Modular forms of classical groups and even unimodular lattices

three lectures, Harvard. Joint work with Jean Lannes
90'

1) Even unimodular lattices

$\mathbb{R}^n$ euclidean space, $(x_i)\cdot(y_i) = \sum_{i=1}^n x_i y_i$

$L \subset \mathbb{R}^n$ lattice. L is even if $\forall x \in \mathbb{R}^n$, $x \cdot x \in 2\mathbb{Z}$ ($\Rightarrow$ integral)
unimodular if $\mathrm{covol}\, L = 1$

Ex: $\mathbb{Z}^n$ unim. not even, $D_n = \{(x_i) \in \mathbb{Z}^n, \sum x_i \equiv 0\ (2)\}$ even

$e = \frac{1}{2}(1,\dots,1)$, $e \cdot e = \frac{n}{4}$, $E_n := D_n + \mathbb{Z}e$ even unim. iff $n \equiv 0\,(8)$
(unique e.u.l $\supset D_n$ up to isom.)

Set $\mathcal{L}_n$ = set of eul in $\mathbb{R}^n$

classical facts $\mathcal{L}_n \neq \emptyset \Leftrightarrow n \equiv 0\ (8)$
$O(\mathbb{R}^n) \backslash \mathcal{L}_n := X_n$ finite set

$X_8 = \{E_8\}$, $X_{16} = \{E_{16}, E_8 \oplus E_8\}$, $\# X_{24} = 24$, $\# X_{32} > 10^9$
Mordell, Witt, Niemeier, King

Motivations to study those lattices?

- $L \in \mathcal{L}_n$, $q_L : L \to \mathbb{Z}$, $x \mapsto \frac{x \cdot x}{2}$ q. form$/\mathbb{Z}$ pos. def. (non deg. $/\mathbb{Z}$)
  X_n is the set of iso. class of such forms
- relations to modular forms $SL_2(\mathbb{Z})$, $Sp_{2g}(\mathbb{Z})$ (theta series), orthogonal gp. ...

[generalizations of all what follows exist for higher det, get more and more complicated; covol $L=1$ good compromise in high dim.]

A word on classif

L integral latt. , $R(L) = \{ x \in L,\ x\cdot x = 2 \}$ "roots"

always ADE root system in $\mathrm{Vect}_{\mathbb{R}} R(L)$

e.g. $R(E_n) = R(D_n)$ if $n > 8$, $\underline{E_8}$ $n=8$ $\Rightarrow E_{16} \not\simeq E_8 \oplus E_8$

$R(D_n) = \underline{D_n}$ $\forall n \geq 2$

ϑ series $L \subset \mathcal{L}_n$, $\vartheta(L) = \sum_{x \in L} q^{\frac{x\cdot x}{2}} \in M_{\frac{n}{2}}(SL_2(\mathbb{Z}))$

$= 1 + \# R(L) q + \cdots$

$n = 8$, $\vartheta(L) = \vartheta(E_8) = 1 + 240 \sum_{n \geq 1} (\sum_{d|n} d^3) q^n$ as M_4 dim 1

$\Rightarrow \# R(L) = 240 \Rightarrow R(L) \simeq E_8 \Rightarrow L = E_8 = \mathbb{Z}[R(E_8)]$

rk ≤ 8 $\Rightarrow$ Mordell.

Same $n = 16$.

$n = 24$ more compl. , nice arg. of Venkov using M_{14}.

Unique $L \in X_{24}$ with no root : Leech lattice (Conway)

thus $X_{24} - \{\text{Leech}\} \longrightarrow$ root syst. bijection onto equicoxeter

$L \longrightarrow R(L)$ root. sys. of rank 24.

(23 such.)

(miraculous geometry)

(2) Kneser neighbors

p prime , $L, M \in \mathcal{L}_n$ are p-neighb. if $L \cap M \subset L$ index p.

(Kneser). Useful way to construct e.u.l's.

Construction fix p. $L \in \mathcal{L}_n$. Let $C_L(\mathbb{F}_p)$ be the set of isotropic lines of $L \otimes \mathbb{Z}/p$. If $l \in L \otimes \mathbb{Z}/p$, and if $pL \subset H \subset L$ is s.t. $H/pL = l^{\perp}$, $\exists$ exactly 2 e.u.l containing H, namely L and $n_p(L, l)$. Concretely if $l = \langle x \rangle_{x \in L}$, $\frac{x\cdot x}{2} \equiv 0\ (p^2)$

then $N = H + \mathbb{Z}\frac{x}{p}$.

Fact : $l \longmapsto n_p(L; l)$ bij. between $C_L(\mathbb{F}_p)$ and set of p-neighb. of L.

Ex: • $|C_L(\mathbb{F}_p)| = 1 + p + \cdots + p^{n-2} + p^{\frac{n}{2}-1}$ (hyperbolic q. form).
$(= c_n(p))$

• Kneser $X_n(p)$ connected: every L can be constr. from E_n and $p = 2$

Main question: given $L, M \in \mathcal{L}_n$, can we determine $N_p(L, M)$?
$n \leq 24$. First. inter. case $n = 16$, but before I come to it...

Ex: • Borcherds $X_{24}(2)$ diam 5 graph, Leech and E_{24} distance 5
Leech $-(A_1^{24})^+$ (explain).

• $x = (0, 1, 2, \ldots, 23) \in E_{24}$, $\frac{x \cdot x}{2} = 46 \cdot 47 \equiv 0\ (47)$
$N_{47}(E_{24}, \langle x \rangle) = \text{Leech}$ (Thompson), (Holy-construction of C.S!)

Thm (Ch.-L.) - $X_{24}(p)$ known $\forall p$, e.g. $X_{24}(p)$ complete iff $p \geq 47$

- $N_p(R^+, \text{Leech}) \neq 0 \iff p \geq h(R)$

Thm (Ch.-L.) $N_p(E_8 \oplus E_8, E_{16}) = \frac{405}{691} \frac{p^4 - 1}{p - 1} (1 + p^{11} - \tau(p))$

remarks!

First goal of the lectures - explain (several) pfs of this thm.
- explain 24 dim analogue (need to phrase it differently)

③ Beginning of pf

$n \equiv 0\ (8)$, consider "Hecke op." $T_p : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$
$$[L] \to \sum_{M\ p\text{-neighb. of } L} [M]$$

easy facts i) T_p commute each other.

ii) $N_p(L, M)\, |O(M)| = N_p(M, L)\, |O(L)|$, i.e. T_p self-adj
for $\langle [L], [M] \rangle = \delta_{[L],[M]}\, |O(L)|$

iii) $\Rightarrow e := \sum \frac{[L]}{|O(L)|} \in \mathbb{Q}[X_n]$ "trivial e.v."; $T_p e = c_n(p)\, e$ $\forall p$.

IV

Pb: find common ev + ronep. systems of Hecke e.v.

$\underline{n=16}$ $O(E_{16}) \cong W(D_{16})$, $O(E_8 \oplus E_8) \simeq W(E_8)^2 \rtimes \mathbb{Z}/2$, quot. card $= \frac{286}{405}$

get ev. $405\,[E_{16}] + 286\,[E_8 \oplus E_8]$ and orth. $[E_{16}] - [E_8 \oplus E_8]$.

↑ trivial $\ast$ ↑ ev ?

$\underline{\text{check}}$ this $\Leftarrow$ ev of T_p on $\nearrow$ is $\tau(p)\frac{p^4-1}{p-1} + p^4\frac{p^7-1}{p-1} + p^7$

$\underline{\text{Rk}}$: on recognizes trace of Frob_p on an ℓ-adic Gal. reps. dim 16.

$1+\omega+\cdots+\omega^{14}+\omega^7$ and $\rho_\Delta \otimes (1+\omega+\omega^2+\omega^3) + \omega^7+\omega^4+\cdots+\omega^{10}$

$\underline{\text{general fact}}$ c. ev on $\mathbb{Q}_\ell[X_n]$ will give rise to n-dim.

Gal. rep. un ram. outside ℓ, trace $\mathrm{Frob}_p = \mathrm{tv}\,T_p$, $p \neq \ell$, cris at ℓ,

HT $0, 1, \ldots, n-2, \frac{n}{2}-1$. (+ orth.)

$\underline{n=24}$ T_2 has been comp. with significant effort by Borcherds (+ Nebe-Venkov)

it turns out that $\neq$ ev, all in $\mathbb{Z}$.

I will explain the G.rep. in next lectures.

$\underline{\text{A tool}}$ $\underline{\vartheta\text{-series}}$

$\vartheta : \mathbb{Z}[X_n] \longrightarrow M_{\frac{n}{2}}(SL_2(\mathbb{Z}))$ linear map, $[L] \to \vartheta(L)$

relevance: Eichler com. relation, $\vartheta \circ T_p = \left(p\frac{p^{n-3}-1}{p-1} + T(p^2)\right) \cdot \vartheta$

"relation between Hecke ev. on both sides"

$n=16$; $[E_{16}] - [E_8 \oplus E_8]$ generates kernel, says nothing.

$n=24$, gives e.v $\tau(p)^2 - p^{11} + p + \cdots + p^{21}$

to go further, Siegel ϑ series, $g \geq 1$

$\vartheta^{(g)} : \mathbb{Z}[X_n] \longrightarrow M_{\frac{n}{2}}(Sp_{2g}(\mathbb{Z}))$ $\frac{1}{2}(v_i \cdot v_j)$

$[L] \longrightarrow \sum_{(v_1, \ldots, v_g) \in L^g} q$

we can view it as generating series of Gram matrices of g-vectors in L.

Siegel modular form follows from Poisson formula again

Again a form of Eichler rel. formula with respect to certain Hecke op. on Siegel mod. forms (V)

map $\vartheta^{(g)}$ is injective for $g \geq n$, surjective $g \leq \frac{n}{4}$

pb. difficult to say something on Siegel side when g grows.

compatibility $\Phi : M_k(Sp_{2g}(\mathbb{Z})) \longrightarrow M_k(Sp_{2g-2}(\mathbb{Z}))$ $g \geq 2$.

Siegel op., $\ker \Phi = S_k(Sp_{2g}(\mathbb{Z}))$ cusp forms, $\Phi\, \vartheta^{(g)} = \vartheta^{(g-1)}$!

$\Rightarrow$ $\ker \vartheta^{(g)}$ descending filtration on $\mathbb{Q}[X_n]$., Hecke stable, graded pieces

$\frac{\ker \vartheta^{(g-1)}}{\ker \vartheta^{(g)}} \hookrightarrow S_{\frac{n}{2}}(Sp_{2g}(\mathbb{Z}))$.

$n = 16$ Witt had conjectured that $\vartheta^{(g)}(E_{16}) = \vartheta^{(g)}(E_8 \oplus E_8)$ $g = 1, 2, 3$!

(easy to see not true in $g = 4$). proved by himself $g = 2$, Igusa $g = 3$.

$F = \vartheta^{(4)}(E_8 \oplus E_8) - \vartheta^{(4)}(E_{16}) \neq 0$ and in $S_8(Sp_8(\mathbb{Z}))$

Igusa: this is Schottky form!, actually has dim 1 (Poor-Yuen).

still not clear why $T(p)$ occurs.

so our pf $\Leftrightarrow$ ev. of Schottky form, still not clear why $T(p)$ occurs.

one way to conclude: use a special construction of cusp forms due to Ikeda

$S_k(SL_2(\mathbb{Z})) \hookrightarrow S_{\frac{k+g}{2}}(Sp_{2g}(\mathbb{Z}))$ sending cusp. to cusp eigenform eigenform

specific prop. "Duke Imamoglu conj", "Saito Kurokawa" for

long story! $\rightsquigarrow$ pf.

Aim to give a simpler argument.

(4) Orthogonal modular forms

$n \equiv 0\ (8)$ $\mathcal{L}_n \supseteq O(\mathbb{R}^n)$, f. many orbits, finite isotropy grps.

$$L^2(\mathcal{L}_n) = \overset{\perp}{\bigoplus_{U \in \mathrm{Irr}(O_n(\mathbb{R}))}} U^* \otimes M_U(O_n), \quad M_U(O_n) = \{ f : \mathcal{L}_n \to U \}$$

$\circlearrowleft O(\mathbb{R}^n)$ $\quad f(gL) = g\, f(L)$, $g \in O(\mathbb{R}^n)$, $L \in \mathcal{L}_n$.

$\circlearrowleft T_p$

$$T_p(f)(L) = \sum_{M\ p\text{-n.}} f(M)$$

ex: $M_{\mathbb{C}}(O_n) = \mathbb{C}[X_n]^* \supset T_p$

$n=8$, $M_U(O_n) \xrightarrow{\sim} U^{O(E_8)}$, not too! diff. to compute dim (with D. Zerad).

$f \longmapsto f(E_8)$

$U = H_d(\mathbb{R}^n) = \{ P : \mathbb{R}^n \to \mathbb{C}$, hom. d° d, harmonic $\}$.

$$\sum_{d \geq 0} \dim M_{H_d(\mathbb{R}^8)}(O_8)\, t^d = \frac{1-t^2}{\prod (1-t^{d_i})} = 1 + t^8 + \dots \qquad d_i = 1+j, \quad 1 \leq j < 30, \quad (j,30)=1$$

$d=8$, first interesting modular form.

Harmonic ϑ-series $d>0$: $P \in H_d(\mathbb{R}^n)$, $L \in \mathcal{L}_n$

$$\vartheta(L;P) = \sum_{x \in L} P(x)\, q^{\frac{x \cdot x}{2}} \in S_{\frac{n}{2}+d}(SL_2(\mathbb{Z}))$$
(Hecke)

may view it as $\mathbb{Z}[\mathcal{L}_n] \otimes H_d(\mathbb{R}^n)/_{O(\mathbb{R}^n)} \longrightarrow S_{\frac{n}{2}+d}(SL_2(\mathbb{Z}))$

get Eichler rel. $\qquad M_{H_d(\mathbb{R}^n)^*}(O_n)^*$.

$n=8$, $d=8$, check isom $\quad M_*(O_8)^* \xrightarrow{\sim} S_{12}(SL_2(\mathbb{Z}))$

nice constr. of Δ.

$\Rightarrow \quad T_p \circlearrowleft \quad \tau(p)^2 - p^{11} + p^9 + p^{10} + p^{11} + p^{12} + p^{13}$

We may use this form to prove thm!

two ingredients: 1 - some more ϑ-series

2 - triality

fact: let O_n be the orth gp scheme $/\mathbb{Z}$ of (E_n, q_{E_n}).

Then $\mathcal{L}_n^{\text{op}} \xrightarrow{\sim} G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}})$

$G = O_n$ indeed, if $L \in \mathcal{L}_n$, then $L \otimes \mathbb{Z}_p \simeq E_n \otimes \mathbb{Z}_p$ $\forall p$, idem $/\mathbb{R}$, $/\mathbb{Q}$

(single genus), so $\exists\, g_\infty \in O_n(\mathbb{R}) = G(\mathbb{R})$, $g_\infty^{-1} L \in E_n \otimes \mathbb{Q}$.

$g_f \in G(\mathbb{A}_f) \qquad \qquad g_\infty^{-1} L = g_f E_n$

Last time $\mathcal{L}_n = \{\text{even u. lattices } L \subset \mathbb{R}^n\}$, $n \equiv 0\ (8)$, discuss class. $n \leq 24$

notion p-neighbors, promised a proof of identity

$$N_p(E_8 \oplus E_8, E_{16}) = \frac{405}{691} \frac{p^4-1}{p-1} (1 + p^{11} - \tau(p))$$

using Eichler rel + Siegel ϑ series, we reduced pb. to study of Hecke e.v on $S_8(Sp_8(\mathbb{Z})) = \mathbb{C} \cdot (\vartheta^{(4)}(E_8 \oplus E_8) - \vartheta^{(4)}(E_{16}))$

then I introduce "orthogonal mod. forms"

$$O(\mathbb{R}^n) \curvearrowright L^2(\mathcal{L}_n) = \bigoplus^{\perp}_{U \in \mathrm{Irr}(O(\mathbb{R}^n))} U^* \otimes M_U(O_m) \quad \circlearrowleft T_p \quad \forall p$$

$$T_p(f)(L) = \sum_{M \text{ p-ng. of } L} f(M)$$

ex: $n = 8$, $L^2(\mathcal{L}_8) \supset$ unique copy of $H_8(\mathbb{R}^8)$
$\circlearrowleft T_p \quad \tau(p^2) + p^9 + \dots + p^{13}$

"why $\perp$ m. forms"?

let $O_m/\mathbb{Z}$ g.p. scheme of quadratic form E_m $(m \equiv 0\ (8))$

claim $\mathcal{L}_n \xrightarrow[\sim]{(*)} O_m(\mathbb{Q}) \backslash O_m(\mathbb{A}) / O_m(\hat{\mathbb{Z}})$, $\mathbb{A}$ = adeles of $\mathbb{Q}$

key $L \in \mathcal{L}_n \Rightarrow L \otimes \mathbb{Z}_p$ hyp. $\forall p \Rightarrow L$ and E_n isom $/\mathbb{Z}_p$ and $\mathbb{Q}$ $\forall p$ ("single genus")

if $L \in \mathcal{L}_n$, $\exists\, g_\infty \in O_m(\mathbb{R})$ s.t. $g_\infty L \subset \underbrace{\{\text{e.u.l in } E_m \otimes \mathbb{Q}\}}_{= \mathcal{L}_n^{\mathbb{Q}}} \circlearrowleft O_m(\mathbb{A}_f)$

but get $\mathcal{L}_n^{\mathbb{Q}} = O_m(\mathbb{A}_f) / O_m(\hat{\mathbb{Z}}) \cdot E_m$ (explain)

finally $*$ is $L \mapsto (g_\infty, g_f)$ where $g_\infty L = g_f E_m$.

observe $(*)$ is equiv. with $O_m(\mathbb{R})$, Hecke op. $\leftrightarrow$ aut. Hecke ev. T_p (see later)

$\rightsquigarrow$ nice gp. theoretic description of $\mathcal{L}_n$.

can go further: $\mathcal{L}_n \xrightarrow{\sim} G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}})$ for $G = O_m, SO_m, PGSO_m$ and PGO_m! (explain)

back $n = 8$ case $PGSO_8$ has a triality aut $/\mathbb{Z}$, e.g. use Cox. octonion int. structure on E_8 to define it.

$\rightsquigarrow$ interesting order 3 ~~isoms~~ permutat. of $\mathcal{L}_8$ fixing E_8. (II)

e.g. $\{p\text{-neigh. of } E_8\} \xrightarrow{\sim} \{\frac{1}{\sqrt{p}} M,\ pL \subset M \subset L$ and $M/pL \subset L \otimes \mathbb{F}_p$ lag. of type $\pm\}^{\pm}$

(half of them)

before I go further, let

$$H_{d,g}(\mathbb{R}^n) = \{P : (\mathbb{R}^n)^g \to \mathbb{C}\} \circlearrowleft O(\mathbb{R}^n) \text{ irred.}$$

harm, $P \circ \gamma = \det \gamma^d P$, $\gamma \in GL_g(\mathbb{C})$; on $SO_n(\mathbb{R})$ irred if $g < \frac{n}{2}$, otherwise $H^{\pm}_{d,g}$

$1 \leq g \leq n/2$

It allows to define harm. Siegel ϑ-series.

$$L \in \mathcal{L}_n,\ P \in H_{d,g}(\mathbb{R}^n),\ d \geq 0,\quad \vartheta^{(g)}(L,P) = \sum_{(v_1,\dots,v_g) \in L^g} P(v_1,\dots,v_g)\, q^{\frac{1}{2}(v_i \cdot v_j)} \in S_{d+\frac{n}{2}}(Sp_{2g}(\mathbb{Z}))$$

Then $n = 8$, $H_8(\mathbb{R}^8) \xleftrightarrow{\text{trivial}} H_{4,4}(\mathbb{R}^8)^{\pm}$!

going back to our example $\rightsquigarrow$ subspace $\subset L^2(\mathcal{L}_8) \simeq H_{4,4}(\mathbb{R}^8)^{\pm}$

compute ϑ series $\rightsquigarrow$ element in $S_8(Sp_8(\mathbb{Z}))$! check $\neq 0$.

relates Schottky form to $\tau(p)$ directly.

We omit details but $\rightsquigarrow$ pf theorem about $N_p(E_8 \oplus E_8, E_{16})$ $\square$

Conclusions

- we have seen nice and non trivial examples of orthogonal modular forms, with useful adelic description. There are many other ones ! See memoir with Lannes. (ex. $M_{\det}(O_{24})$!), as well as my book with Renard.
- Want to explain now big picture. E.g how to explain ubiquitous appearance of $\tau(p)$? (in small weights) precise formulas ? (powers of $p \dots$) Key solution given by Arthur-Langlands philosophy.

(6) Crash course on (conductor 1) automorphic forms

- fix G a semisimple gp. scheme $/\mathbb{Z}$, e.g. PGL_n, Sp_{2g}, any split$/\mathbb{Z}$, SO_n $n \equiv 0(8)$ ...

$$\mathcal{A}(G) = L^2\left(G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}}) \right) \supseteq G(\mathbb{R}) \text{ unit. rep.}$$

if p prime, it also has an action of ring of Hecke op. $H_p(G)$:

one way to define it is to set $X = G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$; $H_p(G) = \operatorname{End}_{\mathbb{Z}[G(\mathbb{Q}_p)]}(\mathbb{Z}[X])$

facts (i) f. generated commutative ring
(ii) H_p's and $G(\mathbb{R})$: commuting actions.
(iii) H_p act by normal end. of $\mathcal{A}(G)$.

ex: $G = SO_n$, $X \simeq \{ \text{e.u.l in } E_n \otimes \mathbb{Z}[\frac{1}{p}] \} \simeq \{ \text{self dual } \mathbb{Z}_p\text{-integral } L \subset E_n \otimes_{\mathbb{Z}} \mathbb{Q}_p \}$

$T_p \in H_p(SO_n)$ but this ring is much bigger! (rel dim $\frac{n}{2}+1$)

- Spectral decomposition

$$\mathcal{A}(G) = \underbrace{\left[\widehat{\bigoplus_{U \in \operatorname{Irr}(G(\mathbb{R}))_{\text{(unit)}}}}^{\perp} U \otimes \mathcal{A}_U(G) \right]}_{\mathcal{A}_{disc}(G)} \overset{\perp}{\oplus} \mathcal{A}_{cont}(G)$$

finite dim. space.
$\rightsquigarrow$ $H_p(G)$ acts codiag.

set $\Pi(G) = \{ \pi = (\pi_v),$ π_∞ unit. rep. of $G(\mathbb{R})$, $\forall p$, $\pi_p : H_p(G) \longrightarrow \mathbb{C}$ ring hom $\}$.

then $\mathcal{A}_{disc}(G) = \overset{\perp}{\bigoplus_{\pi \in \Pi(G)}} m(\pi)\, \pi$, $m(\pi) < \infty$

$\Pi(G) \supset \Pi_{disc}(G) = \{ \pi,\ m(\pi) \neq 0 \} \supset \Pi_{cusp}(G)$ (explain)

Main pb: describe $\Pi_{disc}(G) \subseteq \Pi(G)$ (countable subset!)

- Langlands' way of thinking about $\Pi(G)$ (parameterization)

let $\hat{G}$ be the ss. complex alg. group with dual based-root-datum as $G(\mathbb{C})$. (explain)

e.g. $\widehat{PGL_n} = SL_n(\mathbb{C})$, $SO_n(\mathbb{C}) \simeq \widehat{SO_n}$ $(n \equiv 0(8))$, $\widehat{Sp_{2g}} \simeq SO_{2g+1}(\mathbb{C})$

<u>Satake isomorphism</u> $\forall p$. $\mathrm{Hom}_{\text{ring}}(H_p(G), \mathbb{C}) \xrightarrow[\text{can}]{\sim} \widehat{G}_{ss} = \{\text{ss. conj classes in } \widehat{G}\}$
"systems of Hecke ev"

e.g: $G = SO_n$, e.v of T_p gives $p^{\frac{n}{2}-1} \times$ trace of Satake param in standard rep. on $\mathbb{C}^n$

<u>Harish-Chandra isom</u> $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{z}(U(\mathrm{Lie}\, G(\mathbb{C}))), \mathbb{C}) \xrightarrow[\text{can}]{\sim} (\mathrm{Lie}\,\widehat{G})_{ss}$
means center

if $H_{/\mathbb{C}}$ red. alg. gp, $\mathcal{X}(H) = \{(c_v)_v,\ \begin{matrix} c_\infty \in \mathrm{Lie}\, H_{ss} \\ c_p \in H(\mathbb{C})_{ss}\ \forall p \end{matrix}\}$.

<u>param. map</u>: $c: \quad \Pi(G) \longrightarrow \mathcal{X}(\widehat{G})$
$\qquad\qquad \pi \longmapsto (c_v(\pi_v))$

<u>where</u> $c_p(\pi_p)$ is ass. to π_p by Satake $\forall p$.
$c_\infty(\pi_\infty)$ is the inf. character of π_∞. Namely, if $U := \pi_\infty$ then known that $U(\mathrm{Lie}\, G(\mathbb{C})) \circlearrowright U^\infty$ (smooth vectors) has a central character, hence an element in $\mathrm{Lie}\,\widehat{G}_{ss}$ via H-Ch. isom. (discuss)

<u>Ex</u>: $G = PGL_2$, $U =$ discrete series of $G(\mathbb{R})$ "weight $k \geq 2$"
then $\mathcal{A}_U(G) \simeq S_k(SL_2(\mathbb{Z}))$ (can. up to $\times$ scalar)
$\Pi_{disc}(G) \ni \pi_F \quad \leftrightsquigarrow \quad F = \sum a_n q^n$ normalized Hecke eigenform
"generated" rep.
<u>then</u> $c_\infty(\pi_F) \sim \begin{pmatrix} \frac{k-1}{2} & 0 \\ 0 & -\frac{k-1}{2} \end{pmatrix} \in M_2(\mathbb{C})$

$c_p(\pi_F) \sim \begin{pmatrix} x_p & \\ & x_p^{-1} \end{pmatrix} \in SL_2(\mathbb{C})$, $\quad x_p + x_p^{-1} = \dfrac{a_p}{p^{\frac{k-1}{2}}}$
"Ramanujan Normalisation"

(7) <u>Arthur-Langlands conjectures</u>

some natural operations on parameters:

- $r: H \to H'$ $\mathbb{C}$-morph, induced $\mathcal{X}(H) \xrightarrow{r} \mathcal{X}(H')$

$\oplus, \otimes$ operations on $\mathcal{X}(SL_n)$: $\mathcal{X}(SL_n) \times \mathcal{X}(SL_m) \longrightarrow \begin{cases} \mathcal{X}(SL_{n+m}) \\ \mathcal{X}(SL_{nm}) \end{cases}$

Arthur element $e \in \mathcal{X}(SL_2)$ defined by $e_\infty = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ (V)

actually, $e = c(1_{PGL_2})$! $e_p = \begin{bmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{bmatrix}$

Notation if $d \geq 1$ and $c \in \mathcal{X}(SL_n)$, define

$$c[d] := c \otimes \mathrm{Sym}^{d-1} e \in \mathcal{X}(SL_{nd})$$

Main conjecture (Arthur-Langlands) $\forall \pi \in \Pi_{disc}(G)$, $\forall r : \hat{G} \to SL_n(\mathbb{C})$

there is a unique collection of triples (n_i, π_i, d_i), $i = 1 \dots k$,

s.t. $n = \sum_{i=1}^{k} n_i d_i$, $\pi_i \in \Pi_{cusp}(PGL_{n_i})$, $d_i \geq 1$,

$$r \circ c(\pi) = \bigoplus_{i=1}^{k} c(\pi_i)[d_i]$$

Rmks i) given Ramanujan's conj ($c(\pi_i)$ have norm 1 e.v.), beautifully explain general defect to Ram. conj.

ii) Uniqueness part is known (and deep!), due to Jacquet-Shalika

iii) existence can be "explained" via yoga of Langlands group, as well as "converse result". (ALMF)

Theorem (Arthur) conj. holds for (π, St) where $\pi \in \Pi_{disc}(G)$ and $r = St : \hat{G} \to SL_n(\mathbb{C})$ is the standard rep., if G is either Sp_{2g} or a split $SO_* / \mathbb{Z}$. [Moreover, he proves a form of AMF. and has announced a ff for any $SO_* / \mathbb{Z}$].

Rk: Uses a lot of diff. results (Langlands, Kottwitz, Arthur, Ngô, Waldspurger...) Shelstad, Mezo - Moeglin ...

• We shall see how to use this theorem next time. Remaining time, explain why the examples we encountered so far fit the theory.

① $1_G \in \Pi_{disc}(G)$. Let $r : SL_2 \to \hat{G}$ be a principal SL_2; Trivially fits AL conj.

then Satake $\Rightarrow$ $c(1_G) = r(e)$.

$G = SO_n$, $n \equiv 0 (8)$, $St \circ r = 1 \oplus \mathrm{Sym}^{n-2} \mathbb{C}^2$, $St\, c(1_G) = 1[n-1] \oplus 1[1]$

$\longrightarrow$ recover trivial eigenvalue $1 + p + \dots + p^{n-2} + p^{\frac{n}{2}-1}$ (given form for Satake(T_p))

(ii) consider $G = SO_{16}$, let $\pi \in \Pi_{disc}(G)$ be the unique non-trivial one s.t. $\pi_\infty \simeq \mathbb{C}$. I claim that $\quad$ VI

$$St\, c(\pi) = c(\Pi_\Delta)[4] \oplus [7] \oplus [1], \text{ which}$$

strengthen thm about $E_8 \oplus E_8$ and E_{16}. This is of course reminiscent with Galois theoretic interp. suggested in lect. 1, but actually Galois rep. play no role here (or in Arthur's theory).

<u>Eichler's relations</u> start with $M_V(SO_m)^* \xrightarrow{\vartheta} M_V(Sp_{2g}(\mathbb{Z}))$, $\circlearrowleft H_p(SO_m)$, $\circlearrowleft H_p(Sp_{2g})$

assume $F \in M_V(SO_m)^*$ eigenform for $H_p(SO_m)\ (\forall p)$ and $\vartheta(F) \neq 0$

<u>then</u> $\vartheta(F)$ eigenform for $H_p(Sp_{2g})\ (\forall p)$, and if $\pi \in \Pi_{disc}(SO_m)$ and $\pi' \in \Pi(Sp_{2g})$ are the ass. rep. then

(<u>Rallis' theorem</u>)
$$\begin{cases} \cdot\ St\, c(\pi) = St\, c(\pi') \oplus [m-2g-1] & \text{if } m > 2g \\ \cdot\ St\, c(\pi') = St\, c(c(\pi)) \oplus [2g+1-m] & m \leq 2g \end{cases}$$

see what happened for $m = 8$?

a) we first constructed $\pi \in \Pi_{disc}(SO_8)$ st. $St\, c(c(\pi)) = Sym^2 c(\Pi_\Delta) \oplus [5]$

b) $\mathcal{A}(SO_8) = \mathcal{A}(PGSO_8) \rightsquigarrow$ refine π as a π' $\quad$ (isogeny $SL_2(\mathbb{C}) \to PGL_2(\mathbb{C})$, $= \widehat{PGL_2}$, $= \widehat{Sp_2}$)

st. $St\, c(\pi') = St\, c(\pi)$ but $Spin^{\pm} c(\pi') = c(\Pi_\Delta)[4]$

(slight strengthening of Rallis rel.).

c) apply triality to $\pi' \xrightarrow{\text{get}} \pi''$, $St\, (c(\pi'')) = c(\Pi_\Delta)[4]$

and $\{Spin^{\pm} c(\pi'')\} = \{c(\Pi_\Delta)[4], Sym^2 c(\Pi_\Delta) \oplus [5]\}$.

d) another ϑ-series, $\pi'' \to \pi'''$ on Sp_8 (Schottky form)

<u>get</u> $St(\Pi_{Schottky}) = c(\Pi_\Delta)[4] \oplus [1]$.

e) another ϑ-series $\to \pi^{iv}$ on SO_{16}, $St\, c(\pi) = c(\Pi_\Delta)[4] \oplus (1) \oplus [7]$ $\quad$ ∎

We have seen several instances of the AL conj.

More in next lecture! (with SO_{24})

Last lecture stated A-L. conj: G ss $/\mathbb{Z}$, $\pi \in \Pi_{disc}(G)$, $r : \hat{G} \to SL_n(\mathbb{C})$

there exists unique coll. of triples $(\pi_i, m_i, d_i)_{i=1..k}$, $\pi_i \in \Pi_{cusp}(PGL_{m_i})$

s.t. $r(c(\pi)) = \bigoplus_{i=1}^{k} c(\pi_i)[d_i]$ $\to$ means $c(\pi_i) \otimes \mathrm{Sym}^{d_i - 1} e$

Arthur's theorem conj holds for $G = Sp_{2g}/\mathbb{Z}$, $G =$ split $SO_*/\mathbb{Z}$

and $r = St =$ standard rep.

Goal today what are the $St\, c(\pi)$ of the π's in $\Pi_{disc}(SO_{24})$ with $\pi_\infty = \mathbb{C}$? slight refinement of original pb, X_n replaced with

$\tilde{X}_n = \mathcal{L}_n / SO_n(\mathbb{R})$ "proper iso. classes". $\tilde{X}_n \to X_n$ bij. $n \leq 16$, $\#\tilde{X}_{24} = 25$

because Leech 2 orientations.

main curiosity what are the π_i's supposed to occur in them?

need first to say more about possibilities.

8 Algebraic selfdual rep. of PGL_m

2 observations (1) If $r : \hat{G} \to SL_n(\mathbb{C})$ is selfdual, then

$r\, c(\pi) = r(c(\pi))^{-1} = \bigoplus_{i=1}^{k} c(\pi_i)^{-1}[d_i]$ where $\pi_i^\vee$ contragredient of π_i

"inverse class" $c(\pi_i)^{-1} = c(\pi_i^\vee)$

uniqueness prop $\Rightarrow$ $\pi_i = \pi_i^\vee$ (selfduality)

[concretely, in the picture $\mathcal{A}(PGL_m) = L^2(\{\text{lattices in } \mathbb{R}^m\}/\text{hom})$ the selfdual π's are the ones stable by invol. $g \mapsto {}^t g^{-1}$ or $f(L) \to f(L^\#)$]

some perfect pairings on $\mathbb{Z}^m$

(2) Assume $c_\infty(\pi)$ is the inf. char. of a finite dim. rep. of $G(\mathbb{C})$ (e.g. discrete series, cohom. rep. ...)

then $c_\infty(\pi)$ easy to compute ($\sim \lambda + \rho$) & we see that

λ = dom. wt.

the e.v of $\mathfrak{r}(C_\infty(\pi))$ are in $\frac{1}{2}\mathbb{Z}$, congruent mod $\mathbb{Z}$.

II

this property is inherited ~~from~~ by the π_i's !

<u>Definition</u> $\pi \in \Pi_{\mathrm{cusp}}(PGl_m)$ is called <u>algebraic</u> if the ev. $\lambda_1 \dots \lambda_m$ of $C_\infty(\pi)$ are in $\frac{1}{2}\mathbb{Z}$ and $\lambda_i - \lambda_j \in \mathbb{Z}$ $\forall i,j$.

The λ_i's are called the weights of π, $\lambda_{max} - \lambda_{min} =: w(\pi) \geq 0$ $\in \mathbb{Z}$ called the motivic weight of π

<u>Facts</u> i) if $\lambda_1 \geq \dots \geq \lambda_m$ then $\lambda_i + \lambda_{m+1-i} = 0$ (Clozel purity lemma)
(so $w(\pi) = 2\lambda_1$).

ii) $n=2$, $\lambda_1 \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$ (see eg. class, avoid trivial) $= \frac{k-1}{2}$ $k \geq 0$ even
and π_∞ the disc. series wt. k. ($\pi \leftrightarrow$ normalized eigenform for $SL_2(\mathbb{Z})$)

iii) important relation with Gal. rep. (Langlands - Fontaine - Mazur)

fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, $\exists$ bij $\{\pi \text{ alg}\} \longleftrightarrow \{$ irred. cont. $\rho : G_{\mathbb{Q}} \to Gl_m(\overline{\mathbb{Q}}_\ell)$, unram outside ℓ, crist at ℓ, HT = the $\lambda_i + \frac{w(\pi)}{2}$

$\pi \longleftrightarrow \rho_\pi$

λ_i $\quad \det(X - c(\pi_p)\, p^{\frac{w(\pi)}{2}}) \stackrel{i}{=} \det(X - \rho_\pi(\mathrm{Frob}_p))$ $\quad p \neq \ell$

existence of ρ_π known in many cases, e.g π <u>regular</u> ($\neq$ weights)

$\Rightarrow$ two imp things
i) Ramanujan known for reg. selfdual π's (<u>alg</u>)
ii) get for free from AL. conj existence of ass. Gal. rep. in many cases.

Important problem: can we classify those π's ? (even reg. selfdual ones ?)

In recent works of Ch-Lannes, Ch-Renard, Taïbi, interesting new results on this pb: see later.

(9) $G = PGSp_4 \simeq SO_{3,2}$ (split $/\mathbb{Z}$) : an example (III)

let $w > v > 0$ odd integers

$\exists!$ hol dis_crete series $U_{w,v}$ of $G(\mathbb{R})$ with ∞ char c $Sp_4(\mathbb{C})$ with e.v $\pm w/2$, $\pm v/2$.

fact: $A_{U_{w,v}}(G) \xrightarrow{\sim} S_v(Sp_4(\mathbb{Z}))$ = space of genus 2 Siegel mod forms (cusp.) for $Sp_4(\mathbb{Z})$, vector coeff. $\mathrm{Sym}^{\frac{v-1}{2}} \mathbb{C}^2 \otimes \det^{2+\frac{w-?}{2}}$

its dim. has been computed by R. Tsushima, we observe on table that for $w \leq 21$, it is always 0 or 1., $1 \Leftrightarrow (w,v)$ is in foll. list

$(17,1)$ $(19,7)$ $(21,1)$ $(21,5)$ $(21,9)$ $(21,13)$

in each case, $\exists!$ $\pi_{w,v} \in \Pi_{disc}(PGSp_4)$ with $(\pi_{w,v})_\infty \simeq U_{w,v}$.

Question what is $St\, c(\pi_{w,v}) \in \mathcal{X}(SL_4)$ (exists by Arthur)

only possibilities: $c(\pi_1)$, $c(\pi_1) \oplus 1[2]$, $c(\pi_1) \oplus c(\pi_2)$, ~~[4]~~ (even dim 2 ...), $c(\pi_1)[2]$ (odd m. wt.)

but $S_k(SL_2(\mathbb{Z})) = 0$ for $k = 2, 6, 8, 10, 14$!

so if $v \neq 1$ then $St\, c(\pi_{w,v}) = c(\Delta_{w,v})$, $\Delta_{w,v} \in \Pi_{cusp}(PGL_4)$

$v = 1$, might be $c(\Delta_w) \oplus [2]$ where $\Delta_w \Leftrightarrow$ unique $S_{w+1}(SL_2(\mathbb{Z}))$

and it is indeed ... long story (Saito-Kurokawa cusp form, beginning of "e")

in any case: we found 4 new cusp form PGL_4!

we can go further and analyse AMF here $\rightsquigarrow$ explains $\uparrow$ + gives exact # of $\pi \in \Pi_{cusp}(PGL_4)$ of weights $\pm w/2$, $\pm v/2$.

Similar story can be played each time you can compute dim. formula & understand AMF. With D. Renard, found explicit form of AMF all cases of interest here. Compute dim - very hard pb historically, using exc. isogenies, deal with $SO_{2,2}$, $SO_{2,1}$, Sp_4, SL_2, $SO_{3,2}$ and form. for $SL_2(\mathbb{Z})$ and Tsushima's. Ch-Renard: non split SO_8, SO_7 get inter. rslts but cond. Recent breakthrough of Taïbi on this pb.

$\rightsquigarrow$ Thm explicit formulas (4 tables) for # of self. alg. π of PGL_m (IV)
with weights $\lambda_1 > \lambda_2 > \dots > \lambda_m$ (regular), for $m \leq 14$.
results are still cond. (but expected) if 2 λ_i's are consecutive
(i.e $\lambda_i - \lambda_{i+1} = 1$).

see our table!

(10) Main theorem

recall $\rightsquigarrow$ char of trivial of $SO_{24}(\mathbb{R})$ is $\sim$ diag $(\pm 1, \pm 2, \dots, \pm 11, 0$ twice$)$

Fact: there are exactly 24 elements $c \in \mathcal{X}(SL_{24})$ st:

i) c_∞ has e.v. $\pm 1, \pm 2, \dots, \pm 11$, 0 twice

ii) c is a sum of $c(\pi)[d]$ and π is in the foll. list
$\{1, \Delta_{11}, \Delta_{15}, \Delta_{17}, \Delta_{19}, \Delta_{21}, \mathrm{Sym}^2\Delta_{11}, \Delta_{19,7}, \Delta_{21,5}, \Delta_{21,9}$ and $\Delta_{21,13}\}$

(show list!)

Thm: those 24 elements are exactly the St $c(\pi)$ where $\pi \in \Pi_{disc}(SO_{24})$
and $\pi_\infty \simeq \mathbb{C}$; with $c(\Delta_{11})[12]$ occuring twice

Several remarks a) 20/24 had already been found by ingenious constructions
by T. Ikeda, namely the ones not containing the $\Delta_{w,v}$. His work
is based on works Ibukiyama, Borcherds-Freitag-Weissauer, Böcherer,
Duke-Imamoglu conj + sol. Miyawaki conj.

Nebe-Venkov + his sol. of Nebe-Venkov: part of this set of work is to understand ker $\vartheta^{(g)}$ filtration on $\mathbb{C}[X_{24}]$,
analogue of Witt's conj. to Niem. lattices. A few things were missing
to have complete picture (Nebe-Venkov's conj.), they are proved
by thm. We even show (much harder)

$1 \leq g \leq 12$ $\quad \vartheta^{(g)}: \mathbb{C}[X_{24}] \longrightarrow M_{12}(Sp_{2g}(\mathbb{Z}))$ surjective, iso for $g = 12$.

Erokhin, $\vartheta^{(12)}$ injective, BFW $\vartheta^{(11)}$ kernel dim 1,
$\rightsquigarrow$ existence of Arthur parameters for the π's. (SO_{24} not split!)

b) $C_p(\Delta_{w,v})$ unknown a priori. We found a way to compute them using direct analysis of $N_p(R^+, \text{Leech})$ + Ramanujan bounds.
$\Rightarrow$ could compute $\mathrm{tr}\, C_p(\Delta_{w,v})$ up to $p = 113$! We obtain as corollaries the results on the graphs $X_{24}(p)$ stated in lect. 1.
Our comp. confirm work of Faber & v.d. Geer, computed differently $\mathrm{tr}\, C_p(\Delta_{w,v})$ for $p \leq 37$. (using enumeration of genus 2 curves!)

d) Once we know AMF (if you admit it is proved), case-by-case check shows mult. $\neq 0$, 1 or 2 $\Leftarrow$ only $c(\Delta_{11})[12]$, conclude by $\# X_{24}^2 = 25$! Unfortunately cond. proof (but the best one!)
We argue differently.

(11) Some ingredients

Theorem Let $\pi \in \Pi_{cusp}(PGL_m)$ be algebraic of motivic weight ≤ 21 (but m arbitrary). Then either $m \leq 2$ or π is one of the 4 $\Delta_{w,v}$ found in (9).

Rk: $w(\pi) < 11$, then $\pi = 1$. This was known to Serre-Mestre (compare with Minkowski or Fontaine, $w(\pi) \in \{0,1\}$). still finite $w(\pi) \leq 23$.

- Let us see first how to apply this!
Let $\pi \in \Pi_{disc}(Sp_8)$ ass. to Siegel mod. cusp form wt $\underline{8}$ (e.g Schottky form)
what can $\mathrm{St}\, c(\pi)$ be? inf char: ev. $\pm 7, \pm 6, \pm 5, \pm 4, 0$
write $\mathrm{St}\, c(\pi) = \bigoplus_{i=1}^{k} c(\pi_i)[d_i]$, observe $w(\pi_i) \leq 14$
$\Rightarrow \pi_i \in \{1, \Delta_{11}\}$. Only possibility $c(\Delta_{11})[4] \oplus [1]$!
get first thm (and $\dim S_8(Sp_8(\mathbb{Z})) \leq 1$!)

VI

- we do the same with Siegel cusp forms wt. 12. Unfortunately, a lot of extra compl. : need $w(\pi)=22$, need to eliminate several possible π's (I believe unique π of mot. weight 22, $\mathrm{Sym}^2 \Delta_{11}$)

What about pf of Theorem?

first. ingredient novel use of Riemann-Weil explicit formula, in the spirit of works of Odlyzko, Mestre, Fermigier & Miller.

Assuming π exists, try to limit possibilities for π_∞, using $L(s, \pi \times \pi^\vee)$. Works for $w(\pi) \leq 17$ perfectly.

For higher $w(\pi)$, get only regular possibilities.
A nice trick of Taïbi allow to see they are selfdual
Use (deep lists). In proof of thm for Niemeier lattices we don't need Taïbi's difficult computations though. $\blacksquare$