On level 1 cuspidal algebraic aut. rep. of $GL_m / \mathbb{Q}$

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(I) Consider cuspidal aut. rep. π of $GL_m / \mathbb{Q}$ s.t.

(i) π_p unramified for all primes p ← level 1

(ii) π_∞ algebraic, i.e. with integral weights $k_1 \leq \dots \leq k_m$

↳ Galois rep.

Ex: $m=1$, $\pi = |\cdot|^{k_1}$, $m=2$, $k_1 = 0$ then $k_2 > 0$ and $\{\pi\} \Leftrightarrow$ eigenfcns in $S_{k_2+1}(SL_2(\mathbb{Z}))$

Harish-Chandra: $\forall m$, only finitely many for given $k_1, \dots k_m$

Pb: unknown number for $m=3$ already

A few motivations

Pb 1 Is there $m>1$ s.t. $H^*_{cusp}(SL_m(\mathbb{Z}), \mathbb{Q}) \neq 0$? Miller $m \geq 27$

$\Longleftrightarrow$ existence? of π with $(k_1, \ldots, k_n) = (0, 1, 2, \ldots, m-1)$

Pb 2 Can we guess autom. expr. for $\zeta(s, \overline{\mathcal{M}}_{g,n})$?

$H^*_{ét}(\overline{\mathcal{M}_{g,n}}, \mathbb{Q}_\ell)$ is an ℓ-adic Galois rep. unram outside ℓ & crist. at ℓ.

its irred. summands $\underset{\text{FM. Langlands}}{\overset{\text{conj}}{\rightsquigarrow}}$ π

<u>Pb 3</u> For G a reductive group $/\mathbb{Z}$, can we determine the dim of spaces of level 1 automorphic forms for G of given "integral" weight?

<u>Ex:</u>

dimension of spaces of Siegel cuspforms $S_{28}(\mathbb{Z})$?

number of isom. classes of even unim. lattices in $\mathbb{R}^{32}$?

Conjecturally (known for classical group) those forms are built from such π's on GL_m (Langlands, Arthur)

$m \geq 1$

II two results

thm 1 There are only finitely many level 1 algebraic c.a.r. π of GL_m ($\forall m \geq 1$) with weights $\subset \{0, \dots, 23\}$

rmk: holds for any Artin conductor ; $23 \to 24$ under GRH

thm 2 (Ch-Lannes) there are exactly 11 such π with $\begin{matrix} k_1 = 0 \\ k_m \leq 22 \end{matrix}$

1, Δ_w generated by $S_{w+1}(SL_2(\mathbb{Z}))$ $w = 11, 15, 17, 19, 21$, $Sym^2 \Delta_{11}$, $\left.\begin{matrix} \Delta_{19,7}, \Delta_{21,5}, \\ \Delta_{21,9} \text{ \& } \Delta_{21,13} \end{matrix}\right\} m=4$.

<u>Rmks</u> (i) $n \leq 4$ in all cases, selfdual regular

(ii) for $k_n = 0$ asserts "no Artin level 1 aut. rep",
aut. version of Hermite-Minkowski theorem

(iii) $k_n = 1$ fits "no $\neq 0$ ab. variety $/\mathbb{Z}$" of Fontaine

(iv) with <u>Taïbi</u>: partial classification for $k_n = 23$ & 24.

<u>OPEN Pb</u>: prove Galois theoretic analogue of <u>thm 2</u>.

III Idea of proof

In the lead of many works (Serre, Stark, Odlyzko, Mestre, Fermigier, Miller...) to show a π with given weights cannot exist, we show that some $L(s, \pi \times \pi')$ do not exist.

↖ existing π'

In previous works, $\pi' = 1$ or $\pi' = \pi^\vee$ used, but we discovered with Lannes new developement using all π', simplified then in another work with Taïbi.

Set $W_{\mathbb{R}}^{alg} = W_{\mathbb{R}} / \mathbb{R}_{>0}$, $1 \to S^1 \to W_{\mathbb{R}}^{alg} \to \mathbb{Z}/2 \to 1$

(compact group)

its Grothendieck ring is $K_\infty := \underbrace{\mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z} I_1 \oplus \mathbb{Z} I_2 \oplus \cdots \oplus \mathbb{Z} I_w}_{K_\infty^{\leq w}} \oplus \cdots$

$W_{\mathbb{R}}^{alg}$

with $I_w = \operatorname{ind}_{S^1} x \mapsto x^w$

<u>Closed purity lemma</u>: if π alg, $k_1 = 0$, $\pi \rightsquigarrow \pi |\,|^{-\frac{k_m}{2}}$ unitary norm.

<u>then</u> $L(\pi_\infty) \in K_\infty^{\leq k_m}$

We define a collection of q. forms on K_∞

$F :=$ "test function" $\mathbb{R} \to \mathbb{R}$ even, C^2 & compactly supported

$$q^F_\infty : K_\infty \to \mathbb{R}, \quad V \longmapsto -\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2} + 2i\pi t, V \otimes V\right) \hat{F}(t)\, dt$$

$\downarrow$

a real valued quad. form on K_∞

here $\Gamma(s, v)$

standard recipe for Γ. factors

extra assumption on F

(POS) $F(t) \geq 0 \quad \forall t \in \mathbb{R}$

$\mathrm{Re}\, \hat{F}(s) \geq 0, \quad \lim s \leq \frac{1}{4\pi}$

Set $K = \bigoplus_{\pi} \mathbb{R}\pi$, sum over cusp. alg. norm. π of GL_m $\forall m$,

Euclidean space for $\pi \cdot \pi' = \delta_{\pi,\pi'}$.

Natural lin. map $K \longrightarrow K_\infty \otimes \mathbb{R}$, L-param.

Prop (Ch. Taïbi) "Explicit inequality"

$\forall F$ satisfying (POS) $\quad g^F_\infty(L(x)) \le \widehat{F}\left(\frac{i}{4\pi}\right) x \cdot x$

$\forall x \in K$ effective

Prop for suitable F, q_∞^F pos. definite on $K_\infty^{\leq 23}$

$\uparrow$ 24 under GRH

Pf: either explicit computation using Odlyzko's functions

more conceptual arg $F \to 1$, say under GRH, $1 + \frac{1}{2} + \dots + \frac{1}{n} < \log 8\pi + \gamma$

$\iff n \leq 24$.

$\rightsquigarrow$ for $k_1, \dots, k_n \leq 23$ get explicit finite list of possibilities for π_∞ (hence finiteness). We exclude them essentially case by case.

Ex: show no π level 1 cuspidal $GL_8/\mathbb{Q}$

with $L(\pi_\infty) = I_4 + I_{14} + I_{20} + I_{22}$, otherwise

$$x = 0.70\,\pi + 0.30\,\mathbb{1} + 0.65\,\Delta_{21}$$

contradicts ineq. for certain F

Other applications: prove vanishing results for $\geq 10^4$ regular weights.

IV The selfdual regular case

Thm (Ch-Taïbi) For $n \leq 24$, there is an explicit formula (implemented) for number of level 1 resp. aut. π of $GL_n/\mathbb{Q}$ with $\pi^\vee \simeq \pi \cdot |\,|^*$ & weights $k_1 < k_2 < \dots < k_n$.

→ Extend previous works of Ch-Renard $n \leq 8$ ($n \neq 7$), Taïbi $n \leq 15$

Cor solve Pb 3 for $G = Sp_{2g}$, $SO(p,q)$
& discrete series $\quad 2g \leq 22 \quad$ with $p+q \leq 25$

(V) The "effortless method": when we create everything from emptyness

$G_{/\mathbb{Z}}$ is either Sp_{2g} or split SO_n with $n \not\equiv 2\ (4)$

$\rightsquigarrow$ $G(\mathbb{R})$ has discrete series

let V_λ be f.-dim. rep. of $G(\mathbb{C})$, irred highest weight λ

Arthur "L^2-Lefschetz" trace formula:

$$EP\left(\mathcal{A}^2(G)^{\widehat{G(\mathbb{Z})}} \otimes \check{V}_\lambda\right) = T_{geom}(G,\lambda)$$

$$\overset{!!}{=}\ T_{spec}(G;\lambda)$$

$\in \mathbb{Z}$

$$T_{geom}(G;\lambda) = T_{ell}(G;\lambda) + \text{other terms}$$

$\shortparallel$

function of $T_{ell}(\text{Levi}, \lambda')$

$$T_{ell}(G;\lambda) = \sum_{\gamma} v_{\gamma}\, O_{\gamma}(\mathbb{1}_{G(\hat{\mathbb{Z}})})\, \mathrm{tr}(\gamma \mid V_{\lambda})$$

↖ conj. class of finite order el. in $G(\mathbb{Q})$

we just write $$= \sum_{c} m_c \,\mathrm{tr}(c \mid V_{\lambda})$$

↖ "mass of c", a constant

↗

$G(\overline{\mathbb{Q}})$ - conj class of ...

masses for SO_n

n	3	4	5	7	8	9	11	12	13	15	16	17
#	3	6	12	34	40	99	244	211	598	1339	992	2948

masses for Sp_{2g}

$2g$	2	4	6	8	10	12	14	16
#	3	12	32	92	219	530	1157	2521

to understand $T_{spec}(G;\lambda)$ use Arthur's endo. classification + works of Aranalia-Moeglin-Renard. Say $G = SO_n$ n odd

$$T_{spec}(G;\lambda) = ct \times \# \left\{ \begin{array}{l} \text{cusp. aut. rep. of } GL_{n-1} \\ \text{selfdual symplectic with} \\ \text{weight } k_\lambda \end{array} \right\}$$

explicit

+ endoscopic contributions.

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Assume there is no cusp aut. rep. of weight k_+.

$\Longrightarrow$ the spectral side is purely endoscopic,

by induction we can compute it. hence $T_{ell}(G,\lambda)$.

$\longrightarrow$ a linear relation between masses of G.

If enough λ + luck $\Rightarrow$ inverse the linear system

$\Rightarrow$ get all masses $\Rightarrow$ get # cusp. rep wt k_J $\forall J$!

$\forall \lambda$.

works for Sp_{2g}, $2g \leq 14$, SO_m, $m \leq 15$.

THANKS!