The infinite fern of quaternionic Galois representations

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Introduction : The infinite fern of Gouvêa-Mazur

Let[1] S be a finite set of primes and let $G_{\mathbb{Q},S}$ be the Galois group of a maximal algebraic extension of $\mathbb{Q}$ unramified outside S. Fix a prime p, an integer n, and consider the set $\mathfrak{X}$ of continuous, semi-simple, representations

$$\rho : G_{\mathbb{Q},S} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p}),$$

taken up to isomorphisms. This set $\mathfrak{X}$ is actually the $\overline{\mathbb{Q}}_p$-points of a p-adic analytic space, the general structure of which is widely unknown, especially when $n > 2$. Any ρ as above has a natural *residual representation* $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, which actually allows to cut $\mathfrak{X}$ into closed-open subspaces $\mathfrak{X}(\bar{\rho})$, which can be studied by the deformation theory of $\bar{\rho}$, and described in some favorable cases.

The set $\mathfrak{X}$ contains a natural countable subset

$$\mathfrak{X}^{\mathrm{geom}} \subset \mathfrak{X}$$

of representations ρ which are *geometric*, in the sense that they are cut out from the p-adic étale cohomology of projective smooth algebraic varieties over $\mathbb{Q}$. This course adresses the problem to understand how does $\mathfrak{X}^{\mathrm{geom}}$ sit in $\mathfrak{X}$ for a wide class of components $\mathfrak{X}(\bar{\rho})$ when[2] $n = 2$. In this case, a special role is played by the subset $\mathfrak{X}^{\mathrm{mod}} \subset \mathfrak{X}^{\mathrm{geom}}$ of modular (or automorphic) Galois representations. When $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is odd, irreducible, and generic enough, it is known that $\mathfrak{X}(\bar{\rho})$ is the open unit ball of dimension 3 over $\overline{\mathbb{Q}}_p$.

Theorem : (Gouvêa-Mazur) For such a (modular) $\bar{\rho}$, $\mathfrak{X}^{\mathrm{mod}}$ is Zariski-dense in $\mathfrak{X}(\bar{\rho})$.

The main ingredient of their proof is the theory of p-adic families of modular forms, as developped by Coleman, extending pioneering work of Hida. An expression of this theory is the construction of a p-adic analytic curve $\mathcal{E}(\bar{\rho})$, called the eigencurve, that maps to $\mathfrak{X}(\bar{\rho})$ in a sufficiently non-trivial way to be able to show the theorem above : up to twists by characters, each modular point is a double point of the image of $\mathcal{E}(\bar{\rho})$ inside $\mathfrak{X}(\bar{\rho})$. This latter image is a kind of fractal subset of $\mathfrak{X}(\bar{\rho})$ called the infinite fern.

The main aim of this course is to explain as much as possible the notions encountered in this introduction and to prove the theorem above. We will actually not quite rely on the work of Coleman, but on a parallel theory developped by Buzzard that uses modular forms for definite quaternion algebras. The advantage of this point of view is that the p-adic argument is simpler and generalizes well to other situations. A mild drawback is that it excludes some cases of $\bar{\rho}$.

[1] These are preliminary notes: read at your own risk.

[2] The case $n = 1$ is not difficult, we omit its discussion in this introduction.