Modular forms of classical groups and even unimodular lattices

Three lectures, Harvard. Joint work with Jean Lannes.

1. Even unimodular lattices

$\mathbb{R}^n$ euclidean space, $(x_i, y_i) = \sum_{i=1}^{n} x_i y_i$

$\mathbb{Z}$ lattice $\mathbb{L}$. $\mathbb{L}$ is even if $\forall x \in \mathbb{R}^n, x \cdot x \in 2\mathbb{Z}$ ($\Rightarrow$ integral)

unimodular if $\text{vol } \mathbb{L} = 1$

$E_8 \mathbb{Z} \text{ unim. not even, } D_n \mathbb{Z} = \{ (x_i) \in \mathbb{Z}^n, \sum x_i \equiv 0 \pmod{2} \}$ even

$e = \frac{1}{2} (1, \ldots, 1)$, $e \cdot e = \frac{n}{2}$, $E_n = D_n + \mathbb{Z} e$ even unim. if $n \equiv 0 \pmod{8}$

(Unique $e \in \mathbb{L}$)$D_n$ up to isom.)

Set $L_n = \text{set of euil in } \mathbb{R}^n$

Classical facts $L_3 \neq \emptyset \iff n \equiv 0 \pmod{8}$

$O(\mathbb{R}^n) \setminus L_n = X_m$ finite set

$X_8 = \{ E_8 \}$, $X_{16} = \{ E_{16}, E_8 + E_8 \}$, $\# X_{24} = 24$, $\# X_{32} > 10^9$

Motivations to study these lattices?

- $L \in \mathbb{L}_n$, $q_L: L \to \mathbb{Z}$, $x \mapsto \frac{x^2}{2}$ $q$-forms $/\mathbb{Z}$ per def.

$X_m$ is the set of $\text{iso. class of such forms}$ (no def $/\mathbb{Z}$)

- Relations to modular forms $SL_2(\mathbb{Z})$, $Sp_{2g}(\mathbb{Z})$ (theta series), orthogonal...

[generalizations of all what follows exist for higher def, yet more complicated; each def $= 1$ good compromise in high dim.]
A word on classify

L integral latt. $R(L) = L \times L$, $x \cdot x = 2^y$ "no" $\lambda$

always ADE root system in $\text{Vec}_R R(L)$

e.g. $R(D_n) = R(D_m) \quad n > 8$, $E_8 \quad n = 8 \Rightarrow E_6 \neq E_8 \oplus E_8$

$R(D_m) = D_m \quad \forall n \geq 2$

B series $L \subset L_0$, $D(L) = \sum_{x \in L} q^{\frac{2x^2}{2}} \in M_n^2(SL_2(\mathbf{Z}))$

$= 1 + \# R(L)q + \ldots$

$n = 8$, $B(L) = B^*(E_8) = 1 + 240 \sum_{m \geq 1} (\frac{23}{m}) \frac{q^m}{m}$

as $\text{M}_9$ div 1

$= 1 + \# R(L)q + \ldots 

\Rightarrow L = E_8 = \mathbf{Z}[E_8]$

Same $n = 16$

$n = 24$ more compl. nice arg. of Venkov using $\text{M}_{14}$

unique $L \subset X_{24}$ with no root : leech lattice (Conway)

thus $X_{24} \dashv \text{Leech}^3 \rightarrow$ root syst. lifted onto equicoxeter

thus $X_{24} \dashv \text{Leech}^3 \rightarrow R(L) \rightarrow L$ root syst. of name 24

(minuscule geometry)

2) Kneser neighbors

p prime, $L, M, E$ in are p-neighbors if $L \sqsubseteq M \sqsubseteq L$ index $P$

(Kneser). Useful way to construct various lattices.

Construction fix $p, L \in L_n$. Let $C_p(L_{\mathbf{F}_p})$ be the set of isochoric lattices of $\text{Latt}_{L_{\mathbf{F}_p}}$. If $L \in L \sqsubseteq L_{\mathbf{F}_p}$, and if $pL \in H_{L_{\mathbf{F}_p}}$

is st. $H_pL = L_1$, 3 exactly 2 c.w. containing $H$, namely $L$

and $\text{M}_p(L; L)$. Concretely if $L = \mathbf{Z}^2$, $x \cdot x = 0 (p^2)$

then $\text{X} = H + \mathbf{Z} \frac{2}{p}$

Fact: $L \mapsto \text{M}_p(L; L)$ bij. between $C_p(L_{\mathbf{F}_p})$ and set of p-neighbors of $L$. 
Main question: given \( L, M \subseteq L_n \), can we determine \( N_p(L, M) \)?

For \( n \leq 27 \), the first integer case is \( n = 16 \), but before I come to that.

Theorem (enlarged): \( X_{24}(2) \) is a 5-dimensional graph, Leech, and \( E_8 \) distance 5

- Leech - (\( A_24 \)') (explain).

\[ x = (0, 1, 2, \ldots, 23) \in E_{24}, \quad x + \frac{x}{2} = 46, 47. \equiv 0 \pmod{47} \]

\[ N_p(E_{24}, \langle x \rangle) = \text{Leech} \quad (\text{Thompson}) \]

(Holy construction of \( C_{31} \))

- \( N_7(R^4, \text{Leech}) \neq 0 \implies p \gg \text{h}(R) \)

Theorem (enlarged): \( N_p(E_8 \oplus E_8, E_{16}) = \frac{405}{691} \frac{8^{16}}{p-1} (1 + p^{-1} - I(p)) \)

It remains!

First goal of the lecture: explain (several) facts of this theorem.
- Explain 24-dimensional analogue (need to phrase it differently)

3. Beginning of \( T_p \)

For \( n = 0, 8 \), consider "Hecke op.": \( T_p: \mathbb{Z} \{X_n\} \rightarrow \mathbb{Z} \{X_n\} \)

\[ [L] \rightarrow \Sigma [M] \]

\( M \equiv \text{p-nghb. of } L \)

Easy facts:
1. \( T_p \) commute each other.

ii) \( N_p(L, M) | 10(m) | = N_p(M, L) | 10(L) | \), i.e. \( T_p \) self-adj

\[ g_a \langle L, M \rangle = \delta_{a(M)} | 10(L) | \]

iii) \( e \cdot \sum [L] \in \mathbb{Q}[X_n] \) "trivial c.w.\) \( \Rightarrow T_p e = e(p) e \)
find common e.v. + somep systems of Hede e.v.

\[ n = 16 \quad \text{O}([E_{16}]) = W([D_{16}]), \quad \text{O}([E_6 \otimes E_8]) = W([E_6]) \times \mathbb{F}_2 \quad \text{gcd} \text{and} = \frac{2^8 \cdot 67}{405} \]

get e.v.: 405 \([E_{16}]) + 286 \([E_6 \otimes E_8]) \text{ and nth. } \text{[E}_{16}] - \text{[E}_6 \otimes \text{E}_8].

\text{mod } x^5

Check that \( e_v \) is e.v. of \( T_p \) on \( \text{dim } v \quad \text{is } U(p) \frac{p^{r-1}}{p^{r-1} + p^r} + \frac{p^r}{p^r} + 1 \)

\( \text{ecl. on recognizes trace of } \text{Frob}_p \text{ on any } \mathbb{F}_q \text{-adic Gal. reps. dim } 16. \)

16 \( w = 1 + w^3 \) and \( L = (1 + w + w^2) + w^2 + w^3 + \cdots + w^{10} \)

general fact: \( \text{e.v. on } \mathbb{Q}_e \text{ [Xn]} \) will give rise to \( m \)-dim.

\( \text{Gal. rep. commut. with} \) \( \text{trace } \text{Frob}_p = v \cdot T_p, \text{ and at } L, \)

\( HT 0, 1, \ldots, m - 2, \frac{r - 1}{2}. \quad (\text{nth}) \)

\( n = 24 \quad T_{24} \text{ has been very with significant effort by R.} \quad \text{Nelke-Venkert.} \)

It turns out that \( e_v \), all in \( \mathbb{Z} \).

I will explain the comp. in next lecture.

\( \text{A cool } D \text{-series} \)

\( V : \mathbb{Z} \text{ [Xn]} \rightarrow M_{21} \text{ (Sp}_2 \text{ (Z))} \quad \text{linear map, } [L] \rightarrow [L] \)

\( \text{relevance: } \text{Eichler comm. relation } \quad V \circ T_p = \left( \frac{p^{m-1}}{p-1} + T(p^2) \right) \varphi \)

"relation between Hede e.v. on both side".

\( m = 16 \quad [E_{16}]) - [E_6 \otimes E_8] \) generate kernel, say = nothing.

\( n = 24 \), gives e.v. \( \frac{1}{2} \). \( \sum p(t - p^r + p^{2r} - p^{3r}) \)

to go further, \( \text{Siegel D-series, } \quad g > 1 \)

\( V^{(g)} : \mathbb{Z} \text{ [Xn]} \rightarrow M_{16} \text{ (Sp}_{2g} \text{ (Z))} \quad \frac{1}{2}(v_i \cdot v_j) \)

\( T : \left[ \sum \begin{array}{c}
\phi \in \mathbb{G} \\
(v_i, \ldots, v_g) \in \mathbb{C}^g
\end{array} \right] \)

we can view it as generating series of gram matrices of \( g \)-vectors in \( L \).

\( \text{Siegel modular form follows from Poisson formula. again} \)
Again a form of Eichler rel. formula w. respect to certain \( E \)-the ep on Siegel mod. forms

map \( \mathcal{V}^{(g)} \) is injective for \( g \geq n \), surjective \( g \leq \frac{n}{2} \).

It is difficult to say something on Siegel side when \( g \) grows.

Compatibility:

\[
\Phi : M_k(S_g(Z)) \longrightarrow M_k(S_g(S_g(Z)))
\]

\( g \geq 2 \).

Siegel ep. ker \( \Phi = S_k(S_g(Z)) \) rap form s.

\[
\Phi \mathcal{V}^{(g)} = \mathcal{V}^{(g)}_{g+1}.
\]

\( \Phi \) depending filtration on \( \mathbb{Q}[X_n] \), graded piece

\[
\text{ker} \mathcal{V}^{(g)} \quad \text{into} \quad S^{(g)}_2(S_g(Z)).
\]

\( n = 16 \) with had expected that \( \mathcal{V}^{(g)}(E_6) = \mathcal{V}^{(g)}(E_6 \otimes E_4) \) \( g=1,2,3 \).

It is easy to see not true in \( g=4 \), proved by himself \( g=2 \), Igusa \( g=3 \).

\( F = \mathcal{V}^{(g)}(E_6 \otimes E_4) - \mathcal{V}^{(g)}(E_6) \neq 0 \) and \( \text{in } S_g(S_g(Z)) \).

Igusa: this is skeptical form! actually has dim 1 (Par-Yuen).

So our \( g \) is ev. of skeptical form, still not clear why \( T_p \) occurs.

One way to conclude: use a special contraction of cusp forms due to Ikeda

\[
S_k(S_g(Z)) \longrightarrow S^{(g)}_2(S_g(Z)) \text{ sending ev. to ev.}
\]

Specific prop. "Duke immemorial cong", "Suite kershaw" for long story! \[ \therefore \]

Aim to give a simple argument.

\[ \Phi \]

Orthogonal modular forms

\( n = \phi(8) \)

\[
L_m^2(2n) = \bigoplus U^* \otimes \mathcal{M}_n(On), \quad \mathcal{M}_n(On) = \bigcup_{T \in \text{Fm}(On(1n))} \mathcal{U}_T
\]

\[
\mathcal{T}_p(h)(L_m) = \sum \mathcal{M}_1
\]

\( \Phi \)

\[ \therefore \]
\( M_{0} \left( \mathcal{O}_{n} \right) \cong \mathbb{C}[x_{m}]^{*} \cong T_{p} \)

\( n = 8, \quad M_{0} \left( \mathcal{O}_{n} \right) \xrightarrow{\sim} U(\mathbb{E}_{8}) \), not difficult to compute this, (with D. Zehndler).

\( U = H_{d}(\mathbb{R}^{n}) = \int P : \mathbb{R}^{n} \to \mathbb{A}, \text{ hom. } d \text{ dim, harmonic} ) \)

\[ \sum_{d > 0} \text{dim } M_{H_{d}(\mathbb{R}^{n})} \mathbb{C}^{*} \mathbb{C} = \frac{1 - \epsilon^{2}}{\Pi(1 - \epsilon^{d})} = 1 + 0 + \cdots \]

\( d = 8, \text{ first interesting modular form.} \)

Harmonic \( \mathbb{D} \)-series:

\[ \text{let } I \in H_{d}(\mathbb{R}^{n}), \quad L \in L_{n}, \quad \frac{a_{2}}{2} \in \frac{\mathbb{Z}_{d} + \mathbb{H}(SL_{2}(\mathbb{Z}))}{2 \mathbb{Z}} \]

may view it as:

\[ \mathbb{Z}[L_{n}] \otimes H_{d}(\mathbb{R}^{n}) \xrightarrow{\text{Hecke}} \mathbb{S}_{2d}(SL_{2}(\mathbb{Z})) \]

get Eichler sel.

\( n = 8, d = 8 \), check isom of \( \Delta \)

\[ \implies T_{p} = \mathbb{T}(p) \cong p + p^{10} + p + p^{12} + p^{13} \]

We may use this form to prove this!

two ingredients

1. some more \( \mathbb{D} \)-series

2. Invariance

Fact: let \( \mathcal{O}_{n} \) be the \( n \)th gp scheme of \( \text{Em. } q_{E_{n}} \).

\( \text{then } L_{n}^{G_{0}} \cong G(\mathbb{C}) \setminus G(\mathbb{A}) \setminus G(\mathbb{Z}) \)

\( G = \text{G} \text{em. indeed. if } L \subset \mathcal{O}_{n}, \text{ then } L_{0} z_{p} = E_{n} z_{p} \)

\( \mathbb{Q}_{p}, \text{ idev. } \mathbb{Q}_{p} / \mathbb{Q} \text{ (single genus), so } 3 \gamma_{0} \in \mathcal{O}_{n}(\mathbb{R}) = G(\mathbb{R}) \).

\( \gamma_{1} \in G(\mathbb{A}) \)

\( \gamma_{1} \in \text{Em} @ \mathbb{Q} \)
Last time

L_n = L even \mu, lattices L \subset \mathbb{R}^n, discuss class: n \geq 24

mention \mu-neighbors, promised a proof of identity

N_\mu (E_8 \oplus E_8, E_{16}) = \frac{405}{2501} \frac{p^{n-1}}{p-1} (1 + p^n - \mu(p))

using Eichler + Siegel \mu-series, we reduced prob to study of Hecke opn

S_8 (\text{Sp}_8 (\mathbb{Z}) \to \mathbb{C} \cong \mathbb{R}^8 \cong \text{Hom}(E_8, E_8) \to \text{Hom}(E_8, E_6) \to \mathbb{C}^8)

then I introduce "orthogonal mod forms"

$$\mathcal{L}^2 (L_n) = \bigoplus_{\mu \in M_{\mu} (\text{O}_m)} T_{\mu} V_{\mu}$$

where:

- \mu \in M_{\mu} (\text{O}_m)
- \mathbb{C}^8 = \text{Hom}(E_8, E_6)
- \mathbb{R}^8 = \text{Hom}(E_8, E_8)
- \mathbb{C} = \text{Hom}(E_6, E_6)

then I introduce the concept of "why I'm using forms"?

Let \text{O}_m / \mathbb{Z} be sheaf of quadratic form \text{En}_m \cong \text{O}_m (\mathbb{Z})

claim

$$L_n \cong \text{O}_m (\mathbb{Z}) / \text{O}_m (\mathbb{Q}) \quad \text{Ad. of } \text{O}_m (\mathbb{Q})$$

key: \text{L} \in L_n \Rightarrow \mathbb{Z} \otimes \mathbb{Q_p} \Rightarrow \text{L} \otimes \mathbb{Q_p} \Rightarrow \text{L} \text{ and En}_m / \mathbb{Z}_p \text{ and } \mathbb{Q_p}

let get

$$L_{\text{un}} = \text{O}_m (\mathbb{A}) / \text{O}_m (\mathbb{Z})$$

finally + is \text{L} \to (\theta_x, \theta_y) \text{ where } \theta_x L = \theta_y \text{ En}

and observe \# is equiv with \text{O}_m (\mathbb{Q}) \text{ Hecke opn } T_{\mu} \text{ (see later)}

we use gp theoretic description of \text{L}_n

can go further:

$$L_n \cong \text{G} (\mathbb{Q}) \backslash \text{G}(\mathbb{A}) / \text{G}(\mathbb{Z})$$

for \text{G} = \text{O}_m, \text{Sp}_m, \text{PGSp}_m

Each \text{G} (\mathbb{Z}) \text{ PGSp}_8 \text{ has a tripled root } 1 / 2, \text{ e.g. use Cox}

octonion inv. structure on \text{E}_8 \text{ to define } \text{G}.
Before I go further, let
\[ H_{d,g}(12^g) = \frac{1}{d} \sum_{i=1}^{\frac{d}{2}} P \circ \tau \circ \sigma = \text{det}^3 \sigma \]
\[ \tau \in GL_3(\mathbb{R}) \]
\[ \gamma \in G_2(\mathbb{R}) \]
\[ \text{if } g < 2, \text{ otherwise } H_{d,g} = H_{d,2g} \]

It allows to define \( \text{harm. Siegel } \gamma \)-seives.

\[ L \in d \text{, } P \in H_{d,g}(12^g), \tau_{(L)}(L,P) = \sum_{\nu, \nu, \nu, \nu, \nu} P(\nu, \nu, \nu) \cdot q \]
\[ \nu, \nu, \nu, \nu, \nu \in S_{d,2}(G_{1/2}(\mathbb{R})) \]

Then \( n = 8 \), \( H_8(12^8) \subset \text{Int} H_{4,4}(12^8) \).

Going back to our example \( \Rightarrow \) subspace \( C \subset L^2(\mathbb{R}) \subset H_{4,4}(12^8) \).

Compute \( \gamma \)-semi-invariant element in \( S_8(\mathbb{R}_3(\mathbb{R})) \). Check if

\[ \text{relate Schottky form to } C(P) \text{ directly.} \]

We omit details but we put theorem about \( N_0(E_8 \otimes E_8, E_8) \).

Continuing, we have seen nice and nontrivial examples of orthogonal modular forms, with useful adellic description. There are many other ones! See memoir with Lannes.

(Exc. \( \det(\mathbb{Q}_2) \)) as well as my book with Renard.

I want to explain now big picture. E.g. how to explain ubiquitous appearance of \( C(P) \), (in small weight), precise formulas? (powers of \( P \), ...) Key solution given by Arthur-Langlands philosophy.
Fix $G$ a semisimple group scheme over $\mathbb{Z}$, e.g., $PGl_n$, $Sp_{2g}$, split $\mathbb{Z}$, $SO_n$, $GSp_{2g}$.

$A(G) = L^2(\mathbb{G}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}) / \mathbb{G}(\mathbb{Z})) \otimes \mathbb{G}(\mathbb{R})$ unit rep.

If $p$ prime, it also has an action of the ring of Hecke op. $H_p(G)$.

One way to define it is to set $X = \mathbb{G}(\mathbb{Q}_p) / \mathbb{G}(\mathbb{Z}_p)$; $H_p(G) = \text{End}(\mathbb{Z}[X]) / \mathbb{Z}[\mathbb{G}(\mathbb{Q}_p)]$.

**Facts:**
- Generated commutative ring
- $H_p$'s and $\mathbb{G}(\mathbb{R})$: commutative algebra.
- $H_p$ acts by normal end of $A(G)$.

**Example:** $G = SO_n$, $X = \text{Sel} \text{u.l. in } E_{\text{Sel}} \otimes \mathbb{Z}[rac{1}{p}] \simeq \{ \text{self dual } L \otimes E_{\text{Sel}} \otimes \mathbb{Q}_p \}$

$T_p \in H_p(SO_n)$ but this ring is much bigger! (rel div $\frac{\alpha}{2}$)

**Spectral decomposition**

$$A(G) = \bigoplus_{\pi \in \Pi(G)} \pi \otimes \text{disc} \pi \circ \text{Ad} \circ \text{Ad}(G)$$

Set $\Pi(G) = \bigoplus_{\pi \in \Pi}(\pi)$. The unit rep of $\mathbb{G}(\mathbb{R})$

Then $A(G) = \bigoplus_{\pi \in \Pi(G)} \pi \otimes \text{disc} \pi \circ \text{Ad} \circ \text{Ad}(G)$

$$\Pi(G) \supset \text{disc} \pi \circ \text{Ad} \circ \text{Ad}(G) \subset \Pi(G)$$

**Main pb:** describe $\text{disc} \pi \circ \text{Ad} \circ \text{Ad}(G)$ (countable subset?)

Langlands' way of thinking about $\Pi(G)$ (parametrization)

Let $\hat{G}$ be the ss. complex alg. group with dual based-root datum as $G(C)$. (explain)

E.g., $\text{PGL}_n = SL_n(C)$, $SO_n(C) \cong \text{SO}(n)$, $\text{PSp}_{2g} \cong SO_{2g+1}(C)$, $(n \equiv \text{even})$. (explain)
Satake isomorphism \( \nu_p \) \( \Hom_{\text{ring}}(H_p(G), \mathcal{C}) \xrightarrow{\text{can}} \widehat{G}_{ss} \cong \text{SS cong classe in } \widehat{G} \)

\( e.g. \ G = \text{SO}_n \) \( e \nu \) of \( T_p \) gives \( p^{\frac{n-1}{2}} \times \) trace of Satake param in standard rep on \( \mathcal{C} \)

Harish-Chandra isom \( \Hom_{\text{alg}}(\text{Lie } G(G)), \mathcal{C}) \xrightarrow{\text{can}} (\text{Lie } \widehat{G})_{ss} \)

\( \text{param map} : \mathcal{C} : \Pi(G) \xrightarrow{\text{can}} \text{Lie } \widehat{G} \)

where \( \text{can}(T_p) \) is ass to \( T_p \) by Satake \( \nu_p \).

\( \text{can}(T_u) \) is the inf. character of \( T_u \). Namely, if \( U = \text{I} \mathfrak{t} \mathfrak{a} \mathfrak{e} \) then known that \( U(\text{Lie } \mathfrak{g}(\mathfrak{g})) \subset \mathfrak{u}^\circ \) (smooth vector) has a central character, hence an element in \( \text{Lie } \widehat{G}_{ss} \) via H.C. isom. (discuss)

\( \text{Ex: } G = \text{PGO}_2 \) \( U = \text{discrete series of } \mathcal{O}(R) \) "weight \( k \geq 2" \)

\( \text{then } \chi_U(G) \cong S_k(SL_2(2)) \) (can. up to scalar)

\( \Pi_{\text{disc } (G)} \cong \Pi_F \) "generated rep."

\( \text{Then } \text{can}(\Pi_F) \sim (0 \ 0 \ k-1 \ k^2) \in M_2(G) \)

\( \text{can}(\Pi_F) \sim \left( \begin{array}{cc} \lambda & 0 \\ \lambda^{-1} & 1 \end{array} \right) \in SL_2(C) \) \( \lambda \chi_F + \chi_F^{-1} = \frac{\lambda}{\lambda^2} \)

"Ramanujan Normalisation"

7) Arthur-Langlands conjectures

some natural operations on parameters:

\( \phi : H \rightarrow H' \) \( \sigma \)-map, induced \( \chi(H) \rightarrow \chi(H') \)

\( \boxtimes \) operation on \( X(SL_n) \): \( X(SL_n) \times X(SL_m) \rightarrow \left\{ X(SL_{nm}) \right\} \)

\( \otimes \) operation on \( X(SL_n) \): \( X(SL_n) \times X(SL_m) \rightarrow \left\{ X(SL_{nm}) \right\} \)
Arthur element $e \in X(SL_2)$ defined by

$$e_{\infty} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$e_{\rho} = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}$$

Notation: if $d \geq 1$, and $e \in X(SL_n)$, define
$$c(d) = c \otimes \text{Sym}^{d-1} e \in X(SL_{md})$$

Main conjecture (Arthur-Langlands)

There is a unique collection of triples $(n_i, \pi_i, d_i)$, $i = 1 \ldots k$, $s.t.\ n = \sum_{i=1}^{k} d_i n_i$, $\pi_i \in \text{Rep}(\mathbb{PGL}_n)$, $d_i \geq 1$,

$$n \circ c(\pi) = \bigotimes_{i=1}^{k} c(\pi_i)[d_i]$$

Remarks:

1) Given Ramanujan's conjeture $(c(\pi_i)$ have norm 1 even), beautiful

2) General defect to Ramanujan

3) Uniqueness part is known (and deep!), due to Jacquet-Shalika

4) Existence can be "explained" via yoga of Langlands groups,
   as well as "converse" result... (ALMF)

Theorem (Arthur) conj. holds for $(\pi, St)$ where $\pi \in \Pi_{\text{disc}}(SL)$

and $n = St: \mathbb{C} \to SL_n(C)$ is the standard rep, if $G$ is either

$Sp_{2g}$ or a split $SO_{x/2}$.

Moreover, he proves a form of AMF.

and has announced a $H$ for any $SO_{x/2}$

Remarks:

1. Uses a lot of diff. results (Langlands, Kottwitz, Arthur, Ngo, Waldspurger...)

2. We shall see how to use this theorem next time. Remaining time,
   explain why the examples we encountered so far fit the theory.

1. $G \in \Pi_{\text{disc}}(SL)$. Let $n: St \to G$ be a principal $SL_2$;

   Then $\text{Satash} G = c(\Gamma) = \pi(c(1))$, trivially fits $AC$ conjeture.

   $G = SO_n$, $St = n = 1 \otimes \text{Sym}^{n-2} \mathbb{C}^2$.

   $n = \Theta(8) \rightarrow$ recover miracle eigenvalue

   $\lambda_1^{1 \ldots p + q + 2 \ldots -1}$

   given from $m$th Satellite($T_n$).
consider $G = \text{SO}_{16}$, let $\Pi \in \Pi_{\text{disc}}(G)$ be the unique non-trivial one st. $\Pi \cong \mathbb{T}$. I claim that

$$\text{St } c(\Pi) = c(\Pi_{\Delta}) [4] \otimes [7] \otimes [7],$$

which strengthens them about $E_6 \otimes E_6$ and $E_{16}$. This is of course reminiscent with Galois theorems inter alia suggested in lec 1, but actually Galois rep. play no rule here (or in Arthur's theory).

Eichler's relations start with

$$\begin{array}{ccc}
\text{M}_u(\text{SO}_n)^* & \xrightarrow{\Psi} & \text{M}_v(\text{Sp}_{2g}(\mathbb{R}))^* \\
\uparrow H_0(\text{SO}_n) & & \uparrow H_0(\text{Sp}_{2g}(\mathbb{R})) \\
\Downarrow \Psi(F) \neq 0 & & \Downarrow \Psi(F) \neq 0 \\
\text{then } \varpi(F) \text{ eigenform for } H_p(\text{Sp}_{2g}) & & \text{and if } \varpi \in \Pi_{\text{disc}}(\text{SO}_n) \\
\text{and } \Pi \in \Pi(\text{Sp}_{2g}) & & \text{are the ass. rep. then}
\end{array}$$

(Rallis' theorem)

$$\begin{cases}
\text{St } c(\Pi) = \text{St } c(\Pi') \oplus [\text{m-2g-1}] & \text{if } \Pi \neq \Pi' \\
\text{St } c(\Pi') = \text{St } c(\Pi') \oplus [\text{2g+1-n}] & \text{if } \Pi \neq \Pi'
\end{cases}$$

see what happened for $n=8$?

c) we first considered $\Pi \in \Pi_{\text{disc}}(\text{SO}_8)$ st. $\text{St } c(\Pi_{\Delta 1}) = \text{Sym}^2 c(\Pi_{\Delta 1}) \otimes [5]$

t) $\Delta(\text{SO}_8) = \Delta(\text{PSO}_8) \simeq \Pi_{\text{disc}}(\text{SO}_8)$ st.

\begin{align*}
\text{St } c(\Pi_{\Delta 1}) &= \text{St } c(\Pi) \\
\text{but: } c(\Pi_{\Delta 1}) &= c(\Pi_{\Delta 1}) [4]
\end{align*}

(slight strengthening of Rallis rel.)

c) apply locality to $\Pi \rightarrow \Pi''$ st. $\text{St } c(\Pi'') = c(\Pi_{\Delta 1}) [4]$

and $\Psi_{\text{Sp}_{2g}}^{-1} c(\Pi'') = c(\Pi_{\Delta 1}) [4]$, $\text{Sym}^2 c(\Pi_{\Delta 1}) \otimes [5]$

d) another $\Psi$-seq's, $\Pi'' \rightarrow \Pi'''$ on $\text{Sp}_{2g}$ (Schottky form)

get $\text{St } c(\Pi_{\text{Schottky}}) = c(\Pi_{\Delta 1}) [4] \otimes [7]$

e) another $\Psi$-seq's $\rightarrow \Pi''''$ on $\text{SO}_{16}$, $\text{St } c(\Pi_{\Delta 1}) = c(\Pi_{\Delta 1}) [7] \otimes [7] \otimes [7]$

We have seen several instances of the AC conj.

More in next lecture! (with $\text{SO}_{24}$)
Last lecture stated A-L conj: $G < \mathfrak{sl}_n / \mathbb{Z}, \pi \in \Pi_{disc}(G), \pi \colon G \rightarrow \text{Sym}^d \mathbb{C}$

there exists unique coll. of triples $(\pi_i, m_i, d_i)_{i \in k}$, $\pi_i \in \Pi_{disc}(\text{PGl}_m)$

s.t. $\pi \circ \tilde{c}(\pi_i) = \bigoplus_{i \in k} \bar{c}(\pi_i)[d_i]$ - means $c(\pi_i) \otimes \text{Sym}^d \mathbb{C}$

Ananth's theorem only holds for $G = \text{Sp}_{2g} / \mathbb{Z}$, $G = \text{split} \text{so}_{2m} / \mathbb{Z}$

and $\pi \cong \text{St} = \text{standard rep.}$

Good today. What are the $\text{St} \circ \tilde{c}(\pi_i)$ of the $\pi_i$'s in $\Pi_{disc}(\text{Sp}_{2g})$ with $\pi_0 = \varepsilon$? Slight refinement of original pb: $x_m$ replaced with $\tilde{x}_m = \begin{pmatrix} x_m & m \leq 1, \# x_m = 2S \end{pmatrix}$

because need $\mathbb{Z}$ coefficients.

Main question: what are the $\tilde{\pi}_i$'s supposed to occur in them?

Need first to say more about possibilities.

8. Algebraic selfdual rep. of $\text{PGl}_m$

2 observations

1. If $\pi : \hat{G} \rightarrow \text{St}_\mathbb{C} \mathfrak{sl}_m$ is selfdual, then

\[ \pi(\tilde{c}(\pi_i)) = \tilde{c}(\pi_i)^{-1} = \bigoplus_{i \in \mathbb{Z}} \bar{c}(\pi_i)[d_i] \]

where $\bar{c}(\pi_i)$ "inverse class" of $\pi_i$

uniqueness proof $\Rightarrow \tilde{\pi}_i = \pi_i^\vee$ (selfduality)

[concretely, in the picture $\mathcal{O}(\text{PGl}_m) = \mathbb{L}^2 (3 \text{ lattices in } \mathbb{R}_m^3 / \text{hom} \rightarrow \mathbb{L}^2)$

the selfdual $\pi_i$'s are the one, stable by invol. $g \mapsto g^\vee$ or $\mathfrak{sl}(2) \rightarrow \text{SL}(2)$ some perfect pairing on $\mathbb{L}_m$]

2. Assume $\text{co}(\tilde{\pi})$ is the inf. char. of a finite dim rep. of $G(\mathbb{C})$ (e.g. discrete series, cohom up...)

Then $\text{co}(\tilde{\pi})$ easy to compute ($\sim \chi + \psi$) - & we see that

$\chi$ dominant.
the ev of \( \pi(\rho) \) are in \( \frac{1}{2} \mathbb{Z} \), congruent mod \( \mathbb{Z} \).

This property is inherited from the \( \pi_i \)'s!

**Definition** \( \pi \in \text{Rep}(\text{PGl}(m)) \) is called algebraic if the ev. \( \lambda_1, \ldots, \lambda_m \)
of \( \rho(\pi_i) \) are in \( \frac{1}{2} \mathbb{Z} \) and \( \lambda_i - \lambda_j \in \mathbb{Z} \) if \( i < j \).

The \( \lambda_i \)'s are called the weights of \( \pi \) and \( \lambda_{\text{max}} - \lambda_{\text{min}} = \omega(\pi) \geq 0 \).

Called the arith. weight of \( \pi \).

**Facts**

i) \( \lambda_1 \geq \ldots \geq \lambda_m \) then \( \lambda_i \lambda_{m+1-i} = 0 \) (closed formula lemma)

\( \text{so } \omega(\pi) = 2 \lambda_1 \).

ii) \( m = 1, \lambda_1 \in \frac{1}{2} \mathbb{Z} - \mathbb{Z} \) (see e.g. class, avoid trivial) \( \frac{k-1}{2} \lambda_1 \). \( \text{and } \pi \text{ is the disc. series wr. } \mathbb{R} \).

\( \pi \leftrightarrow \text{normalized eigenform } \Psi \) for \( \mathfrak{S}_2(\mathbb{Z}) \).

iii) important relation with Gal. rep. (Langlands - Fontaine - Mazur)

\[ \pi \leftrightarrow \psi \]

fix \( \xi, \psi \) \( \text{alg } \xi \leftarrow \text{linear cont. } \psi : G_{\mathbb{Q}} \to \text{Gl}(m(\mathbb{C})), \text{unram.} \]

\( \pi \leftrightarrow \psi \)

\( \lambda_i \) det \( (x - c(i) \psi) \), \( p \neq k \)

existence of \( \psi \) known in many cases \( \text{e.g. } \pi \text{ regular } (\neq \text{weights}) \)

\( \Rightarrow \text{two imp. things} \)

1) Ramanujan known for reg. selfdual \( \pi_i \) (alg).

2) get far free from \( \mathbb{Q} \). e.g. existence of assoc. Gal. rep. in many cases.

**Important problem:** can we classify these \( \xi \)'s? (even reg. selfdual ones?)

In recent works of Ch-Lannes, Ch-Renard, Taibi, interesting new results on this pl: see later.
$G = \text{PGSp}_4 \cong \text{SO}_{3,2}$ (split Z) - an example

let $\omega > \nu > 0$ odd integers
3D hol dins rate series $\omega, \nu$ of $G(Z)$ with a char $C_{\text{Sp}_4}(t)$ with $t \equiv \pm \frac{1}{2}, \pm \frac{1}{2}$ (up)

Fact: $\chi_{\omega, \nu}(G) \to \chi_{\omega, \nu}(\text{Sp}_4(Z)) = \text{space of genus 2 Siegel mod forms}
\text{for } \text{Sp}_4(Z), \text{vector coeff. Sym}^2 \otimes \text{det}

its dim. has been computed by R. Tsushima, we close on table, that for $\omega \leq 21$, it is always 0 or 1. $1 \cong (\omega, \nu)$ is in foll. list

(17,1) (19,7) (21,1) (21,5) (21,9) (21,13)
in each case, 3! $\Pi_{\omega, \nu} \in \text{Pdisc}(\text{PGSp}_4)$ with $(\Pi_{\omega, \nu})_\omega \otimes U_{\omega, \nu}$

Okahara what is $\text{St}(\Pi_{\omega, \nu}) \in \text{D}(\text{SL}_4(Z))$ (exists by Arthur)

only possibilities:
$c(T\Pi_1)$, $c(T\Pi_1) \otimes [23]$, $c(T\Pi_1) \otimes c(T\Pi_2)$, $[23]$, $c(T\Pi_1) \otimes [23]
$

but $\chi_{\omega, \nu}(\text{SL}_4(Z)) = 0$ for $\omega = 2, 6, 8, 10, 14$

so if $\nu = 1$, then $\text{St}(\Pi_{\omega, \nu}) \in \triangle_{\omega, \nu} \in \text{Pdisc}(\text{PGSp}_4)$

$v = 1$, might be $c(\Delta_{\omega}) \otimes [23]$ where $\Delta_{\omega} \cong$ unique $\text{Sw}_{\omega, \nu}(\text{SL}_4(Z))$

and it is indeed . . . log story. (Saito-Kakunuma with form, begining of "e")

in any case: we found 4 new cup form PGSp!

we can go further and analyze AMF here -- explains + gives

exact # of $\Pi \in \text{Pdisc}(\text{PGSp}_4)$ of weights $\pm \frac{1}{2}, \pm \frac{1}{2}$.

Similar story can be played each time you can compute dim. formula & understand AMF. With D. Renard, found explicit form of AMF all cases of interest here. Compute dim. very hard ph. historically,

using etc. isogenies, deal with $\text{SO}_{2,2}, \text{SO}_{2,1}, \text{Sp}_4, \text{SL}_2, \text{SO}_3$,

and form $\text{Sw}_{\omega, \nu}(\text{SL}_4(Z))$ and Tsushima's. Ch-Renard: non split $\text{SO}_8, \text{SO}_7$

get inter. eqns but cond. RECENT BREAKTHROUGH OF TAIIBI ON THIS PH.
As then explicit formulas (cf. table) for # of self. alg. Π of PGCM with weights λ₁ > λ₂ > ... > λₙ (regular), for m ≤ 14.

results are still cond. (but expected) if 2 λᵢ's are consecutive (i.e. λᵢ - λᵢ₊₁ = 1).

see our table!

10 Main Theorem

recall s char of trivial of $SO_{2n}(12)$ is a diag. $(±1, ±2, ..., ±n)$, 0

Fact: There are exactly 24 elements $c ∈ \mathbb{H}(SL_{2n})$ st:

i) $c\alpha$ has t.v. ±1, ±2, ..., ±n, 0 twice

ii) $c$ is a sum of $c(\Delta_i) [12]$ and $\Pi$ is in the foll. list

$\{ 1, \Delta_{11}, \Delta_{15}, \Delta_{17}, \Delta_{19}, \Delta_{21}, \text{Sym}^2 \Delta_{11}, \Delta_{19,17}, \Delta_{21,19}, \Delta_{21,23}, \Delta_{23,21}, \}$

(show list!)

Thm: These 24 elements are exactly the $SL(\Pi)$ where $\Pi ∈ \text{Triv}_{[12]}(SO_{2n})$

and $\Pi c \equiv c$ with $c(\Delta_{11}) [12]$ occurring twice.

Several remarks

a) 24/24 had already been found by ingenious constructions by T. Ikeda, namely the ones not containing the $\Delta_{19,21}$, His work is based on works Frechkin, Borchardt-Freitag-Weissauer, Bokler, Nebe-Venkove + its sel. of Duke-Jamann's conj + sel. Miyawaki conj.

part of this set of work is to understand ker from filtration on $A[X_{2n}]$, analogue of Witt's conj. to Niem. lattice. A few things were missing to have complete picture (Nebe-Venkove's conj.), they are proved by tm. We even show (much harder)

$k ≤ 12$ $\forall k$: $A[X_{2n}] → \frac{M_{2k}(\mathbb{Z})}{12}$ surjective, iso for $k = 12$. 

Enshrinin, $D^{(w)}$ injective. BEW $D^{(w)}$ kernel dim 1.

Existence of Arthur parameters for $H_i$'s. (So $O_{2n}$ not split!)

6) $C_p(\Delta_{w\cdot})$ unknown a priori. We found a way to compute them using direct analysis of $N_p(R^n \setminus \text{Leech}) + \text{Ramanujan bounds}$. We could compute $C_p(\Delta_{w\cdot})$ up to $p = 113$. We obtain as corollaries the results on the graphs $X_{2n}(p)$ stated in lec 1.

Our comp. confirm work of Falas & v.d. Geer, computed differently to $C_p(\Delta_{w\cdot})$ for $p \leq 37$. (using enumeration of genus 2 curves!)

7) Once we know AMF (if you admit it is proved), case-by-case check shows m = 0, k = 1, $z_{\pi} = 0$ only $(\Delta_{1,1}) \otimes (2)$, conclude by $\# X_{2m} = 25$. Unfortunately cond. proof (but the best one!)

We argue differently.

11 Some ingredients

Theorem Let $\Pi \in \Pi^{\text{rigid}}(\text{PGl}_m)$ be algebraic of arch. weight $\leq 21$ (but m arbitrary). Then either $m \leq 2$ or $\Pi$ is one of the 4 $\Delta_{w\cdot}$

found in 9.

Ref. $w(\Pi) \leq 11$, then $\Pi = 1$. This was known to Serre-Hecke

(compare with Mannweiler, Fontaine, $w(\Pi) \leq 10, 11$). Still finite $w(\Pi) \leq 23$.

Let us see first how to apply this:

Let $\Pi \in \Pi^{\text{rigid}}(\text{Sp}_8)$ ass. to Siegel mod. form $\mathfrak{m}$ (e.g.营销 form)

What can $St \mathfrak{c}(\Pi)$ be? Ind. char: ev. $\pm 7, \pm 6, \pm 5, 4, 0$

write $St c(\Pi) = \otimes c(\Pi_i) \otimes [1]$, observe $w(c(\Pi_i)) \leq 14$

$\Rightarrow \Pi_i \subset \Delta_{1,1}$. Only possibility $c(\Delta_{1,1}) \otimes [1]$

get first thin (and dim $S_8(\text{Sp}_8(2)) \leq 1$)
Unfortunately, a lot of extra comple. : need \( \omega(\pi) = 22 \) need to eliminate several possible \( \pi \)'s (I believe unique \( \pi \) of cond. weight 22, \( \text{Sym}^2 \Delta_n \))

What about pf of theorem?

First ingredient novel use of Riemann-Weil explicit formula, in the spirit of works of Odlyzko, Neumeier, Feit, and Miller.

Assuming \( \pi \) exists, try to limit possibilities for \( \pi \), using \( L(s, \pi \times \pi') \). Works for \( \omega(\pi) \leq 17 \) perfectly.

For higher \( \omega(\pi) \), get only regular possibilities.

A nice trick of Taishi allow to see they are selfdual by (deep list). In proof of them for Niemeier lattices we don't need Taishi's difficult computations though.