to $p.(p^n t)$, realizes the closed disc of radius p^n as an affinoid subdomain of the closed disc of radius p^{n+1} (defined by $|t| \leq p^n$! it is a special case of Weierstrass/Laurent domain : $\mathbb{Q}_p\langle (p^{n+1}t)\rangle\langle u\rangle/(u - p^n t) = \mathbb{Q}_p\langle (p^n t)\rangle$).

Remark. In general, strict inequalities will not define affinoid subdomains. For instance the open unit ball $|t| < 1$ is not an affinoid subdomain of the closed unit disc, although the closed disc $|t| \leq 1/p$ and the "circle" $|t| = 1$ are (Laurent domains).

Another important but difficult result is that *any open affinoid is a finite union of rational domains* (Gerritzen-Grauert, the converse is not true however). The main theorem is the following sheaf property for finite coverings by affinoid subdomains.

[2] Here is the argument. Let $f : A \to B$ a morphism in An such that $f(\mathrm{Max}(B)) \subset U = \{x, |f_i(x)| \leq |g(x)| \forall i\}$. Note that g does not vanish on U by as $(f_1, \ldots, f_r, g) = A$, so $f(g)$ does not vanish on $\mathrm{Max}(B)$: $f(g) \in B^\times$ by the nullstellensatz. The elements $t_i = f(f_i)/f(g) \in B$ are well-defined and their sup-norm is less than 1 by definition of U : they are power bounded in B. It follows that $A \to B$ extends to $A\langle t_1, \ldots, t_r\rangle/(gt_i - f_i)$ if we send t_i to t_i. The other direction is trivial. A similar argument works for Weierstrass/Laurent domains.

THEOREM 1.11. *(Tate's acyclicity theorem) For any finite covering of* Max(A) *by open affinoids U_i, the natural sequence*

$$0 \to \mathcal{O}(X) \to \prod \mathcal{O}(U_i) \to \prod \mathcal{O}(U_i \cap U_j)$$

is exact.

Note that the requirement on the covering allows to avoid locally constant functions. For instance we see that the closed unit disc is not the disjoint union of finitely many affinoid subdomains as the Tate-algebra is a domain ! We see also that the open unit disc is not an affinoid subdomain.

Exercise: Show that if A is an affinoid and e is an idempotent of A, then $e = 0$ and $e = 1$ are affinoid subdomains of Max(A). Show that Spec(A) is connected iff Max(A) is not the union of two disjoint affinoid subdomains.

1.12. Affinoids and Rigid analytic spaces. We now define the "Tate topology" or G-topology on Max(A). Strictly speaking it is *not* a topology, but rather a special case of Grothendieck-topology.

DEFINITION 1.13. *Let A be an affinoid algebra and $X =$* Max(A)*. A subset $U \subset X$ is called an* admissible open *if it has a set-theoretic covering by affinoid subdomains $U_i \subset X$, such that for any $f : A \to B$ with $f(\text{Max}(B)) \subset U$, the covering $f^{-1}(U_i)$ of* Max(B) *admits a finite subcovering (recall that each $f^{-1}(U_i)$ is an affinoid subdomain).*

A collection U_j of admissible open subsets of X is an admissible covering *of its set-theoretic union U if for any $f : A \to B$ with $f(\text{Max}(B)) \subset U$, the covering $f^{-1}(U_j)$ of* Max(B) *admits a refinement which is a finite covering by open affinoids of B. (Such a U is necessarily an admissible open)*

Finite union of open affinoids are obviously admissible open subsets (but not necessarily affinoid subdomains), but there are many more. For instance the Zariski-open subsets of Max(A) are admissible open (by the maximum modulus principle).

Admissible open and admissible coverings turn out to form a G-topology on Max(A). A G-topology on a set X is a collection of subsets $\mathcal{U}$ (the "open subsets"), as well as for each $U \in \mathcal{U}$ a set Cov$(U) \in \mathcal{P}(U)$ of set-theoretic coverings of U by elements of $\mathcal{U}$, satisfying properties similar to the ones satisfied by the open subsets of the usual topologies:

(T1) $\mathcal{U}$ is stable by finite intersections and contains $\emptyset$,
(T2) $\{U\} \in \text{Cov}(U)$ for $U \in \mathcal{U}$,
(T3) if $\{U_i\} \in \text{Cov}(U)$ and $V \subset U$ then $V \in \mathcal{U}$ if and only if $V \cap U_i \in \mathcal{U}$ for all i,
(T4) if $\{U_i\} \in \text{Cov}(U)$ and $\{V_{i,j}\}_{j \in J(i)} \in \text{Cov}(U_i)$ then $\{V_{i,j}\} \in \text{Cov}(U)$.

There is then an obvious notion of sheaf for such a topology, and by *a locally ringed G-space* we shall mean a triple $(X, \mathcal{T}, \mathcal{O})$ where X is a set, $\mathcal{T}$ is a G-topology on X, and $\mathcal{O}$ is a sheaf of $\mathbb{Q}_p$-algebra on $\mathcal{T}$ whose stalks at any $x \in X$ are local rings[3]. It is then more or less formal to deduce from Tate's acyclicity theorem that

[3]The stalk of $\mathcal{O}$ at x, often denoted by $\mathcal{O}_x$, is the inductive limit of the $\mathcal{O}(U)$ over all the open subsets $U \in \mathcal{U}$ containing x (this is a directed set as open subsets are stable by finite intersections).

Definition-Proposition 1.14. *Admissible opens and admissible covering form a G-topology on* Max(A)*.* $U \mapsto \mathcal{O}(U)$*, for* $U \subset X$ *an affinoid subdomain, extends uniquely to a sheaf of* $\mathbb{Q}_p$*-algebra on* Max(A) *for this G-topology above. The obtained locally ringed G-space is the affinoid* Sp(A)*.*

Note that if $(X, \mathcal{T}, \mathcal{O})$ is any locally ringed G-space and U is an open subset of X, then we obtain by restriction a structure of locally ringed G-space $(U, \mathcal{T}_{|U}, \mathcal{O}_{|U})$ (keep only the open subsets of U). Locally ringed G-space form also a category in an obvious way.

Definition 1.15. *A rigid analytic space is a locally ringed G-space* $(X, \mathcal{T}, \mathcal{O})$ *admitting a covering* $\{U_i\} \in \mathrm{Cov}(X)$ *such that for each* i*,* $(U_i, \mathcal{T}_{|U_i}, \mathcal{O}_{|U_i})$ *is isomorphic to an affinoid. A morphism* $X \to Y$ *between two rigid analytic spaces is a morphism between the associated locally ringed G-spaces.*

Just as for schemes one shows that the functor $A \mapsto$ Sp(A) is fully-faithful from An to the category Rig of rigid analytic spaces.

A way to construct rigid anaytic space is to glue a collection of affinoids X_i along admissible opens $X_{i,j}$. We omit the rather obvious details here (see BGR). For instance, if we have a collection of affinoids X_n, $n \geq 1$, and morphisms $X_n \to X_{n+1}$ such that X_n is an affinoid subdomain of X_{n+1} for each n, then there is a unique rigid analytic space X which is the admissible increasing union of the X_n. In the next lectures, most of the non-affinoid rigid analytic spaces that we shall encounter will be actually be of this type. For such a space $\mathcal{O}(X)$ is the projective limit of the $\mathcal{O}(X_n)$ for $n \geq 1$ (apply the sheaf property to the covering $\{X_n, n \geq 1\}$).

Examples (i) The rigid analytic affine space $\mathbb{A}^n$ is the admissible increasing union of the closed balls of radius p^n for all $n \geq 0$.

(ii) The open unit ball is the admissible increasing union of the closed balls of radius $1/p^{1/n}$, for all $n \geq 1$.

(iii) The rigid analytic $\mathbb{G}_m$ is the admissible increasing union of annuli $1/p^n \leq |t| \leq p^n$ for all $n \geq 1$.

See BGR for other standard and useful considerations : closed immersions, finite morphisms, coherent sheaves, proper morphisms ... To really go beyond the definitions and prove serious theorems in rigid analytic geometry one often needs to use other approaches to rigid analytic geometry, like Raynaud's approach via formal geometry or Berkovich's theory.

2. The p-adic (pseudo)-character variety of a profinite group

Let G be a profinite group, n an integer and p a prime. We assume that (*G) holds and that $p > n$ (this condition might be relaxed, see my paper "The p-adic analytic space of pseudo-characters of a profinite group", but we shall be content here with that case).

Define a covariant functor $E : \mathrm{An} \to \mathrm{Sets}$ as follows : for an affinoid $\mathbb{Q}_p$-algebra A, $E(A)$ is the set of continuous pseudo-characters $T_A : G \to A$ of dimension n, and if $T_A \in E(A)$ and $f \in \mathrm{Hom}_{\mathrm{An}}(A, B)$ then $E(f)(T_A) = T \otimes_A B$ as usual.

A very related functor is the functor $D : \mathrm{An} \to \mathrm{Sets}$ associating to A the isomorphism classes of continuous representations $\rho_A : G \to \mathrm{GL}_n(A)$. The rule $\rho_A \mapsto \mathrm{trace}(\rho_A)$ induces a morphism $D \to E$.

THEOREM 2.1. *E is representable by a rigid analytic space $\mathfrak{X}$ over $\mathbb{Q}_p$.*

We call $\mathfrak{X}$ *the p-adic character variety of G in dimension n*. The statement means that $\mathrm{Hom}_{\mathrm{Rig}}(\mathrm{Sp(A)}, \mathfrak{X}) = \mathrm{E(A)}$ (functorially in A), which determines $\mathfrak{X}$ uniquely (just as a scheme is uniquely determined by its points in affine schemes).

This theorem can be precised as follows. If $T_A \in E(A)$, for all $x \in X = \mathrm{Max}(A)$ we denote by $T_x = T \otimes_A k(x)$ (where $k(x)$ is the residue field of A at x, a finite extension of $\mathbb{Q}_p$). In particular T_x is the trace of a unique continuous semi-simple representation $\rho_x : G \to \mathrm{GL}_n(\overline{k(x)})$, and it is good to think of T_A as family of representations indexed by $\mathrm{Max}(A)$ which is analytic in the sense that $\mathrm{trace}(\rho_x(g))$ is an analytic function of x (i.e. belong to A) for all $g \in G$.

We can go a bit further. By compactness of G, $T_x(G)$ necessarily falls inside $\mathcal{O}_{k(x)}$ (prove it!), and in particular we can reduce it and get some $\overline{T}_x : G \to k_x$ where k_x is the residue field of $\mathcal{O}_{k(x)}$. We say that T is *residually constant and equal to some fixed pseudo-character $\overline{T} : G \to \overline{\mathbb{F}}_p$* if for all $x \in X$, $\overline{T}_x \otimes_{k_x} \overline{\mathbb{F}}_p = \overline{T}$ for some embedding $k_x \to \overline{\mathbb{F}}_p$. We define $E_{\overline{T}}$ as the subfunctor of E parametrising the T_A which are residually constant and equal to $\overline{T}$ (we leave it as an exercise to check that it is indeed a subfunctor). As $\overline{T}$ is the trace of a unique semi-simple $\bar{\rho}$ we also denote $E_{\overline{T}}$ by $E_{\bar{\rho}}$ without confusion.

If A is an affinoid and $\rho_A : G \to \mathrm{GL}_n(A)$ is a continuous representation, we can define as above ρ_x and $\bar{\rho}_x$ for all $x \in X$ ("evaluation at x and semi-simplified reduction of the evaluation"), as well as $D_{\bar{\rho}} \subset D$ the subfunctor parametrising the ρ_A such that $\bar{\rho}_x \otimes_{k_x} \overline{\mathbb{F}_p} \simeq \bar{\rho}$ for some fixed semi-simple $\bar{\rho} : G \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and some embedding $k_x \to \overline{\mathbb{F}}_p$. The morphism $D \to E$ induces $D_{\bar{\rho}} \to E_{\bar{\rho}}$.

THEOREM 2.2.
- (i) *For any continuous semi-simple $\bar{\rho} : G \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, $E_{\bar{\rho}}$ is representable by a rigid-analytic space $\mathfrak{X}(\bar{\rho})$ over $\mathbb{Q}_p$.*
- (ii) *The space $\mathfrak{X}$ is an admissible disjoint union of the $\mathfrak{X}(\bar{\rho})$ over all the possible $\bar{\rho}$ (up to isomorphism and Frobenius action on image).*
- (iii) *$\mathfrak{X}(\bar{\rho})$ is isomorphic*[4] *to Berthelot's generic fiber of $R(\overline{T})$ where $\overline{T} = \mathrm{trace}(\bar{\rho})$.*
- (iv) *When $\bar{\rho}$ is irreducible then the natural morphism $D_{\bar{\rho}} \to E_{\bar{\rho}}$ is an isomorphism (hence $\mathfrak{X}(\bar{\rho})$ represents $D_{\bar{\rho}}$).*

We now sketch a proof of these theorems (we will recall Berthelot's construction in due time).

An affine *model*, or simply a model here, of an affinoid A is a $\mathbb{Z}_p$-subalgebra $\mathcal{A} \subset A$ of the form $\mathbb{Z}_p\langle t_1, \dots, t_m\rangle$ (obvious definition) where each $t_i \in A$ is power bounded and such that $\mathcal{A}[1/p] = A$. Such models always exist, as we see by chosing a surjective morphism $\mathbb{Q}_p\langle t_1, \dots, t_m\rangle \to A$. When A is reduced there is a biggest one : the subring of power-bounded elements (it is non-trivial that this is actually a model, but true). Otherwise, there is no canonical one. For instance for $\mathbb{Q}_p[\varepsilon]$ the models have the form $\mathbb{Z}_p \oplus \lambda\varepsilon\mathbb{Z}_p$ for any $\lambda \in \mathbb{Q}_p^\times$. If $\mathcal{A}$ is a model of A, then $p^n\mathcal{A}$ is open in A for all $n \geq 0$ and $\mathcal{A}$ is complete for the p-adic topology.

[4] This requires some precision as $\overline{T}$ is $\overline{\mathbb{F}}_p$-valued see below.

LEMMA 2.3. *Let A be a topological ring and $T : G \to A$ a continuous pseudo-character of dimension n (so $n! \in A^\times$).*

(i) *If A is an affinoid then $T(g)$ is power-bounded for any $g \in G$ and there is a model $\mathcal{A} \subset A$ such that $T(G) \subset \mathcal{A}$,*
(ii) *If $A = \lim_n A/p^nA$ with A/p^nA discrete for each n, then the closure of $\mathbb{Z}[T(G)]$ in A is a semi-local profinite ring,*
(iii) *If A is discrete then $\mathbb{Z}[T(G)]$ is a finitely generated $\mathbb{Z}$-module.*
(iv) *If A is affinoid again, and if $\mathrm{Spec}(A)$ is connected, then T_A is residually constant: $T_A \in E_{\overline{T}}(A)$ for some $\overline{T}$. Moreover, if $k = \mathbb{F}_p[\overline{T}(G)]$ (a finite field), $\mathcal{O} = W(k)$ and if $\overline{T}$ is viewed as a k-valued pseudo-character, there is a unique continuous $\mathbb{Z}_p$-algebra homomorphism $R(\overline{T}) \to A$ such that T factors through the universal pseudo-character $G \to R(\overline{T})$.*

Proof — Let us prove this lemma. We first check (iii). As G is compact, A discrete and T continuous, T factors through a finite quotient of G. Indeed, for each $g \in G$ there is a normal open subgroup $H_g \subset G$ such that $T(gh) = T(g)$ for all $h \in H_g$. If $G = \cup_{i=1}^n g_i H_{g_i}$ then $H = \cap_i H_{g_i}$ is a normal open subgroup such that $T(gh) = T(g)$ for all $h \in H$, and T factors through G/H. As a consequence, we may assume that G is a finite group. In particular, $\mathbb{Z}[T(G)]$ is of finite type as $\mathbb{Z}$-algebra and we may assume that A is of finite type as $\mathbb{Z}$-algebra. Such an A is finite over $\mathbb{Z}$ iff for any minimal prime ideal $P \subset A$ then $K = \mathrm{Frac}(A/P)$ has finite degree over its prime subfield. Fix such a P and consider $T \otimes_A K$. It is the trace of a semi-simple representation $\rho : G \to \mathrm{GL}_n(\overline{K})$ and as G is finite the eigenvalues of each $\rho(g)$ are $|G|$-th roots of unity so each $T(g)$ is algebraic over the prime subfield of K. As the field K is generated by $T(G)$ we are done.

To check part (ii), we apply part (i) to $T \otimes_A A/p^nA$ for each n. If B_n denotes the image of $B = \mathbb{Z}[T(G)]$ in A/p^nA, then (i) shows that B_n is finite for each n. Moreover, if $\widehat{B}$ is the closure of the statement then $\widehat{B} = \mathrm{projlim}_n B_n$ by definition : it is a profinite ring. The radical of $\widehat{B}$ contains the open ideal $pA \cap \widehat{B}$ as A and $\widehat{B}$ are complete, so $\widehat{B}$ is semi-local as $B_1 = \widehat{B}/(pA \cap \widehat{B})$ is (it is a finite ring) : this proves (ii).

Let us check (i). In any affinoid algebra, an element is power bounded if and only if its image in $k(x)$ is in $\mathcal{O}_{k(x)}$ for all $x \in \mathrm{Sp}(A)$ (this is not trivial but true) in which case we have already said that $T_x(G) \subset \mathcal{O}_{k(x)}$, hence the first part of (i). For the second part, pick first any model $\mathcal{A} \subset A$ and consider the compact subset $T(G) \subset A$. As $\mathcal{A}$ is open, there are finitely many elements $g_1, \dots, g_s \in G$ such that

$$T(G) \subset \bigcup (T(g_i) + \mathcal{A}).$$

As $T(g_i)$ is power-bounded for each i, $\mathcal{A}' = \mathcal{A}\langle T(g_1), \dots, T(g_s)\rangle$ is a model of A and we are done.

Let us check (iv) finally. Denote by $\widetilde{T} : G \to \widehat{B}$ the factorisation of T_A by $\widehat{B} \subset A$. As A is connected, the ring $\widehat{B}$ is local. If k is the residue field of $\widehat{B}$ we set $\overline{T} = \widetilde{T} \otimes_{\widehat{B}} k$. If $x \in X$ we have a natural map $\widehat{B} \to k(x)$ that necessarily falls inside $\mathcal{O}_{k(x)}$. It follows that the residue field k of $\widehat{B}$ maps to k_x and that the residual pseudo-character $\overline{T}_x$ comes by a scalar extension from $\widetilde{T}$. Note that by

Hensel lemma, there is a unique morphism $\mathcal{O} \to \widehat{B}$ that induces the identify on the residue fields. The universal property of $R(\overline{T})$ concludes the proof. $\square$

We now go back to the proofs of theorems 2.1 and 2.2. By (iv) of lemma 2.3, theorem 2.1 and (ii) of 2.2 are consequences of part (i) of Thm. 2.2. Let us check the first part of (iv) of that Theorem. We only have to show that if $T_A \in E_{\overline{T}}(A)$ and if $\overline{T} = \mathrm{trace}(\bar{\rho})$ where $\bar{\rho}$ is irreducible[5], then there is a unique representation $G \to \mathrm{GL}_n(A)$ whose trace is T_A (up to $\mathrm{GL}_n(A)$-conjugation). The existence follows from part (iv) of the Lemma above: if $\rho^u : G \to \mathrm{GL}_n(R(\bar{\rho}))$ is the universal representation, then $\rho^u \otimes_{R(\bar{\rho})} A$ has the required property by that lemma. The uniqueness actually follows from the fact that we have an isomorphism at the level of Cayley-Hamilton quotients

$$A[G]/\mathrm{CH}(T) = (R(\overline{T})[G]/\mathrm{CH}(T)) \otimes_{R(\overline{T})} A \simeq M_n(A)$$

by the precision we gave in the theorem of Nyssen-Rouquier-Procesi, so that the uniqueness follows again from the fact that any A-algebra endomorphism of $M_n(A)$ is the conjugation by some matrix in $\mathrm{GL}_n(A)$.

We know prove (i) and (iii) of Thm. 2.2. We need for this to recall a construction due to Berthelot (see the appendix §7 to DeJong "Crystalline Dieudonné module theory via formal and Rigid geometry"). This construction starts with some local complete noetherian $\mathbb{Z}_p$-algebra R say with (finite) residue field k, and maximal ideal called m and produce some rigid space $\mathfrak{X}_R$. If you choose a presentation

$$\mathcal{O}[[X_1, \dots, X_r]]/J = R$$

this space will simply be the subspace of the open unit ball in r-variables over $\mathbb{Q}_q$ (viewed by restriction over $\mathbb{Q}_p$) defined by the vanishing of the elements of J. In particular, it will simply be the open unit ball when R is formally smooth over $\mathcal{O}$. Here is a canonical (and useful) way to define it abstracly.

PROPOSITION 2.4. *For any R as above, the functor* An $\to$ Sets *associating to an affinoid algebra A the continuous ring homomorphisms $R \to A$ is representable by a rigid analytic space $\mathfrak{X}_R$.*

Explicitely, this space is an admissible increasing union of the affinoids $\mathfrak{X}_{R,n} =$ $\mathrm{Sp}(R_n[1/p])$, *$n \geq 1$, where R_n is the p-adic completion of $R[m^n/p]$. The space $\mathfrak{X}_{R,n}$ is an affinoid subdomain of $\mathfrak{X}_{R,n+1}$.*

Note that this proposition concludes the proof of Theorem 2.2.

Proof — (see also De Jong Lemma 7.1.2 of loc. cit., Prop. 7.1.7). Note that up to replacing R by its biggest p-torsion free quotient (which does not change the functor), we might assume that R has no p-torsion if we like (this is not necessary however). There are three things to check :

(a) $R_n[1/p]$ is an affinoid algebra. For this it is enough to check that $R[m^n/p]/(p)$ is finitely generated over k. But this follows at once from the inclusion

$$m^n R[m^n/p] \subset pR[m^n/p].$$

[5]Note that $\bar{\rho}$ is defined over the field k by a well-known result of Schur (no obstruction to define an absolutely irreducible rep. over its field of trace if $p > n$, this comes in turn from the fact that all finite fields are commutative).

(b) If $f : R \to A$ is a continuous homomorphism to a $\mathbb{Q}_p$-affinoid A and if $x_1, \dots, x_r \in R$ is a system of R-generators of m, then the $f(x_i)$ are topologically nilpotent in A. If $\mathcal{A}$ is any model of A then $f(m^n) \subset p\mathcal{A}$ for n big enough, so that f extends to a homomorphism $R[m^n/p] \to \mathcal{A}$ (for all n big enough). As $\mathcal{A}$ is p-adically complete, this induces a morphism $R_n \to A$, hence $R_n[1/p] \to A$ for all n big enough. It follows from this analysis and from (a) that $R_n[1/p]$ represents the functor An $\to$ Sets of continuous homomorphisms $f : R \to A$ such that each element of $f(m^n)/p$ is power-bounded.

(c) The transition maps are affinoid subdomains. If $x_1, \dots, x_s$ are some R-generators of m^n then it follows from the universal property of $R_n[1/p]$ above that the natural map $R_{n+1}[1/p]\langle X_1, \dots, X_s\rangle(pX_i - x_i) \xrightarrow{\sim} R_n[1/p]$ is an isomorphism.

□

Example : When $R = \mathbb{Z}_p[[T]]$ then $\mathfrak{X}_R$ is the open unit ball over $\mathbb{Q}_p$ of dimension 1. Indeed, for any affinoid A then $\mathrm{Hom}_{\mathrm{An}}(R_n[1/p], A)$ is exactly the set of elements $t \in A$ such that t^n/p is power-bounded, so $R_n[1/p] = \mathbb{Q}_p\langle \frac{t^n}{p}\rangle[U]/(t^n - U^n)$ is the affinoid disc $|t| \leq 1/p^{1/n}$, and $\mathfrak{X}_{R,n} \subset \mathfrak{X}_{R,n+1}$ is the obvious inclusion. It follows that $\mathfrak{X}_R$ is simply the open unit disc $|t| < 1$ over $\mathbb{Q}_p$. From this we deduce at once that if

$$R = \mathcal{O}[[T_1, \dots, T_m]]/J$$

then $\mathfrak{X}_R$ is the locus $J = 0$ in the open unit disc in m-variables $|t_i| < 1$ over $\mathcal{O}[1/p]$ (viewed by restriction over $\mathbb{Q}_p$).

Corollary 2.5. *If $\bar{\rho}$ is absolutely irreducible and $R(\bar{\rho}) \simeq \mathcal{O}[[T_1, \dots, T_m]]$ then $\mathfrak{X}(\bar{\rho})$ is the open unit ball $|t_i| < 1$ of dimension m over $\mathbb{Q}_q$.*

If X is a rigid analytic space over $\mathbb{Q}_p$ we denote by $X(\overline{\mathbb{Q}}_p)$ the union of the $X(L)$ for all the finite extensions L of $\overline{\mathbb{Q}}_p$ inside $\overline{\mathbb{Q}}_p$.

Proposition 2.6. (i) *$\mathfrak{X}(\overline{\mathbb{Q}}_p)$ is in canonical bijection with the set of isomorphism classes of semi-simple representation $G \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$.*

(ii) *The subset $\mathfrak{X}^{\mathrm{irr}} \subset \mathfrak{X}$ parametrising the absolutely irreducible representations is an admissible (Zariski-open) subset.*

(iii) *Moreover, if $x \in \mathfrak{X}$ and if the associated semi-simple representation ρ_x is irreducible and defined over $L \supset k(x)$, then $\mathcal{O}_{\mathfrak{X},x} \otimes_{k(x)} L$ represents the deformation functor of ρ_x to the local complete noetherian L-algebras with residue field L.*

Indeed, (i) follows from the fact that $\mathfrak{X}$ represents E and the first theorem on pseudo-characters that we had in lecture 3. In part (ii), the locus in question is the complement of the closed subset defined by the vanishing of the determinants of all the matrices of the form $(T(g_i g_j))$ where $g_1, \dots, g_{n^2}$ are any n^2 elements of G (this follows from Wedderburn theorem). We could actually show that $\mathfrak{X}^{\mathrm{irr}}$ has a nice moduli property : it is the same to give an A-point of X^{irr} and an isomorphism class of continuous surjective representation $A[G] \to B$ where B is an Azumaya algebra of rank n^2 over A. Part (iii) follows from $E = \mathrm{Hom}(\mathfrak{X}, -)$ applied to the artinian objects of An.