On the dimension of spaces of Siegel cuspforms for $\mathrm{Sp}_{2g}(\mathbb{Z})$

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Dimension formulas

Let $g \geq 1$ and set $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$.

Define $\mathrm{S}_k(\Gamma_g)$ and $\mathrm{S}_{\underline{k}}(\Gamma_g)$ respectively as the spaces of cuspidal Siegel modular forms for Γ_g which are either scalar-valued of weight $k \in \mathbb{Z}$, or more generally vector-valued of weight $\underline{k} = (k_1, k_2, \ldots, k_g)$ in $\mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g$.

Classical problem : Determine $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ (formula ?).

Only general constraints: $\mathrm{S}_{\underline{k}}(\Gamma_g) = 0$ unless $\sum_i k_i \equiv 0 \bmod 2$ (easy) and $k_g \geq g/2$ (Freitag, Reznikoff, Weissauer).

Known results : $g = 1$ and $g = 2$

For $g = 1$, classical modular forms for $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$: well known.

Assume $g = 2$, so $k_1 + k_2$ is even and $k_2 \geq 1$.

Formula in the scalar-valued case due to Igusa (1962) and in the vector-valued case by Tsushima (1984) for $k_2 \geq 5$.

Tsushima's formula also holds for $k_2 \geq 3$, as was conjectured by Ibukiyama, and proved later by Petersen and Taïbi (2015).

Known results : $g = 2$ (continued) and $g = 3$

We have $\mathrm{S}_{\underline{k}}(\Gamma_2) = 0$ for $k_2 = 1$ (Ibukiyama, Skoruppa), but $\dim \mathrm{S}_{\underline{k}}(\Gamma_2)$ still unknown for $k_2 = 2$!

(Known to vanish for all $k_1 \leq 54$ by recent results of Clery, van de Geer and Ch.-Taïbi.)

Many other results known for $g \leq 2$ with higher level that I don't mention here !

For $g = 3$, formula in the scalar-valued case due to Tsuyumine (1984), only quite recently a conjectural formula proposed for $k_3 \geq 4$ by J. Bergström, C. Faber & G. van der Geer (2011).

Taïbi's thesis (2015)

Building on work of Ch.-Renard, Taïbi gives *loc. cit.* an explicit formula for $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ for g arbitrary in the case $k_g > g$.

His formula contains some unknown terms, namely certain orbital integrals at torsion elements of split classical groups over $\mathbb{Q}_p$.

Taïbi developed several case-by-case algorithms to compute those terms efficiently with the help of the computer. He was able to evaluate all of them for $g \leq 7$.

Conclusion: Given any $g \leq 7$ and any $\underline{k}$ with $k_g > g$ and k_1 not too big, the computer and Taïbi's implementation returns $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ in a few seconds.

This proved BFvdG's conjecture in particular, and much more.

Goal today

Goal: Explain a variant of Taïbi's method which reproves his results in a simpler and comparatively "effortless" way : no direct orbital integrals calculation. (Joint-work with Taïbi on arXiv)

Combining both methods, get also a formula in the case $g = 8$.

http://gaetan.chenevier.perso.math.cnrs.fr/levelone/

$\longrightarrow$ tables for $k_1 \leq 16$ and Taïbi's sage scripts allowing computations for general $\underline{k}$ (with $k_g > g$ and $g \leq 8$).

Remarks: (a) Other results in [Ch.-Taïbi] include a computation of $\dim \mathrm{S}_k(\Gamma_g)$ (scalar-valued case) for all $g \geq 1$ in the case $k \leq 13$. I'll show list if time permits.

(b) We do not use any previous computation of dimension of spaces of modular forms, and in the end we seem to recover all known $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ (including works of Witt, Poor-Yuen, Nebe-Venkov, Borcherds-Freitag-Weissauer...)

The three main ingredients

1. Arthur's endoscopic classification specified to the level 1 algebraic cuspforms of all split classical groups over $\mathbb{Z}$, namely Sp_{2g} or split SO_n over $\mathbb{Z}$, which are discrete series at the Archimedean place.
2. The "L^2-Lefschetz" version of Arthur's trace formula.
3. Non-existence results of certain level 1 "algebraic" cuspforms on GL_n (see later).

Let me start with an instructive *baby case* where (1) plays no role.

A (too complicated) way to determine $\dim \mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$

First basic tool, a trace formula.

Trace formula with **simplest geometric side** = the one of Arthur's 1989 paper L^2-*Lefschetz numbers of Hecke operators*.
Drawback : simplified but still complicated spectral side.

I want to describe this trace formula for any split semisimple group scheme G over $\mathbb{Z}$ and the trivial Hecke operator (giving a "dimension").

In this section, I take $G = \mathrm{PGL}_2 \simeq \mathrm{SO}_3$ and fix $k \geq 2$ even, but also prepare for the general case.

The test function

Let $f = \otimes'_v f_v$ be a smooth c.s. function on $G(\mathbb{A})$, and $dg = \prod_v dg_v$ a Haar measure on $G(\mathbb{A})$, such that :

(a) $f_p = 1_{G(\mathbb{Z}_p)}$ and $\mathrm{vol}(G(\mathbb{Z}_p), dg_p) = 1$,

(b) $f_\infty(g_\infty)dg_\infty$ = a signed *pseudocoefficient* for the discrete series representation D_k of weight k for $G(\mathbb{R})$ $(= \mathrm{PGL}_2(\mathbb{R}))$.

Meaning : if U is any *tempered* unitary irrep. of $G(\mathbb{R})$, then

$$\mathrm{trace}(f_\infty(g_\infty)dg_\infty \,|\, U) = \begin{cases} (-1)^{\frac{1}{2}\dim G(\mathbb{R})/K_\infty} = -1 & \text{if } U \simeq \mathrm{D}_k \\ 0 & \text{otherwise} \end{cases}$$

Pseudocoefficients of discrete series exists in general (Clozel-Delorme). Elementary for $\mathrm{PGL}_2(\mathbb{R})$ (Harish-Chandra, Duflo-Labesse).

An important warning

If U is a **non tempered** unitary irrep. of $G(\mathbb{R})$, we may have $\mathrm{trace}(f_\infty(g_\infty)dg_\infty \,|\, U) \neq 0$.

For $G = \mathrm{PGL}_2$, only happens for $k = 2$ and $\dim U = 1$ (trivial or sign) by Bargmann's classification.

Explanation : $f_\infty(g_\infty)dg_\infty$ has trace 0 in any full principal series, and there is a principal series of $\mathrm{PGL}_2(\mathbb{R})$ which is an extension of D_k by the finite dimensional rep. $V_k := \mathrm{Sym}^{k-2}\,\mathbb{C}^2 \otimes \det^{1-k/2}$, so

$$-\mathrm{trace}(f_\infty(g_\infty)dg_\infty \,|\, \mathrm{D}_k) = \mathrm{trace}(f_\infty(g_\infty)dg_\infty \,|\, V_k) = 1.$$

Of course, V_k is unitary only for $k = 2$.

Spectral side of the trace formula

Define $\mathcal{A}^2(G)$ as the space of automorphic forms in the space $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (*square integrable automorphic forms*) and set:

$$\mathrm{T}_{\mathrm{spec}}(G;k) \overset{def}{=} \mathrm{trace}(f(g)dg \,|\, \mathcal{A}^2(G)).$$

Essentially by definition and the above remarks we have

$$\mathrm{T}_{\mathrm{spec}}(\mathrm{PGL}_2;k) = -\dim \mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z})) + \delta_{k,2}$$

(sign does not globally contribute in level 1, by strong approximation and $\mathrm{sign}(\mathrm{PGL}_2(\mathbb{Z})) = \{\pm 1\}$).

Geometric side

Arthur's paper gives another formula for $\mathrm{T}_{\mathrm{spec}}(G;k)$, which also depends only on k, and is denoted $\mathrm{T}_{\mathrm{geom}}(G;k)$.

$$(\mathrm{ATF}): \quad \mathrm{T}_{\mathrm{spec}}(G;k) = \mathrm{T}_{\mathrm{geom}}(G;k).$$

– For general G, $\mathrm{T}_{\mathrm{geom}}(G;k)$ would be a finite sum (of sums) indexed by certain classes of Levi subgroups L of G. Most important term, associated to $L = G$ itself, is called $\mathrm{T}_{\mathrm{ell}}(G;k)$.

– For PGL_2, unique other Levi is $\mathbb{G}_m$ and we can show for all k

$$\mathrm{T}_{\mathrm{geom}}(\mathrm{PGL}_2;k) = \mathrm{T}_{\mathrm{ell}}(\mathrm{PGL}_2;k) + 1/2$$

It remains to explain the *elliptic term* $\mathrm{T}_{\mathrm{ell}}(G;k)$.

Elliptic terms

$$\mathrm{T}_{\mathrm{ell}}(G;k) \stackrel{def}{=} \sum_{\gamma} \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}), dg_\gamma) \cdot \mathrm{O}_\gamma(1_{\mathrm{G}(\widehat{\mathbb{Z}})}\frac{dg}{dg_\gamma}) \cdot \mathrm{trace}(\gamma \,|\, \mathrm{V}_k),$$

where γ runs over the $G(\mathbb{Q})$-conjugacy classes semisimple elements of $G(\mathbb{Q})$ whose $G(\mathbb{Q}_p)$-conjugacy class meets $G(\mathbb{Z}_p)$ for each prime p, and with γ_∞ compact (or better, $\mathbb{R}$-elliptic).

Recall $G \simeq \mathrm{SO}_3$: any such γ has a (degree 3) char. poly. which is a product of cyclotomic polynomials (Kronecker). In particular, **any contributing γ has finite order**.

Remark: rational ss. conjugacy classes are more complicated for classical groups over $\mathbb{Q}$ than for GL_n : infinitely many different classes can have the same char. poly. Nevertheless, only finitely classes contribute non trivially to the sum above.

The masses of G

Each term in $\mathrm{T}_{\mathrm{ell}}(G;k)$ could be computed easily for $G = \mathrm{PGL}_2$, but **painful** when G is replaced by Sp_{2g} or SO_n with high g or n: see Taïbi's thesis for algorithms and numerical applications in small rank. **We choose not to do so and simply write**

$$\mathrm{T}_{\mathrm{ell}}(G;k) = \sum_{c \in \mathrm{C}(G)} m_c \ \mathrm{trace}(c|V_k),$$

where $\mathrm{C}(G)$ is the set of $G(\overline{\mathbb{Q}})$-conjugacy classes of finite order elements in $G(\mathbb{Q})$ (this is possible!). Equivalent to give c in $\mathrm{C}(G)$ and its char. poly. (a product of cyclo. pol.).

Definition : Call m_c the **mass** of the element c of $\mathrm{C}(G)$.

They are absolute constant, i.e. do not depend on k. We can show $m_c \in \mathbb{Q}$ for all c.

$\mathrm{C}(\mathrm{PGL}_2)$

There are 5 possible char. poly.

$$\phi_1^3,\ \phi_1\phi_2^2,\ \phi_3\phi_1,\ \phi_4\phi_1,\ \phi_6\phi_1,$$

hence at most 5 classes, say c_d for $d = 1, 2, 3, 4, 6$ with respective order d. Moreover, for $d > 1$ we have

$$\mathrm{trace}(\, c_d \mid V_k) \;=\; \sin(k\pi/d)/\sin(\pi/d)$$

(must be in $\mathbb{Z}$ for the d above.)

Last key ingredient

Fact: We have $\dim \mathrm{S}_k(\Gamma_1) = 0$ for $k = 2, 4, 6, 8, 10$.

Assume this fact for the moment. The trace formula for those k leads to the linear system :

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & 1 & 2 \\ 5 & 1 & -1 & -1 & 1 \\ 7 & -1 & 1 & -1 & -1 \\ 9 & 1 & 0 & 1 & -2 \end{bmatrix} . \begin{bmatrix} m_{c_1} \\ m_{c_2} \\ m_{c_3} \\ m_{c_4} \\ m_{c_6} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} .$$

Luckily, the matrix on the left-hand side is invertible: we find $m_{c_1} = -\frac{1}{12}$, $m_{c_2} = \frac{1}{4}$, $m_{c_3} = \frac{1}{3}$ and $m_{c_4} = m_{c_6} = 0$.

Consequence: Recover the classical formula for $\dim \mathrm{S}_k(\Gamma_1)$ (for all k), just by proving a few modular forms do not exist.

Remark: Simple explanation for $m_{c_4} = m_{c_6} = 0$ (exercise!).

Proof of the fact, following Mestre

Use an L-function argument first observed by J. F. Mestre in 1986, in the lead of works of Stark, Odlyzko and Serre on discriminant lower bounds for number fields.

Assume $\mathrm{S}_k(\Gamma_1)$ is nonzero : it contains a nonzero Hecke eigenform $f = \sum_{n \geq 1} a_n q^n$. Let

$$\Lambda(s, f) = \Gamma_{\mathbb{C}}(s + (k-1)/2)\mathrm{L}(s + (k-1)/2, f)$$

be its "completed" Hecke L-function. This is an entire function, BVS, with an Euler product and $\Lambda(s, f) = i^k \Lambda(1-s, f)$.

Main idea : show that there is no such function for $k < 12$ by applying the so-called *explicit formula* to $\frac{\Lambda'}{\Lambda}$.

The "explicit formula" following Weil, Poitou and Mestre

Result of a (limit of) contour integration $\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Lambda'}{\Lambda}(s)\Phi_F(s)ds$ for a suitable *test function* F. *(Draw* $\mathcal{C}$*)*.

For us, $F:\mathbb{R}\to\mathbb{R}$ is any even, compactly supported, function of class $\mathcal{C}^2$, and define Φ_F (an entire complex function) by

$$\Phi_F(s)=\int_{\mathbb{R}}F(t)e^{(s-1/2)t}dt=\widehat{F}(\frac{1/2-s}{2i\pi}).$$

Set $L'/L(s)=\sum_{p^k}b_{p^k}p^{-ks}$. Using Cauchy's residue theorem + functional equation + Euler product for $\mathrm{Re}s>1$ + some horizontal estimates, get for each test function F:

$$\int_{\mathbb{R}}\frac{\Gamma_{\mathbb{C}}'}{\Gamma_{\mathbb{C}}}(k/2+2i\pi t)\widehat{F}(t)\mathrm{d}t+\sum_{p^k}b_{p^k}\frac{\log p}{p^{k/2}}F(\log p^k)$$

$$=\frac{1}{2}\sum_{0\le\mathrm{Re}\,\rho\le1}\mathrm{Re}\,\Phi_F(\rho)\quad\mathrm{ord}_{s=\rho}\Lambda(s)$$

Basic inequality

Assume $F \geq 0$, $\mathrm{Re}\,\Phi_F(s) \geq 0$ for $0 \leq \mathrm{Re}\,s \leq 1$, and F vanishes outside $[-\log 2, \log 2]$. For each such F we get the (surprisingly sharp in practice) *basic inequality*:

$$(BI): \quad \int_{\mathbb{R}} \frac{\Gamma'_{\mathbb{C}}}{\Gamma_{\mathbb{C}}}(k/2 + 2i\pi t)\widehat{F}(t)\mathrm{d}t \geq 0.$$

Functions used in practice : recall Odlyzko's function $u(x) =$ twice square convolution of $\cos(\pi x)1_{|x|\leq 1/2}$. Then

$$F_\lambda(x) = u(x/\lambda)/\cosh(x/2)$$

satisfies the 2 positivity assumptions, with support in $[-\lambda, \lambda]$.

Numerical application : for $F = F_{\log 2}$, LHS of (BI) is increasing when k grows : it is $\simeq -0.07$ for $k = 10$ and $\simeq 0.06$ for $k = 12$. $\square$

Higher dimensional variants

Very general method : applies to arbitrary L-functions satisfying suitable analytic properties such as the standard L-functions of cuspidal automorphic representations of GL_m.

As observed by Serre and Miller in the past, even more powerful when applied to the Rankin-Selberg L-function : as the b_{p^k} are ≤ 0, we may use F_λ with arbitrary λ.

Experience shows that trivial looking inequalities such as (BI) are miraculously accurate in small weights and conductor.

Industrial applications: With Lannes and Taïbi, we have used this method (with important improvements that I will ignore here) to prove *the inexistence of several thousands of automorphic eigenforms for* $\mathrm{GL}_m(\mathbb{Z})$ *with say* $m \leq 17$ *and specific Archimedean components (or* Γ*-factors).*

Selfdual level 1 algebraic cusp forms on GL_m

Consider cuspidal automorphic rep's. π of GL_m over $\mathbb{Q}$ such that :

(i) (selfdual) $\pi^\vee \simeq \pi$,

(ii) (level 1) π_p is unramified for each prime p,

(iii) (algebraic) the infinitesimal character $\inf \pi_\infty \subset \mathrm{M}_m(\mathbb{C})$ has eigenvalues $w_1 \geq w_2 \geq \cdots \geq w_m$ with $w_i - w_j \in \mathbb{Z}$ and $w_i \in \frac{1}{2}\mathbb{Z}$ (called the **weights** of π) .

Counting problem : Determine the number $\mathrm{N}_m(w_1, \ldots, w_m)$ of π of weights $w_1, \ldots, w_m$ (finite by Harish-Chandra)

Under (iii) we expect (and actually know) the existence of associated m-dimensional ℓ-adic Galois representations to $\pi|.|^{w_1/2}$.

A few simple properties

1. π has trivial central character (and $\pi = 1$ for $m = 1$).
2. As $\pi_\infty \simeq \pi_\infty^\vee$ we have $w_{m+1-i} = -w_i$ for each i.
3. Archimedean Jacquet-Shalika estimates imply temperedness $\mathrm{L}(\pi_\infty)_{|\mathbb{C}^\times} \simeq \oplus_{i=1}^{m} (z/\overline{z})^{w_i}$ (*Clozel's purity lemma*). So we essentially now $\mathrm{L}(\pi_\infty)$ from knowledge of the w_i.
4. For $k > 0$ even we have $\mathrm{N}_2(\frac{k-1}{2}, -\frac{k-1}{2}) = \dim \mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$.

Definition: Say π is **regular** if $\mathrm{L}(\pi_\infty)$ is multiplicity free. ($\Leftrightarrow$ the w_i are distinct, except possibly two weights 0 for $m \equiv 0 \bmod 4$.)

Fact: A regular π is orthogonal iff its weights are in $\mathbb{Z}$.

Back to the explicit formula methods

Using the explicit formula method, we prove the following key:

Proposition : (Ch.-Taïbi) For several thousands of explicit regular $w = (w_i)_{1 \le i \le m}$ and $m \le 17$ we have $\mathrm{N}_m(w) = 0$.

Remark: The explicit formula method gives at best concrete upper bounds on $\mathrm{N}_m(w)$, but never allows to prove lower bounds.

Review of Arthur's theory for Siegel modular forms I

Assume $F \in \mathrm{S}_{\underline{k}}(\Gamma_g)$ is a cuspidal Hecke eigenform.

Let π be the cusp. aut. representation of $\mathrm{Sp}_{2g}(\mathbb{A})$ generated by F.

– $\pi_p^{\mathrm{Sp}_{2g}(\mathbb{Z}_p)} \neq 0$ for each prime p.

– $\pi_\infty \simeq \mathrm{D}_{\underline{k}}$ (lowest/highest weight module).

Simple but important fact: the $2g+1$ eigenvalues of the infinitesimal character of $\mathrm{D}_{\underline{k}}$ are 0 and the $\pm(k_i - i)$, $i = 1, \ldots, g$.

They are distinct for $k_g > g$, *i.e.* when $\mathrm{D}_{\underline{k}}$ is (hol.) discrete series.

Review of Arthur's theory for Siegel modular forms II

Let $\psi = \oplus_{j=1}^{s} \pi_j[d_j]$ the global Arthur parameter of π. Then:

(a) ψ_p is unramified for each prime p (i.e. each π_j has level 1).

(b) ψ_∞ has the same inf. character as π_∞.

Definitely assume $k_g > g$. Assertion (b) has two consequences :

– (**weights condition**) π_j is algebraic regular for each j and

$$\{0, \pm(k_i - i)\, i = 1, \dots, g\} = \{w + a\}$$

with $w \in \mathrm{Weights}(\pi_j)$ and $a \in \frac{1}{2}\mathbb{Z}$ s.t. $|a| \leq d_j$ and $a \equiv d_j \bmod \mathbb{Z}$.
– ψ_∞ is an Adams-Johnson parameter, i.e. $\Pi(\psi_\infty)$ is an Adams-Johnson packet (AMR).

Review of Arthur's theory for Siegel modular forms III

Most important case: $s = 1$ and $d_1 = 1$, i.e. $\psi = \varpi$ with ϖ a level 1, orthogonal, cusp. aut. rep. of GL_{2g+1} with reg. weights

$$w_{\underline{k}} = (k_1 - 1, k_2 - 2, \dots, k_g - g, 0, \dots)$$

(Trivial) special case of (AMF) : Conversely, any level 1, selfdual orthogonal, algebraic regular ϖ appears this way, for a unique F up to scalars.

If $s > 1$ or $d_1 > 1$, the form F is usually called *endoscopic*.

Review of Arthur's theory for Siegel modular forms IV

In general, there is a unique $j_0 \in \{1, \ldots, s\}$ such that π_{j_0} has odd dimension (i.e the weight 0).

Further observations (Ch.-Renard, AMR):

1. We have $d_{j_0} = 1$, otherwise $\Pi(\psi_\infty)$ does not contain π_∞.
2. $\langle -, \mathrm{D}_{\underline{k}} \rangle$ is **always** $\epsilon_2 + \epsilon_4 + \epsilon_6 + \cdots + \epsilon_{2[g/2]}$.

$\longrightarrow$ Allows to find all further restrictions on the weights of the π_j by applying (AMF) (parity, relative ordering).

No other constraints : conversely, using (AMF) we are thus able to determine all possible endoscopic contributions ("lifts"). See Ch.-Lannes for list of concrete formulas.

Conclusion : (Key Fact A) In order to determine $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$, enough to know $\mathrm{N}_m(w)$ for all $m \leq 2g+1$ and $w_1 \leq k_1 - 1$.

Statement of main theorems

Main Theorem with Taïbi: (i) Computation of all masses for Sp_{2g} with $g \leq 8$ and for split SO_n with $n \leq 17$.

(ii) "Concrete" and implemented formulas for $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ for $g \leq 8$ and $k_g > g$, including contributions of all possible endoscopic lifts.

(iii) "Concrete" and implemented formulas for $\mathrm{N}_m(w)$ for any $m \leq 16$ and regular w.

See webpage for many table.

Inductive proof : even if we are interested only in Sp_{2g}, we are forced to consider as well all $\mathrm{Sp}_{2g'}$ with $g' < g$ and all split $\mathrm{SO}_{n'}$ with $n' < 2g+1$.

Back to trace formula for Sp_{2g}

Fix $\underline{k} = (k_1, k_2, ..., k_g) \in \mathbb{Z}^g$ with $k_g > g$.

Let $\Pi_{\underline{k}}$ be the set of 2^g discrete series of $\mathrm{Sp}_{2g}(\mathbb{R})$ with same inf. character as $\mathrm{D}_{\underline{k}}$ (discrete series L-packet).

We apply Arthur's formula to $G = \mathrm{Sp}_{2g}$ and test function $f(g)dg$:

– same $f_p(g_p)dg_p$ as before,

– to get a formula with a nice geom. side Arthur is forced to choose for $f_\infty(g_\infty)dg_\infty$ the sum of "the" pseudocoefficients of all the elements of $\Pi_{\underline{k}}$ (with signs $(-1)^{\frac{g(g+1)}{2}}$).

$f_\infty(g_\infty)dg_\infty$ is an Euler-Poincaré function

Set $V_{\underline{k}}$ = finite dim. irrep. of $\mathrm{Sp}_{2g}(\mathbb{C})$ with same inf. char. as $\mathrm{D}_{\underline{k}}$.

Clozel-Delorme: For any irr. unitary rep. of $G(\mathbb{R})$ we have

$$\mathrm{trace}(f_\infty(g_\infty)dg_\infty|U) = \sum_{i\geq 0}(-1)^i \dim \mathrm{H}^i(\mathfrak{g}, K; U\otimes V_{\underline{k}}^\vee) =: \mathrm{EP}(U, \underline{k}).$$

– Only depends on $\underline{k}$.

– Only regular cohomological representations with same inf. char. as $\mathrm{D}_{\underline{k}}$ contribute (discrete series & many nontempered in gen.).

Spectral side

Still $\mathrm{T}_{\mathrm{spec}}(G;\underline{k}) = \mathrm{trace}(f(g)dg \,|\, \mathcal{A}^2(G))$. We have thus

$$\mathrm{T}_{\mathrm{spec}}(G;\underline{k}) = \mathrm{EP}(\mathcal{A}^2(G),\underline{k}) \in \mathbb{Z}$$

Fairly complicated alternating sum and much work needed to understand it. In much the same way I explained $\mathrm{S}_{\underline{k}}(\Gamma_g)$ may be reconstructed from selfdual alg. regular level 1 algebraic π's, Arthur's endoscopic classification (using (AMF) and AMR) imply:

Key fact B: $\mathrm{T}_{\mathrm{spec}}(G;\underline{k}) = 2^g(-1)^{g(g+1)/2}\mathrm{N}_{2g+1}(w_{\underline{k}})$ + an explicit function of the $\mathrm{N}_m(w)$ for $w_1 \leq k_1 - 1$ and $m < 2g+1$.

See Taïbi's AENS paper for the precise recipe.

Geometric side

Arthur's trace formula still takes the form:

$$\mathrm{T}_{\mathrm{spec}}(G;\underline{k}) = \mathrm{T}_{\mathrm{geom}}(G;\underline{k}) = \mathrm{T}_{\mathrm{ell}}(G;\underline{k}) + \mathrm{T}_{\mathrm{nonell}}(G;\underline{k})$$

where:

– $\mathrm{T}_{\mathrm{ell}}(G,\underline{k})$ is defined exactly as before : just replace k by $\underline{k}$.

– $\mathrm{T}_{\mathrm{nonell}}(G;\underline{k})$ may be explicitly deduced from the $\mathrm{T}_{\mathrm{ell}}(L;\underline{k}')$ for the so-called cuspidal Levi subgroups L of G (there are products of GL_1, GL_2 (close to PGL_2), and of $\mathrm{Sp}_{2g'}$ with $g' < g$). Very concrete general formulas for these non elliptic terms are given by Taïbi.

The masses

Still as before we write :

$$\mathrm{T}_{\mathrm{ell}}(G;k) = \sum_{c \in \mathrm{C}(G)} m_c \ \mathrm{trace}(c|V_{\underline{k}}),$$

with the unknown masses $\mathrm{m}_c \in \mathbb{Q}$. Rather easy to determine $\mathrm{C}(G)$, say mod $x \mapsto -x$ (degree $2g$ products of cycl. pol.).

G	Sp_2	Sp_4	Sp_6	Sp_8	Sp_{10}	Sp_{12}	Sp_{14}	Sp_{16}
$\lvert \mathrm{C}(G)/\sim \rvert$	3	12	32	92	219	530	1157	2521

Last argument

Assume we know the masses of $\mathrm{Sp}_{2g'}$ for $g' < g$, and of $\mathrm{SO}_{n'}$ for $n' < 2g+1$.

Induction: We know the $\mathrm{T}_{\mathrm{geom}}(G', \underline{k}')$ for those G', hence the $\mathrm{T}_{\mathrm{spec}}(G', \underline{k}')$ as well by trace formula. By Key Fact B we have a formula for $\mathrm{N}_m(w)$ for all $m \leq 2g$ and regular w.

Assume we also know $\mathrm{N}_{2g+1}(w_0) = 0$ for some regular w_0, e.g. using explicit formula. Write $w_0 = w_{\underline{k}}$. We deduce $\mathrm{T}_{\mathrm{spec}}(G; \underline{k})$ hence **get a linear relation among the masses** m_c.

Miracle: We proved enough vanishing results to get enough relations this way up to $g = 7$ to invert the linear system !

This being done, we get all masses m_c, hence $\mathrm{N}_{2g+1}(w)$ for all w (by Key Fact B), hence formulas for $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ (by Key fact A) and all endoscopic contributions (by AMF).

For $g = 8$, not enough relations, but compute enough "easy m_c" by Taïbi's method. $\square$