$\mathrm{Spin}(7)$ IS UNACCEPTABLE

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ABSTRACT. We classify the pairs of group morphisms $\Gamma \to \mathrm{Spin}(7)$ which are element conjugate but not globally conjugate. As an application, we study the case where Γ is the Weil group of p-adic local field, which is relevant to the recent approach to the local Langlands correspondence for G_2 and PGSp_6 in [GS22]. As a second application, we improve some result in [KS] about GSpin_7-valued Galois representations.

1. INTRODUCTION

Let G be a compact group and let Γ be an arbitrary group. A group morphism $r : \Gamma \to G$ is called *unacceptable* if there is a morphism $r' : \Gamma \to G$ such that:

(U1) for all $\gamma \in \Gamma$, $r(\gamma)$ and $r'(\gamma)$ are conjugate in G,

(U2) r' is not G-conjugate to r,

Otherwise, we say that r is *acceptable*. If r is unacceptable, then so is any G-conjugate of r. Following Larsen [LAR94] we also say that G is *acceptable* if every G-valued group morphism is acceptable. If r and r' satisfy condition (U1) we also say that they are *element conjugate* (in G). Since the characters of (finite dimensional, continuous) representations of G separate its conjugacy classes, condition (U1) is also equivalent to:

(U1)' for all representations $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, the representations $\rho \circ r$ and $\rho \circ r'$ of Γ are isomorphic.

Beyond the group theoretic legitimacy of these notions, a motivation for their study comes from the theory of automorphic representations. Indeed, it was observed long ago by Langlands that, for a given reductive group H over a number field, the unacceptability of (the compact form of) the L-group of H creates serious local-global difficulties in the study of automorphic representations of H, and in particular non multiplicity one phenomena (see *e.g.* [BLA94] for the first instance of such a phenomenon). In a similar vein, acceptability questions typically arise when one tries to characterize a representation of the absolute Galois group of

The first author is supported by the C.N.R.S. and by the project ANR-19-CE40-0015-02 (COLOSS). The second author is partially supported by a Singapore government MOE Tier 1 grant R-146-000-320-114. The authors thank Michael Larsen and Jun Yu for some useful discussions concerning their articles [LAR94] and [YU21].

$\mathbb{Q}$ with values in a reductive group by its Frobenius conjugacy classes. We also mention that these notions have applications to constructions of isospectral, non isometric, Riemannian manifolds (see the introduction in [Lar94]).

It is a folklore result that for each integer $n \geq 1$, the classical compact groups $\mathrm{U}(n)$, as well as $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$, are all acceptable [Lar94]. It follows that $\mathrm{SU}(n)$ and that $\mathrm{SO}(2n+1)$ are acceptable as well. Also, it is not difficult to show that $\mathrm{SO}(4)$ is acceptable, and results of Griess in [Gri95] also imply that G_2 is acceptable. It follows from these facts and standard exceptional isomorphisms that $\mathrm{Spin}(n)$ is acceptable for $n \leq 6$. Our first result is that this fails for $n = 7$.

Proposition 1.1. *The group* $\mathrm{Spin}(n)$ *is not acceptable for* $n \geq 7$

A classification of the compact connected acceptable Lie groups was initiated by Larsen in [Lar94] and [Lar96], and more recently pursued by J. Yu [Yu21]. For the little story, both [Lar94] and the first version of [Yu21] contained two different incorrect "proofs" that Spin(7) is acceptable! Some problem in Larsen's proof was discovered by the second author in 2017, after he wrote a manuscript on a local Langlands correspondence for PGSp_6 over p-adic fields, and in which the acceptability of Spin(7) played a crucial role.[1] A counterexample was then found by the first author. Our aim in this paper is not only to explain this counterexample, by partially reproducing a letter we sent some time ago to Larsen and Yu [CG18], but also to give a classification of the unacceptable Spin(7)-valued morphisms, with in view some applications to the local Langlands correspondence for G_2 and PGSp_6 [GS22]. We end by stating a remarkable result proved by Yu in [Yu21]: a compact connected Lie group is acceptable if, and only if, its derived subgroup is a direct product of the aforementioned acceptable groups. Also, we mention a study in [Wan15] of the unacceptable continuous morphisms $\Gamma \to \mathrm{SO}(2n)$ with Γ compact and connected.

A first general result we prove in Section 3, which holds for all odd n, is the following. We denote by E the standard representation of $\mathrm{Spin}(n)$, an n-dimensional *real* representation, and by S a Spin representation, a 2^k-dimensional *complex* representation with $n = 2k+1$. If $r : \Gamma \to \mathrm{Spin}(n)$ is a given morphism, it allows to view E as an $\mathbb{R}[\Gamma]$-module, and S as a $\mathbb{C}[\Gamma]$-module.

Theorem 1.2. *A morphism* $r : \Gamma \to \mathrm{Spin}(n)$*, with* n *odd, is unacceptable if, and only if, there is an order* 2 *character* $\eta : \Gamma \to \{\pm 1\}$ *such that:*

(i) *we have* $S \simeq S \otimes \eta$ *as* $\mathbb{C}[\Gamma]$*-modules,*
(ii) *no* $\mathbb{R}[\Gamma]$*-submodule of* E *has determinant* η*.*

Of course, if Γ has no order two character, then any morphism $\Gamma \to \mathrm{Spin}(n)$ is acceptable (an easier fact). Although the theorem above is interesting, it does not explain the shape of the unacceptable morphisms (nor why they only exist for

[1] The content of that manuscript just appeared as the Appendix C of the preprint [GS22].

$n \geq 7$). Our second type of results are thus more specific to the case $n = 7$. A subgroup of Spin(7) is called a Spin(1, 6)-*subgroup* if it is obtained as the inverse image, via the canonical map Spin(7) $\to$ SO(7), of the stabilizer of a line in $\mathbb{R}^7$. All these subgroups are conjugate in Spin(7) and are semi-direct products of $\mathbb{Z}/2\mathbb{Z}$ by Spin(6) $\simeq$ SU(4); however, they are not isomorphic to Pin(6). The main result of Section 4 is:

Theorem 1.3. *Assume* $r : \Gamma \to \mathrm{Spin}(7)$ *is unacceptable. Then its image* $r(\Gamma)$ *is contained in a* Spin(1, 6)*-subgroup of* Spin(7)*, or equivalently, the* $\mathbb{R}[\Gamma]$*-module* E *contains a stable line.*

We stress that the necessary condition above is by no mean sufficient for being unacceptable: the inclusion of a Spin(1, 6)-subgroup in Spin(7) is acceptable. Nevertheless, if $r : \Gamma \to \mathrm{Spin}(7)$ is unacceptable then E contains some character $\chi : \Gamma \to \{\pm 1\}$. We say that r is *of type* I if we may take $\chi = 1$. Otherwise, we let $\Gamma_0 \subset \Gamma$ be the kernel of χ and say that r is *of type* II (with respect to χ) if $r_{|\Gamma_0}$ is unacceptable, and *of type* III otherwise (the precise definitions are slightly more constraining: see Definitions 6.1 and 7.2.) The main goal of the remaining parts of the paper is to give a classification of the unacceptable morphisms of each type. Essentially, we first give in each case some examples and then show that they are universal in some precise sense. Three specific compact subgroups of Spin(7), that we denote by $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{I}$ in the paper, and whose given embeddings into Spin(7) are all unacceptable, play an important role. They are extensions of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by $\mathrm{SO}(2) \times \mathrm{SO}(2)$, $\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ and SO(4) respectively.

In Section 5, we prove that up to conjugacy, any unacceptable morphism of type I factors through $\mathcal{G}$ (Theorem 5.8). Conversely, we give two necessary and sufficient conditions for a morphism into $\mathcal{G}$ to give rise to an unacceptable morphism into Spin(7) (Propositions 5.10 and 5.13). In Section 6, we reduce the study of type II morphisms to that of type I ones. Then we prove in Section 7 that up to conjugacy, any unacceptable morphism of type III factors through either $\mathcal{H}$ or $\mathcal{I}$ (see Theorems 7.4 and 7.7). The two corresponding situations are called type IIIa and IIIb respectively. A few lemmas needed in the proofs are gathered in the Appendix. A simple corollary of these results is:

Corollary 1.4. *Assume* $r : \Gamma \to \mathrm{Spin}(7)$ *is unacceptable. Then up to conjugating* r *if necessary, the image of* r *is either included in* $\mathcal{G}$*,* $\mathcal{H}$ *or* $\mathcal{I}$*, or has an index 2 subgroup included in* $\mathcal{G}$*.*

We leave as an open problem the question of classifying the unacceptable Spin(n)-valued morphisms for $n > 7$.

In Section 8 we finally study the special case where Γ is the Weil group W_F of a finite extension F of $\mathbb{Q}_p$. The general question, of inverse Galois theory flavour, is to understand to what extend the variety of general examples discussed above does occur for W_F. For instance, as we shall see, type I unacceptable morphisms do always exist. We also discuss the more restrictive case of *discrete* and *stable* morphisms, which are quite meaningful from the point of view of the local Langlands correspondence. We show that there is no type I discrete unacceptable morphisms, nor type II ones for $p > 2$. Moreover, although type III discrete unacceptable morphisms turn out to always exist, we show that there is no stable unacceptable discrete morphism for p odd. All these results show that, for several natural families of Langlands parameters for $\mathrm{PGSp}_6(F)$, the *weak* equivalence class appearing in the local Langlands correspondence in [GS22, App. C, Thm. 12.6] coincides with the familiar, and stronger, one (given by conjugacy by the dual group).

In the final Sections 9 and 10, we explain how our results can also be applied to study the acceptability of $\mathrm{GSpin}(n)$-valued morphisms, and as an example, we give an application to the acceptability of certain GSpin_7-valued ℓ-adic Galois representations which improves some result in [KS].

Remark 1.5. (A general remark on topology) *In the study of unacceptable morphisms, we would not lose much in restricting to injective and continuous morphisms $\Gamma \to G$ from compact groups Γ. Indeed, assume $r_1, r_2 : \Gamma \to G$ are two element conjugate morphisms, with G a compact group and Γ an arbitrary group. Consider the morphism $r_1 \times r_2 : \Gamma \to G \times G$. Up to replacing Γ by its image under $r_1 \times r_2$, we may assume $\Gamma \subset G \times G$ and that r_1 and r_2 are the two natural projections. Define Γ' as the (compact) closure of Γ in $G \times G$, and $r'_1, r'_2 : \Gamma' \to G$ as the two (continuous) projections. As $\{(g, hgh^{-1}) \,|\, g, h \in G\}$ is closed in $G \times G$, the morphisms r'_1 and r'_2 are element conjugate; they are conjugate in G if, and only if, r_1 and r_2 are. In particular, for $(\gamma_1, \gamma_2) \in \Gamma'$ we have $\gamma_1 = 1$ if, and only if, $\gamma_2 = 1$, as γ_1 and γ_2 are conjugate, so r_1 and r_2 are both injective.*

Contents

2. General notations on Spin groups

Let $n \geq 1$ be an integer and E the standard Euclidean space $\mathbb{R}^n$ with inner product denoted by $x.y$ for $x, y \in E$. We denote by $\mathrm{O}(n) = \mathrm{O}(E)$ the orthogonal group of E, by $\mathrm{SO}(n) = \mathrm{SO}(E)$ its special orthogonal group, and by $\mathrm{Cl}(E)$ the Clifford algebra of E. We have a natural inclusion $E \subset \mathrm{Cl}(E)$. For $e \in E$ with $e.e = 1$, we have $e^2 = 1$ in $\mathrm{Cl}(E)$ and the conjugation by e in $\mathrm{Cl}(E)$ preserves the subspace E and induces the opposite of the Euclidean reflection of E about e. Our convention is that $\mathrm{Pin}(n) = \mathrm{Pin}(E)$ is the subgroup of $\mathrm{Cl}(E)^\times$ generated by the elements $e \in E$ with $e.e = 1$. The $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathrm{Cl}(E)$ defines a group morphism $\deg : \mathrm{Pin}(n) \to \mathbb{Z}/2\mathbb{Z}$ sending any such e to $1 \in \mathbb{Z}/2\mathbb{Z}$, and whose kernel is by definition $\mathrm{Spin}(n) = \mathrm{Spin}(E)$. We denote by

$$\pi : \mathrm{Pin}(n) \to \mathrm{O}(n)$$

the group morphism defined for all $\gamma \in \mathrm{Pin}(n)$ and all $v \in E$ by the equality

$$\pi(\gamma)(v) \,=\, (-1)^{\deg(\gamma)}\,\gamma v \gamma^{-1}$$

in $\mathrm{Cl}(E)$. The morphism π is surjective as Euclidean reflections generate $\mathrm{O}(E)$. Its kernel is a central subgroup of order 2, generated by an element of $\mathrm{Spin}(n)$ denoted -1. We have $\det \circ \pi = (-1)^{\deg}$ on $\mathrm{Pin}(n)$. Also, $\mathrm{Pin}(n)$ is a compact subgroup of the invertible elements of the finite dimensional $\mathbb{R}$-algebra $\mathrm{Cl}(E)$ and for this topology the morphism π is continuous.

Assume now n is odd and write $n = a + b$ with a odd and $b \neq 0$ even. The a-dimensional subspaces of $E = \mathbb{R}^n$ form a single orbit under $\mathrm{SO}(E)$. Fix such a subspace $A \subset E$ and set $B = A^\perp$. The stabilizer in $\mathrm{SO}(E)$ of A, hence of B, is

$$\mathrm{S}(\mathrm{O}(A) \times \mathrm{O}(B)) := \{(g,h) \in \mathrm{O}(A) \times \mathrm{O}(B) \mid \det g = \det h\}.$$

It is isomorphic to $\mathrm{SO}(a) \times \mathrm{O}(b)$. We denote by $\mathrm{Spin}(A,B)$ the inverse image of $\mathrm{S}(\mathrm{O}(A) \times \mathrm{O}(B))$ in $\mathrm{Spin}(E)$. As B is nonzero, we have a natural order 2 character $\mathrm{S}(\mathrm{O}(A) \times \mathrm{O}(B)) \to \{\pm 1\}$ sending (g,h) to $\det g = \det h$. Composing this character with $\pi : \mathrm{Spin}(A,B) \to \mathrm{S}(\mathrm{O}(A) \times \mathrm{O}(B))$ defines a character

$$\kappa : \mathrm{Spin}(A,B) \to \{\pm 1\}. \tag{1}$$

The kernel of κ, the inverse image of $\mathrm{SO}(A) \times \mathrm{SO}(B)$ in $\mathrm{Spin}(E)$, coincides with

$$\mathrm{Spin}(A) \cdot \mathrm{Spin}(B) \simeq \mathrm{Spin}(A) \times \mathrm{Spin}(B)/\langle(-1,-1)\rangle.$$

Let e and f be elements of A and B respectively, with $e.e = f.f = 1$. Then $\pi(ef)$ acts by $-\mathrm{id}$ on $\mathbb{R}e \perp \mathbb{R}f$, and by $+\mathrm{id}$ on its orthogonal. So ef is an element of $\mathrm{Spin}(A,B)$ with $\kappa(ef) = -1$. It satisfies $(ef)^2 = efef = -eeff = -1$. As a is odd, the center of $\mathrm{Spin}(A) \cdot \mathrm{Spin}(B)$ is the center of $\mathrm{Spin}(B)$, which has four elements since b is even. This center contains -1, but also the element

$$z_B := f_1 f_2 \cdots f_b \in \mathrm{Spin}(B) \tag{2}$$

where f_i is some orthonormal basis of B, since z_B anti-commutes with any f_i. Note that $\pi(z_B)$ is the element $(\mathrm{id}_A, -\mathrm{id}_B)$ of $\mathrm{S}(\mathrm{O}(A) \times \mathrm{O}(B))$. It follows that $\pm z_B$ does not depend on the choice of the orthonormal basis f_i of B. We also have $z_B^2 = (-1)^{b/2}$ and $z_B\, f \ = \ -\,f\, z_B$ for all $f \in B$.

In the standard case $A = \mathbb{R}^a$, $B = \mathbb{R}^b$, $E = A \perp B = \mathbb{R}^{a+b}$, we simply write $\mathrm{Spin}(a, b)$ for $\mathrm{Spin}(A, B)$. Also, by a $\mathrm{Spin}(a, b)$-*subgroup* of $\mathrm{Spin}(n)$ we mean a subgroup of the form $\mathrm{Spin}(A, B)$ with $\dim A = a$ and $\dim B = b$; they form a single conjugacy class under $\mathrm{Spin}(n)$.

Lemma 2.1. *Assume we have $E = A \perp B$ with $a = \dim A$, $b = \dim B$, and $b > 0$ even. Define $\kappa : \mathrm{Spin}(A, B) \to \{\pm 1\}$ as in Formula (1), and $\pm z_B \in \mathrm{Spin}(A, B)$ by Formula (2). Then for all $g \in \mathrm{Spin}(A, B)$ we have $z_B\, g\, z_B^{-1} \ = \ \kappa(g)\, g$.*

Proof. For $\kappa(g) = 1$, this follows as z_B is in the center of $\ker \kappa = \mathrm{Spin}(A){\cdot}\mathrm{Spin}(B)$. For $g = ef$ with $e \in A$, $f \in B$ and $e.e = f.f = 1$, we have $z_B\, g\, z_B^{-1} \ = \ -g$ since z_B commutes with e (as b is even) and anti-commutes with f (as we have seen). We conclude as ef and $\ker \kappa$ generate $\mathrm{Spin}(A, B)$. ☐

Remark 2.2. *Lemma 2.1 implies that the center of $\mathrm{Spin}(A, B)$ is $\{\pm 1\}$.*

3. The conditions (U1) and (U2) for $\mathrm{Spin}(n)$ with general odd n

In all this section, $n \geq 1$ is an *odd* integer and Γ is a group.

Proposition 3.1. *Assume $r, r' : \Gamma \to \mathrm{Spin}(n)$ are element conjugate (i.e. satisfy (U1)). There is $g \in \mathrm{Spin}(n)$ and a character $\eta : \Gamma \to \{\pm 1\}$ such that for all $\gamma \in \Gamma$ we have $g\, r'(\gamma)\, g^{-1} \ = \ \eta(\gamma)\, r(\gamma)$.*

Proof. As n is odd, we know that the group $\mathrm{SO}(n)$ is acceptable. Since $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is surjective with kernel $\{\pm 1\}$, it follows that there is $g \in \mathrm{Spin}(n)$ such that for all $\gamma \in \Gamma$ there is $\eta(\gamma) \in \{\pm 1\}$ with $g\, r(\gamma)\, g^{-1} \ = \ \eta(\gamma)\, r(\gamma)$. This defines a map $\eta : \Gamma \to \{\pm 1\}$, which is necessarily a group morphism. ☐

Definition 3.2. *Consider group homomorphisms $r : \Gamma \to \mathrm{Spin}(n)$ and $\eta : \Gamma \to \{\pm 1\}$. We say that (r, η) satisfies (U1) (resp. (U2)) if this property holds with $r' := \eta r$. We say that (r, η) is unacceptable if it satisfies both (U1) and (U2).*

Proposition 3.1 asserts that for all unacceptable r there is η such that (r, η) is unacceptable. Of course, if (r, η) is unacceptable then we have $\eta \neq 1$. Another simple property is the following.

Proposition 3.3. *Let r and η be as in Definition 3.2 and assume (r, η) satisfies (U1). Then η is trivial on the kernel of the morphism $\pi \circ r : \Gamma \to \mathrm{SO}(n)$.*

Proof. Assume $\gamma \in \Gamma$ satisfies $\pi(r(\gamma)) = 1$, *i.e.* $r(\gamma) = \pm 1$. By (U1), $r(\gamma)$ is conjugate to $\eta(\gamma)r(\gamma)$ in $\mathrm{Spin}(n)$. As 1 and -1 are not conjugate, this forces $\eta(\gamma) = 1$. $\square$

Definition 3.4. *Let $r : \Gamma \to \mathrm{Spin}(n)$ be a group morphism. We denote by $\mathrm{X}(r)$ the set of group morphisms $\chi : \Gamma \to \{\pm 1\}$ such that there is $g \in \mathrm{Spin}(n)$ satisfying*

$$\forall \gamma \in \Gamma, \;\; g\, r(\gamma)\, g^{-1} = \chi(\gamma) r(\gamma).$$

This set $\mathrm{X}(r)$ is a subgroup of $\mathrm{Hom}(\Gamma, \{\pm 1\})$.

By definition, if (r, η) satisfies (U1), then (r, η) is unacceptable if, and only if, $\eta \notin \mathrm{X}(r)$.

Remark 3.5. *Let us denote by $\mathrm{E}(r)$ the set of morphisms $\eta : \Gamma \to \{\pm 1\}$ such that (r, η) is unacceptable. By Proposition 3.1, r is unacceptable if, and only if, $\mathrm{E}(r)$ is nonempty. By Definition 3.4, if we have $\eta \in \mathrm{E}(r)$ and $\chi \in \mathrm{X}(r)$, then $\eta\chi$ also belongs to $\mathrm{E}(r)$, so that $\mathrm{X}(r)$ acts freely by multiplication on $\mathrm{E}(r)$.*

Our first aim now is to give an alternative description of $\mathrm{X}(r)$. If $r : \Gamma \to \mathrm{Spin}(n)$ is a given morphism, it has a natural n-dimensional real representation $\pi \circ r : \Gamma \to \mathrm{SO}(n)$ on the Euclidean space $E = \mathbb{R}^n$. We shall simply denote by E this $\mathbb{R}[\Gamma]$-module. This is a semi-simple $\mathbb{R}[\Gamma]$-module : for each $\mathbb{R}[\Gamma]$-submodule $V \subset E$, we have the Γ-stable decomposition $E = V \oplus V^\perp$. For any such V, we also denote by $\det_V : \Gamma \to \{\pm 1\}$ its determinant character. We have $\det_E = 1$, hence the equality $\det_V = \det_{V^\perp}$.

Proposition 3.6. *Let $r : \Gamma \to \mathrm{Spin}(n)$ be a group morphism. The subgroup $\mathrm{X}(r)$ of $\mathrm{Hom}(\Gamma, \{\pm 1\})$ is the subset of characters of the form $\det_V$ where V runs among the $\mathbb{R}[\Gamma]$-submodules of E.*

Proof. Assume we have an $\mathbb{R}[\Gamma]$-stable decomposition $E = A \perp B$. Up to exchanging A and B, we may assume $b := \dim B$ is even and > 0, and we set $a = n - b = \dim A$. We have $r(\Gamma) \subset \mathrm{Spin}(A, B)$. The restriction to Γ of the character κ of Formula (1) is $\det_A = \det_B$ by construction. Lemma 2.1 thus shows that $\det_A$ and $\det_B$ are in $\mathrm{X}(r)$.

In order to prove the proposition, it is enough to show that $\mathrm{X}(r)$ is generated by the elements of the form $\det_V$ with $V \subset E$ an $\mathbb{R}[\Gamma]$-submodule. Indeed, this assertion implies first that $\mathrm{X}(r)$ is generated by those $\det_V$ with V irreducible, and then using that these characters have order ≤ 2, we deduce that any element of $\mathrm{X}(r)$ has the form $\det_V$ with $V \subset E$ an $\mathbb{R}[\Gamma]$-submodule (non necessarily irreducible).

Define $\mathrm{C}(r)$ as the centralizer of $\pi(r(\Gamma))$ in $\mathrm{SO}(n)$, and set $\mathrm{D}(r) = \pi^{-1}(\mathrm{C}(r))$. In other words, we have

$$\mathrm{D}(r) = \{g \in \mathrm{Spin}(n) \mid \forall \gamma \in \Gamma, \; g\, r(\gamma)\, g^{-1} \; = \pm\, r(\gamma)\}.$$

This is a closed subgroup of $\mathrm{Spin}(n)$ containing $\{\pm 1\}$, and that sits in the exact sequence $1 \to \{\pm 1\} \to \mathrm{D}(r) \xrightarrow{\pi} \mathrm{C}(r) \to 1$. Fix $g \in \mathrm{D}(r)$. For each $\gamma \in \Gamma$, we have a unique sign $\mathrm{e}_g(\gamma) \in \{\pm 1\}$ such that

$$g\, r(\gamma)\, g^{-1} \;=\; \mathrm{e}_g(\gamma) r(\gamma).$$

This formula shows that $\mathrm{e}_g : \Gamma \to \{\pm 1\}, \gamma \mapsto \mathrm{e}_g(\gamma)$, is a character, which clearly belongs to $\mathrm{X}(r)$, and also that $\mathrm{D}(r) \to \mathrm{X}(r), g \mapsto \mathrm{e}_g$, is a group morphism. This latter morphism is surjective. Indeed, for $\chi \in \mathrm{X}(r)$ the element g given by Definition 3.4 lies in $\mathrm{D}(r)$ and satisfies $\mathrm{e}_g = \chi$.

For a fixed $\gamma \in \Gamma$, the morphism $\mathrm{D}(r) \to \{\pm 1\}, g \mapsto \mathrm{e}_g(\gamma)$, is continuous, and trivial on the subgroup $\{\pm 1\}$ of $\mathrm{D}(r)$, so it only depends on the image of g in $\mathrm{C}(r)/\mathrm{C}(r)^0$. Here, $\mathrm{C}(r)^0$ denotes the neutral connected component of $\mathrm{C}(r)$, so that we also have $\mathrm{C}(r)/\mathrm{C}(r)^0 \simeq \pi_0(\mathrm{C}(r))$. As as a consequence, we have proved that the (surjective) morphism $\mathrm{D}(r) \to \mathrm{X}(r), g \mapsto \mathrm{e}_g$, induces a (surjective) morphism

$$\overline{\mathrm{e}} : \mathrm{C}(r)/\mathrm{C}(r)^0 \longrightarrow \mathrm{X}(r),$$

such that for all $g \in \mathrm{D}(r)$ we have $\overline{\mathrm{e}}(\pi(g)) = \mathrm{e}_g$.

For each $\mathbb{R}[\Gamma]$-stable decomposition $E = A \oplus B$ as in the first paragraph of the proof, with $\dim B = b$ even and > 0, we have $r(\Gamma) \subset \mathrm{Spin}(A, B)$ and Lemma 2.1 precisely states that the element z_B *loc. cit.* satisfies

$$z_B \in \mathrm{D}(r) \quad \text{and} \quad \overline{\mathrm{e}}(\pi(z_B)) \;=\; \det{}_B. \tag{3}$$

In order to conclude it remains to show that the $\pi(z_B)$, for B an even dimensional $\mathbb{R}[\Gamma]$-submodule of E, do generate $\pi_0(\mathrm{C}(r))$. Consider for this the isotypical decomposition

$$E = \mathop{\perp}_{i \in I} E_i,$$

of the $\mathbb{R}[\Gamma]$-module E. The centralizer of $\pi \circ r$ in $\mathrm{O}(E)$ (rather than in $\mathrm{SO}(E)$) is the direct product of the centralizer C_i of $(\pi \circ r)_{|E_i}$ in $\mathrm{O}(E_i)$. Write $E_i \simeq U_i^{\oplus n_i}$ with U_i an irreducible $\mathbb{R}[\Gamma]$-module and $n_i \geq 1$. According to Schur's lemma, there are three well-known possibilities: (see *e.g.* [BtD, p. 96])

(a) $\mathrm{End}_{\mathbb{R}[\Gamma]}(U_i) = \mathbb{R}$. In this case, the $\mathbb{C}[\Gamma]$-module $U_i \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible (we say that U_i is *absolutely irreducible*), we have $C_i \simeq \mathrm{O}(n_i)$ and $(\det_{E_i})_{|C_i} = \det^{\dim U_i}$.

(b) $\mathrm{End}_{\mathbb{R}[\Gamma]}(U_i) \simeq \mathbb{C}$ and $C_i \simeq \mathrm{U}(n_i)$ (unitary group).

(c) $\mathrm{End}_{\mathbb{R}[\Gamma]}(U_i) \simeq \mathbb{H}$ and $C_i \simeq \mathrm{Sp}(n_i)$ (compact symplectic group).

For $i \in I$ of type (b) or (c) the group C_i is connected, so we have $C_i \subset \mathrm{SO}(E)$ and $C_i \subset \mathrm{C}(r)^0$. It follows that we have

$$\mathrm{C}(r)/\mathrm{C}(r)^0 \simeq (\mathbb{Z}/2\mathbb{Z})^m$$

where m or $m+1$ is the number of $i \in I$ of type (a). More precisely, choose for each $i \in I$ of type (a), an $\mathbb{R}[\Gamma]$-stable decomposition $E_i = F_i \perp G_i$ with F_i irreducible (so $F_i \simeq U_i$), and consider the element $\sigma_i \in \mathrm{O}(E)$ acting by $-\mathrm{id}$

on F_i and by id on G_i and each E_j with $j \neq i$. This element is in $C_i \setminus C_i^0$, and has determinant $(-1)^{\dim U_i}$. It follows that $\mathrm{C}(r)/\mathrm{C}(r)^0$ is generated by the images of the σ_i with $\dim U_i$ even, and by the images of the $\sigma_i\sigma_j$ with $i \neq j$ and $\dim U_i \equiv \dim U_j \equiv 1 \bmod 2$. Each of these elements has the form $\pi(z_B)$ for $B = F_i$ or $B = F_i \perp F_j$. □

Corollary 3.7. *$\mathrm{X}(r)$ is generated by those $\det_V$ with V irreducible.*

Proof. Immediate from Proposition 3.6. □

Corollary 3.8. *Assume we have morphisms $r : \Gamma \to \mathrm{Spin}(n)$ and $\eta : \Gamma \to \{\pm 1\}$. Then (r, η) satisfies (U2) if, and only if, there is no $\mathbb{R}[\Gamma]$-stable subspace V of E with $\det_V = \eta$.*

Proof. Clear from the definition of $\mathrm{X}(r)$ and Proposition 3.6. □

We will say that a group embedding $\mathrm{Spin}(n) \to \mathrm{Spin}(m)$, with $m \geq n$, is *standard* if the restriction to $\mathrm{Spin}(n)$ of the standard representation of $\mathrm{Spin}(m)$ is isomorphic to $E \oplus 1^{m-n}$ (with 1 the trivial representation).

Corollary 3.9. *Assume $r : \Gamma \to \mathrm{Spin}(n)$ is unacceptable (recall n is odd). Let $m \geq n$ be an integer and $\rho : \mathrm{Spin}(n) \to \mathrm{Spin}(m)$ a standard embedding. Then $\rho \circ r : \Gamma \to \mathrm{Spin}(m)$ is unacceptable.*

Proof. By Proposition 3.1, we may choose η such that (r, η) is unacceptable. Note that have $\rho \circ (\eta r) = \eta\,(\rho \circ r)$. Its thus clear that $(\rho \circ r, \eta)$ satisfies (U1). In order to check that it satisfies (U2), we may of course increase m, hence assume m odd as well. We conclude by Corollary 3.8 applied to (r, η) and $(\rho \circ r, \eta)$. □

We now give a few equivalent conditions for (U1). Recall that we denote by E the standard representation of $\mathrm{Spin}(n)$ (an n-dimensional real representation) and by S a Spin representation (a $2^{(n-1)/2}$-dimensional complex vector space). If $r : \Gamma \to \mathrm{Spin}(n)$ is a given morphism, it allows to view E as an $\mathbb{R}[\Gamma]$-module and S as a $\mathbb{C}[\Gamma]$-module.

Lemma 3.10. *Assume γ is in $\mathrm{Spin}(n)$ with n odd. The following are equivalent:*

(i) *γ is conjugate to $-\gamma$ in $\mathrm{Spin}(n)$,*

(ii) *γ admits the eigenvalue -1 on E,*

(iii) *the trace of γ is 0 on S.*

Proof. We first show the equivalence between (i) and (ii). This could be proved directly, but we rather deduce it from Proposition 3.6. Consider $\Gamma = \mathbb{Z}$ and $r : \Gamma \to \mathrm{Spin}(n)$ sending $m \in \mathbb{Z}$ to γ^m. Define $\eta : \mathbb{Z} \to \{\pm 1\}$ by $\eta(m) = (-1)^m$. Since $\mathbb{Z}$ is generated by 1, assertion (i) holds if and only if we have $\eta \in \mathrm{X}(r)$. By Proposition 3.6 this is equivalent to ask that E has a γ-stable subspace V with $\det_V(\gamma) = -1$, *i.e.* that γ has the eigenvalue -1 in E.

Let us now prove the equivalence between (i) and (iii). Note that the representation ring of the simply connected group $\mathrm{Spin}(n)$ is generated by its fundamental representations (over $\mathbb{C}$), which are S and some others which all factor through $\mathrm{SO}(n)$ (since n is odd). It follows that two elements γ_1 and γ_2 of $\mathrm{Spin}(n)$ are conjugate if, and only if, (a) their image are conjugate in $\mathrm{SO}(n)$, and (b) γ_1 and γ_2 have the same trace in S. We conclude by applying this remark to $\gamma_1 = \gamma$ and $\gamma_2 = -\gamma$, since -1 acts by $-\mathrm{id}$ on S. $\square$

Remark 3.11. *For all n, an element γ in $\mathrm{Spin}(n)$ is conjugate to $-\gamma$ if and only if γ admits the eigenvalues 1 and -1 on E.*

We now give a few equivalent properties to (U1). For $r : \Gamma \to \mathrm{Spin}(n)$, we have already defined the $\mathbb{R}[\Gamma]$-module E with $\dim_{\mathbb{R}} E = n$. We also define the $\mathbb{R}[\Gamma]$-module

$$\Lambda^\sharp E = \oplus_{i=1}^k \Lambda^i E, \quad \text{with} \quad n = 2k+1. \tag{4}$$

The Γ-modules E, S and $\Lambda^\sharp E$ are semi-simple, since they extend to the compact group $\mathrm{Spin}(n)$. Both E and $\Lambda^\sharp E$ are defined over $\mathbb{R}$, and S is defined over $\mathbb{C}$.

Proposition 3.12. *Assume $r : \Gamma \to \mathrm{Spin}(n)$ is a group morphism (recall n is odd) and let $\eta : \Gamma \to \{\pm 1\}$ be a morphism. The following are equivalent:*

(i) *(r, η) satisfies* (U1),

(ii) *for all $\gamma \in \Gamma$, γ has the eigenvalue $\eta(\gamma)$ on E,*

(iii) *there is an isomorphism $\Lambda^\sharp E \simeq \Lambda^\sharp E \otimes \eta$ of $\mathbb{R}[\Gamma]$-modules,*

(iv) *there is an isomorphism $S \simeq \eta \otimes S$ of $\mathbb{C}[\Gamma]$-modules.*

Proof. Since 1 is an eigenvalue of each element of $\mathrm{SO}(n)$ for odd n, the equivalence of (i) and (ii) follows from that of Lemma 3.10. But (ii) is equivalent to $\det(\eta(\gamma) - r(\gamma)) = 0$ for all γ in Γ, or equivalently,

$$\sum_{i=0}^{n} (-1)^i \ \mathrm{trace}(\gamma \,|\, \Lambda^i E) \ \eta(\gamma)^{n-i} = 0 \tag{5}$$

for all γ in Γ. Set $X_+ = \bigoplus_{i \text{ even}} \Lambda^i E$ and $X_- = \bigoplus_{i \text{ odd}} \Lambda^i E$. As an $\mathbb{R}[\mathrm{SO}(n)]$-module we have $E \simeq E^\star$ and $\det E = 1$, so $\Lambda^i E \simeq \Lambda^{n-i} E$ for all i. This shows

$X_+ \simeq X_- \simeq \Lambda^\sharp E$ as n is odd. As $\Lambda^\sharp E$ is semi-simple, (5) is thus equivalent to $\Lambda^\sharp E \simeq \Lambda^\sharp E \otimes \eta$, and we have proved $(ii) \iff (iii)$.

We end by proving $(i) \iff (iv)$. Set $r' = \eta r$. The equivalence (i) $\iff$ (iii) of Lemma 3.10 shows that r and r' are element conjugate if, and only if, S and $\eta \otimes S$ have the same trace on Γ. We conclude by semi-simplicity of the $\mathbb{C}[\Gamma]$-module S. $\square$

Remark 3.13. *Condition (iv) also means that we have* $S \simeq \mathrm{Ind}_{\Gamma'}^{\Gamma} S'$ *for some complex representation* S' *of the index* 2 *subgroup* $\Gamma' = \ker \eta$ *of* Γ.

Of course, Corollary 3.8 and Proposition 3.12 together imply Theorem 1.2 of the introduction.

4. The case $n = 7$

In this section we focus on the case $n = 7$, the first for which there turns out to exist unacceptable morphisms $\Gamma \to \mathrm{Spin}(n)$. Several arguments below follow Larsen's original arguments in [Lar96], and correct the erroneous Lemma 2.3 and Proposition 2.4 *loc. cit.* An important role will be played by the conjugacy class of $\mathrm{Spin}(1,6)$-subgroups of $\mathrm{Spin}(7)$.

Notation: *We fix a decomposition* $\mathbb{R}^7 = L \perp F$ *with* L *a line, and we denote by* N *the associated* $\mathrm{Spin}(1,6)$*-subgroup* $\mathrm{Spin}(L,F)$ *of* $\mathrm{Spin}(7)$.

We have a natural surjective morphism $N \xrightarrow{\pi} \mathrm{S}(\mathrm{O}(L) \times \mathrm{O}(F)) \simeq \mathrm{O}(F)$. The inverse image of $\mathrm{SO}(F) \simeq \mathrm{SO}(6)$ in N is isomorphic to $\mathrm{Spin}(6)$, has index 2 in N, hence coincides with the connected component N^0 of the identity in N. The choice of a unitary half-spin representation of $N^0 \simeq \mathrm{Spin}(6)$ identifies it with $\mathrm{SU}(4)$. We definitely fix such an identification and allow ourselves to write $N^0 = \mathrm{SU}(4)$ accordingly. The precise structure of N is the following.

Lemma 4.1. *The group* N *is the semi-direct product of* $\mathbb{Z}/2\mathbb{Z}$ *by its subgroup* $\mathrm{SU}(4)$ *with respect to an order* 2 *symmetric outer automorphism.*

The meaning of the statement is the following. Recall that for $n \geq 2$ *even*, any order 2 outer automorphism θ of $\mathrm{SU}(n)$ has the form $g \mapsto p\overline{g}p^{-1}$ with $p \in \mathrm{SU}(n)$ satisfying ${}^t p = \pm p$; we say that θ is *symmetric* if p is, and *antisymmetric* otherwise. The subgroup of fixed points in $\mathrm{SU}(n)$ of such a θ is isomorphic to $\mathrm{SO}(n)$ in the symmetric case, and to $\mathrm{Sp}(n/2)$ in the antisymmetric case. Also, in the natural semi-direct product $\mathrm{SU}(n) \rtimes \langle\theta\rangle$, the elements of the form $\vartheta = h\theta$ with $h \in \mathrm{SU}(n)$ and $\vartheta^2 = 1$ (resp. $\vartheta^2 = -1$) induce by conjugation all the order 2 outer automorphisms of $\mathrm{SU}(n)$ with same (resp. opposite) symmetry type as θ. In particular, the lemma above asserts that there is $\vartheta \in N \smallsetminus N^0$ with $\vartheta^2 = 1$ and $\vartheta g \vartheta^{-1} = \overline{g}$ for all $g \in \mathrm{SU}(4)$.

Proof. Write $L = \mathbb{R}e$ and choose an orthonormal basis $f_1, \ldots, f_6$ of F. Consider the element $\tau = ef_1f_2f_3 \in \mathrm{Spin}(7)$, which satisfies $\tau^2 = 1$. We have $\tau \in N \smallsetminus N^0$, as $\pi(\tau)$ acts by -1 on L, and by a symmetry on F with fixed subspace of dimension 3. This shows $N = N^0 \rtimes \langle\tau\rangle$. The conjugation by τ defines an automorphism of N^0; it is not inner since we have $\tau z_F \tau^{-1} = -z_F$ by Lemma 2.1, whereas z_F is central (of order 4) in N^0. The morphism π induces an exact sequence on fixed points $1 \to \{\pm 1\} \to \mathrm{SU}(4)^{\tau=1} \to \mathrm{O}(6)^{\pi(\tau)=1}$. But the description above of $\pi(\tau)$ shows that the neutral component of $\mathrm{O}(6)^{\pi(\tau)=1}$ is $\mathrm{SO}(3) \times \mathrm{SO}(3)$. The only possibility is thus $\mathrm{SU}(4)^{\tau=1} \simeq \mathrm{SO}(4)$, and we are done. □

Remark 4.2. *A similar argument shows that* $\mathrm{Pin}(6)$ *is also a semi-direct product of* $\mathbb{Z}/2\mathbb{Z}$ *by* $\mathrm{SU}(4)$*, but with respect to an antisymmetric outer automorphism. In particular, it is not isomorphic to* $\mathrm{Spin}(1,6)$.[2] *As the embeddings* $\mathrm{Spin}(6) \to \mathrm{Spin}(7)$ *are unique up to* $\mathrm{Spin}(7)$*-conjugacy, and with normalizers the* $\mathrm{Spin}(1,6)$*-subgroups, it follows that the compact group* $\mathrm{Pin}(6)$ *does not embed into* $\mathrm{Spin}(7)$*.*

We will now give a few properties of the N-valued morphisms.

Remark 4.3. *Let G be a semi-direct product of $\mathbb{Z}/2\mathbb{Z}$ by $\mathrm{SU}(m)$ defined by an outer automorphism. Two morphisms $r, r' : \Gamma \to \mathrm{SU}(m)$ are conjugate in G if, and only if, we have $r \simeq r'$ or $r^* \simeq r'$ as m-dimensional representations of Γ.*

Recall that $\kappa : N \to \{\pm 1\}$ denotes the character of N acting on the line $L \subset E$, or equivalently, the determinant of the action of N on $F \subset E$ (see Formula (1)). We have $\ker \kappa = N^0$. The embedding $N \subset \mathrm{Spin}(7)$ has the following nice property.

Proposition 4.4. *Let $r, r' : \Gamma \to N$ be two group morphisms. The following are equivalent:*

(i) *r and r' are conjugate under N,*

(ii) *r and r' are conjugate in $\mathrm{Spin}(7) \supset N$, and we have $\kappa \circ r = \kappa \circ r'$.*

In particular, two elements of N are conjugate if, and only if, they have the same image in N/N^0 and are conjugate in $\mathrm{Spin}(7)$ (case $\Gamma = \mathbb{Z}$).

Proof. We clearly have (i) $\Longrightarrow$ (ii). Assume (ii) holds. In particular, the two 7-dimensional representations $\pi \circ r$ and $\pi \circ r'$ of Γ on E are isomorphic. They both have the same character on the line L, as this character is $\kappa \circ r = \kappa \circ r'$, hence their restriction to $F = L^\perp$ are isomorphic as well, hence $\mathrm{O}(F)$-conjugate by Lemma A.1. Up to replacing r' by an N-conjugate, we may thus assume that we have $\pi \circ r = \pi \circ r'$, *i.e.* $r' = \eta r$ for some morphism $\eta : \Gamma \to \{\pm 1\}$. By definition and (ii), we have then $\eta \in \mathrm{X}(r)$. Consider an irreducible decomposition $F = \oplus_i F_i$ of the $\mathbb{R}[r(\Gamma)]$-submodule of F. Note that the elements $\pm z_F$ are in N, as well

[2]As an example, the compact form of the Langlands dual group of $\mathrm{PU}(4)$ is isomorphic to $\mathrm{Pin}(6)$, and not to $\mathrm{Spin}(1,6)$.

as the $\pm z_{F_i}$ for $\dim F_i$ even, and the $z_L z_{F_i}$ for F_i odd; with respective image in $\mathrm{Hom}(\Gamma, \{\pm 1\})$ the characters $\det_L = \kappa \circ r$, $\det_{F_i}$ and $\det_L \det_{F_i}$, by Formula (3). Since $\mathrm{X}(r)$ is generated by $\det_L$ and the $\det_{F_i}$ by Corollary 3.7, it follows that there is $n \in N$ such that $nrn^{-1} = \eta r$, which proves (i). $\square$

We have an exact sequence $1 \to \{\pm 1\} \to N \to \mathrm{O}(F) \to 1$. As $\{\pm 1\}$ is central in N, and as $\mathrm{O}(F)$ is acceptable, we have the:

Definition-Proposition 4.5. *The statement of Proposition 3.1 also holds with* $\mathrm{Spin}(n)$ *replaced by* N*, with the same proof. Similarly, Definition 3.2 also makes sense for pairs of morphisms* $r : \Gamma \to N$ *and* $\eta : \Gamma \to \{\pm 1\}$.

This definition does not conflict with Definition 3.2, more precisely:

Corollary 4.6. *Let* $r : \Gamma \to N$ *and* $\eta : \Gamma \to \{\pm 1\}$ *be two morphisms, and* $\widetilde{r} : \Gamma \to \mathrm{Spin}(7)$ *the composition of* r *with the inclusion* $N \subset \mathrm{Spin}(7)$*. Then* (r, η) *satisfies* (U1) *(resp.* (U2)*) if and only if* $(\widetilde{r}, \eta)$ *has this property. In particular,* (r, η) *is unacceptable if, and only if,* $(\widetilde{r}, \eta)$ *is unacceptable.*

Proof. If r and $r' = \eta r$ are Spin(7)-conjugate, they are N-conjugate by the proposition, as we have $\kappa \circ r' = \kappa \circ r$ as $-1 \in N^0$. In the special case $\Gamma = \mathbb{Z}$, this also shows that if $\widetilde{r}$ and $\eta \widetilde{r}$ are element conjugate in Spin(7), then r and ηr are element conjugate in N. The reverse implications are trivial. $\square$

Proposition 4.7. *The group* $\mathrm{Spin}(1,6)$ *(hence* N*) is not acceptable.*

We give below a first example of an unacceptable N-valued morphism. We denote by $\mu_d \subset \mathbb{C}^\times$ the subgroup of d-th roots of unity.

Example 1

Set $\Gamma = \mu_4 \times \mu_4$ and denote by a and b the two order 4 characters of Γ defined by the first and second projections respectively. Consider a morphism $r : \Gamma \to \mathrm{SU}(4)$ which, viewed as a complex 4-dimensional representation of Γ, satisfies

$$r \simeq a \oplus ab^2 \oplus b \oplus ba^2.$$

Such a morphism r is unique up to SU(4)-conjugation by Lemma A.1. Set $\eta = b^2$. We have in particular $\eta r \simeq r \otimes \eta \simeq a \oplus ab^2 \oplus b^{-1} \oplus b^{-1}a^2$. As neither b^{-1} nor a^{-1} appears in r, observe that neither ηr, nor its dual, is isomorphic to r. This shows that r and ηr are not N-conjugate, by Lemma 4.1 and Remark 4.3. Nevertheless, r and ηr are element conjugate: for all $g \in \Gamma$, we have $\eta(g)r(g)$ conjugate in SU(4) to either $r(g)$ or $r(g)^{-1}$. Indeed, this is obvious for g in $\ker \eta$. Moreover, over any of the two subgroups $\ker a^2$ and $\ker (ab)^2$ we have $a \oplus ab^2 \simeq a^{-1} \oplus a^{-1}b^2$, so $r(g)^{-1}$ is conjugate to $\eta(g)r(g)$. We conclude as Γ is the union of $\ker a^2$, $\ker b^2$ and $\ker (ab)^2$. $\square$

Corollary 4.8. *For $n \geq 7$, the group* $\mathrm{Spin}(n)$ *is not acceptable.*

Proof. For $n = 7$, this follows from Example 1 and Corollary 4.6. The general case follows then from the case $n = 7$ and Corollary 3.9. □

Remark 4.9. *The representation of $N^0 = \mathrm{SU}(4)$ on $E \otimes \mathbb{C}$ via π is $1 \oplus \Lambda^2 V$ where $V = \mathbb{C}^4$ is its tautological representation. As a consequence, if $r : \Gamma \to \mathrm{SU}(4) \to \mathrm{Spin}(7)$ is as in Example 1, then we have a $\mathbb{C}[\Gamma]$-module isomorphism*

$$E \otimes \mathbb{C} \simeq 1 \oplus a^2b^2 \oplus a^2b^2 \oplus \left(ab \oplus a^{-1}b^{-1}\right) \oplus \left(a^{-1}b \oplus ab^{-1}\right),$$

as well as $\mathrm{X}(r) = \{1, a^2b^2\}$ and $\eta = b^2$.

Before giving more examples, our goal up to the end of this section will be to prove Theorem 1.3 of the introduction, which is a key result of this paper. Our proof below is inspired by Larsen's analysis in [Lar96]. We start with some general facts about Spin(7). Recall that the spin representation of Spin(7) is well-known to be irreducible, real, and the unique 8-dimensional faithful representation of Spin(7). By Lemma A.1, there is thus a unique O(8)-conjugacy class of embeddings $\mathrm{Spin}(7) \to \mathrm{SO}(8)$. As every automorphism of Spin(7) is inner we obtain:

Lemma 4.10. *There are exactly two conjugacy classes of compact subgroups of* SO(8) *which are isomorphic to* Spin(7)*, and these two classes are permuted transitively under* O(8)*-conjugation.*

The center of each Spin(7) in SO(8) is the center $\{\pm 1\}$ of SO(8). We denote by Z the center of Spin(8); we have $Z \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Recall that any *triality* automorphism of Spin(8) permutes transitively the three order 2 subgroups of Z.

Lemma 4.11. *There are three conjugacy classes of compact subgroups of* Spin(8) *which are isomorphic to* Spin(7)*. These three classes are distinguished by their intersection with Z, which can be any of the three order 2 subgroups of Z.*

Proof. We denote by $\mathcal{A}$ the set of subgroups of Spin(8) which are isomorphic to Spin(7), and by $\mathcal{B}$ the set of subgroups of SO(8) which are isomorphic to either Spin(7) or SO(7). Fix also a surjective morphism $\rho : \mathrm{Spin}(8) \to \mathrm{SO}(8)$. For T in $\mathcal{A}$ we have $\rho(T) \simeq \mathrm{Spin}(7)$ if $(\ker \rho) \cap T = \{1\}$ and $\rho(T) \simeq \mathrm{SO}(7)$ otherwise, hence ρ induces a natural map $\rho_* : \mathcal{A} \to \mathcal{B}, T \mapsto \rho(T)$. We claim that ρ_* is bijective.

Indeed, the injectivity is clear over the $T \in \mathcal{A}$ mapped to SO(7), since then we have $T = \rho^{-1}(\rho(T))$, and follows from $\mathrm{Hom}(\mathrm{Spin}(7), \mu_2) = \{1\}$ for the others. For the surjectivity, note that as Spin(7) is simply connected, any $S \in \mathcal{B}$ which is isomorphic to Spin(7) has the form $\rho(T)$ for some (unique) $T \in \mathcal{A}$. To conclude, we recall the easy fact that the subgroups of SO(8) isomorphic to SO(7) are all conjugate, since they are the stabilizers of the norm 1 elements of $\mathbb{R}^8$, and that their inverse image in Spin(8) are isomorphic to Spin(7). So ρ_* is bijective.

By Lemma 4.10 and the previous sentence, there are three SO(8)-conjugacy classes in $\mathcal{B}$. As ρ is surjective, and as ρ_* is bijective and commutes with conjugacy, there are also three Spin(8)-conjugacy classes in $\mathcal{A}$, and a single one with center $\ker \rho$. We conclude as any order 2 subgroup of Z is the kernel of a suitable ρ, by *triality*. $\square$

As is well-known, the exceptional compact group G_2 has a unique non trivial irreducible represesentation of dimension ≤ 8, and it is faithful of dimension 7. By Lemma A.1, it follows that both SO(7) and SO(8) have a unique conjugacy class of compact subgroups isomorphic to G_2. Using that G_2 is simply connected and $\mathrm{Hom}(\mathrm{G}_2, \mu_2) = \{1\}$, it follows that Spin(7) and Spin(8) also have a unique conjugacy class of compact subgroups isomorphic to G_2, and that we shall call the G_2*-subgroups*. The centralizer in Spin(7) of a G_2-subgroup H is thus the center $\{\pm 1\}$ of Spin(7), and since we have $\mathrm{Out}(\mathrm{G}_2) = 1$, the normalizer of H in Spin(7) is $\{\pm 1\} \times H$. The following proposition corrects [Lar96, Prop. 2.4].

Proposition 4.12. *Let S_1 and S_2 be two different subgroups of* SO(8) *both isomorphic to* Spin(7)*. Then $S_1 \cap S_2$ is one of the following subgroups of*[3] $S_1 \simeq \mathrm{Spin}(7)$*:*

(i) *the normalizer of a* G_2*-subgroup,*

(ii) *a* $\mathrm{Spin}(1,6)$*-subgroup,*

(iii) *the identity component of a* $\mathrm{Spin}(1,6)$*-subgroup.*

We are in the first case if, and only if, S_1 and S_2 are not conjugate under SO(8)*.*

Proof. Fix a surjective morphism $\pi : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ and set $\widetilde{S}_i = \pi^{-1} S_i$ for $i = 1, 2$. By the bijectivity of ρ_* in the previous proof, there are unique subgroups $T_i \subset \mathrm{Spin}(8)$ isomorphic to Spin(7) and with $\pi(T_i) = S_i$. Moreover, we have $T_i \cap (\ker \pi) = 1$, $\widetilde{S}_i = ZT_i$, and the groups S_i are conjugate in SO(8) if, and only if, the centers of T_1 and T_2 coincide. By triality, we may choose a morphism $\rho : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ whose kernel is the center of T_1 (so $\rho \neq \pi$). The subgroup $\rho(\widetilde{S}_1) = \pm\rho(T_1)$ of SO(8) is thus isomorphic to $\mathrm{O}(7) = \mathbb{Z}/2 \times \mathrm{SO}(7)$, namely it is the stabilizer of a line L_1 in the Euclidean $\mathbb{R}^8$. There are two cases:

(a) *S_1 and S_2 are not conjugate under* SO(8). We have then $\rho(T_2) \simeq \mathrm{Spin}(7)$ and the subgroup $\rho(\widetilde{S}_2) = \pm\rho(T_2) = \rho(T_2)$ of SO(8) is thus also isomorphic to Spin(7). But the subgroups of Spin(7) isomorphic to G_2 (resp. $\mathbb{Z}/2\mathbb{Z} \times G_2$) are exactly the stabilizers of nonzero vectors (resp. lines) in the spin representation of Spin(7) by [Ada96, Thm. 5.5]. Applying this to $\rho(\widetilde{S}_2)$ we obtain

$$\rho(\widetilde{S}_1 \cap \widetilde{S}_2) = \rho(\widetilde{S}_1) \cap \rho(\widetilde{S}_2) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathrm{G}_2.$$

As G_2 is simply connected and satisfies $\mathrm{Hom}(G_2, \mu_2) = 1$, we obtain $\pi^{-1}(S_1 \cap S_2) = \widetilde{S}_1 \cap \widetilde{S}_2 = Z \times H$ with $H \simeq \mathrm{G}_2$, hence $S_1 \cap S_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathrm{G}_2$. This is case (i).

[3] The choice of the isomorphism $S_1 \simeq \mathrm{Spin}(7)$ does not matter as $\mathrm{Out}(\mathrm{Spin}(7)) = 1$.

(b) S_1 *and* S_2 *are conjugate under* $\mathrm{SO}(8)$. In this case T_1 and T_2 have the same center, namely ker ρ. The subgroup $\rho(\widetilde{S}_2)$ is then isomorphic to $\mathrm{O}(7)$ as well, *i.e.* it is the stabilizer of a line L_2 in $\mathbb{R}^8$. We have $L_1 \neq L_2$ as $S_1 \neq S_2$. Let Q be the Euclidean plane $L_1 \oplus L_2$ in $\mathbb{R}^8$, $U \subset \mathrm{O}(Q)$ the subgroup fixing pointwise L_1 and preserving L_2, and $I \subset \mathrm{SO}(8)$ the subgroup of elements (g,h) of $U \times \mathrm{O}(Q^\perp)$ with $\det g = \det h$. We have $\rho(\widetilde{S}_1) \cap \rho(\widetilde{S}_2) = \{\pm 1\} \times I$ in $\mathrm{SO}(8)$. As U preserves the orthogonal L_3 of L_1 in Q (a "third" line in Q), there are two sub-cases:

(b1) $L_3 = L_2$, $U = \mathrm{SO}(L_1) \times \mathrm{O}(L_2) \simeq \mathbb{Z}/2\mathbb{Z}$ and $I \simeq \mathrm{O}(Q^\perp) \simeq \mathrm{O}(6)$, or

(b2) $L_3 \neq L_2$, $U = 1$ and $I = 1 \times \mathrm{SO}(Q^\perp) \simeq \mathrm{SO}(6)$.

As the subgroup I fixes pointwise L_1 by construction, we have $I \subset \rho(T_1) = \mathrm{SO}(L_1^\perp)$, and thus its inverse image $I' = \rho^{-1}(I)$ in $\mathrm{Spin}(8)$ is included in $T_1 \simeq \mathrm{Spin}(7)$. If $M \subset T_1$ denotes the stabilizer of L_3, then M is a $\mathrm{Spin}(1,6)$-subgroup of T_1, and we have $I' = M$ in case (b1) and $I' = M^0$ in case (b2). We also have

$$\widetilde{S}_1 \cap \widetilde{S}_2 = \rho^{-1}(\{\pm 1\} \times I) = ZI' \ \text{ and } \ S_1 \cap S_2 = \pi(\widetilde{S}_1 \cap \widetilde{S}_2),$$

so $S_1 \cap S_2 = \pi(ZI') = \pm\pi(I') = \pi(I')$ as $\pi(\ker\rho) = \{\pm 1\}$ and $\ker\rho \subset I'$. It follows that π induces isomorphisms $T_1 \xrightarrow{\sim} S_1$ and $I' \xrightarrow{\sim} S_1 \cap S_2$. This shows that assertions (ii) and (iii) of the statement hold respectively in sub-cases (b1) and (b2). $\square$

Proposition 4.13. *Let* $r : \Gamma \to \mathrm{Spin}(7)$ *be a morphism such that* $r(\Gamma)$ *is included in* $\{\pm 1\} \times H$ *with* $H \simeq \mathrm{G}_2$. *Then* r *is acceptable.*

Proof. We may choose η such that (r, η) satisfies (U1) and (U2). As the spin representation of $\mathrm{Spin}(7)$ is real, we may and do view it as an 8-dimensional $\mathbb{R}[\mathrm{Spin}(7)]$-module. As $\mathbb{R}[H]$-modules, the representations S and E of $\mathrm{Spin}(7)$ satisfy $S \simeq E \oplus 1$. As a $\{\pm 1\} \times H$-module, we have thus $S \simeq \eta \otimes E \oplus \eta$, where η denotes the first projection (whose restriction to Γ is indeed the character already denoted by η), and so the same isomorphism holds as $\mathbb{R}[\Gamma]$-modules. But we also have $S \simeq \eta \otimes S$ by (U1) and Proposition 3.12, and thus

$$E \oplus 1 \simeq \eta \otimes E \oplus \eta.$$

As $\eta \neq 1$, and by semi-simplicity, this forces η to appear as an $\mathbb{R}[\Gamma]$-submodule of E, in contradiction with (U2) and Corollary 3.8. $\square$

We are now able to prove Theorem 1.3.

Proof. (of Theorem 1.3) Assume we have $r : \Gamma \to \mathrm{Spin}(7)$ and $\eta : \Gamma \to \{\pm 1\}$ with (r, η) unacceptable. Set $r' = \eta r$. We may choose a subgroup S_1 of $\mathrm{SO}(8)$ isomorphic to $\mathrm{Spin}(7)$ and assume that r is S_1-valued. By (U1) and the acceptability of $\mathrm{O}(8)$, there is $g \in \mathrm{O}(8)$ such that $gr'g^{-1} = r$. In particular, we have

$$r(\Gamma) \subset S_1 \cap S_2 \ \text{ with } S_2 := g\, S_1\, g^{-1}. \tag{6}$$

Assume first $S_1 = S_2$. Then g is in the normalizer of Spin(7) in SO(8). But this normalizer is Spin(7), as the latter only has inner automorphisms and centralizer $\{\pm 1\}$ in SO(8). So we have $g \in \mathrm{Spin}(7)$, a contradiction as (r, η) is unacceptable. This shows $S_1 \neq S_2$, and so we may apply Proposition 4.12. It implies that inside $S_1 \simeq \mathrm{Spin}(7)$, the subgroup $S_1 \cap S_2$ is either the normalizer of a G_2-subgroup or is included in a $\mathrm{Spin}(1,6)$-subgroup. We conclude as the first case is excluded by Proposition 4.13. □

By Theorem 1.3 and Proposition 4.4, it is equivalent to classify the unacceptable N-valued or Spin(7)-valued morphisms.

Definition 4.14. *For any morphism $r : \Gamma \to \mathrm{Spin}(n)$ we denote by $\mathrm{Y}(r)$ the subset of $\mathrm{X}(r)$ consisting of characters of Γ on the 1-dimensional $\mathbb{R}[\Gamma]$-submodules of E.*

Another formulation of Theorem 1.3 is:

Corollary 4.15. *If $r : \Gamma \to \mathrm{Spin}(7)$ is unacceptable, we have $\mathrm{Y}(r) \neq \emptyset$.*

5. Type I unacceptable morphisms

Definition 5.1. *Let $r : \Gamma \to \mathrm{Spin}(7)$ be unacceptable. We say that r is* of type I *if we have $1 \in \mathrm{Y}(r)$, or equivalently, if the $\mathbb{R}[\Gamma]$-module E contains* 1.

Our goal in this section is to classify the type I unacceptable Spin(7)-valued morphisms. We thus focus on the morphisms $r : \Gamma \to N$ satisfying $r(\Gamma) \subset N^0 = \mathrm{SU}(4)$, or equivalently, on the complex 4-dimensional unitary representations of Γ with determinant 1. We start with an important example, inspired by the analysis in [Che19, §4.6], and that will turn out to be universal.

Example 2

Fix first a complex, non degenerate, quadratic plane $P \simeq \mathbb{C}^2$ and consider its similitude orthogonal group $\mathrm{GO}(P)$. By the choice of an orthonormal basis of P we may and do identify P with $\mathbb{C}^2$ equipped with the quadratic form $(x, y) \mapsto x^2 + y^2$, in which case $\mathrm{GO}(P)$ is the *standard* $\mathrm{GO}_2(\mathbb{C})$. Let $\mu : \mathrm{GO}(P) \to \mathbb{C}^\times$ be the similitude factor and det the determinant. The two isotropic lines of P are permuted by $\mathrm{GO}(P)$, and we let $\epsilon : \mathrm{GO}(P) \to \{\pm 1\}$ be the signature of this permutation representation; its kernel is the subgroup $\mathrm{GSO}(P) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ of proper similitudes. The structure of $\mathrm{GO}(P)$ is clear, for instance we have $\ker \mu = \mathrm{O}(P)$, $\det = \mu\,\epsilon$ and $\mathrm{GO}(P) = \mathbb{C}^\times \cdot \mathrm{O}(P)$. We denote by $\mathrm{GO}(2)$ the (actually unique) maximal compact subgroup of $\mathrm{GO}_2(\mathbb{C})$, and define

$$\mathrm{O}(2)^{\pm} \subset \mathrm{GO}(2) \subset \mathrm{GO}_2(\mathbb{C})$$

as the subgroup of elements $g \in \mathrm{GO}(2)$ with $\mu(g) = \pm 1$. The group $\mathrm{O}(2)^{\pm}$ is generated by the homothety $i1_2$ and $\mathrm{O}(2)$, hence is also isomorphic to the quotient of $\mu_4 \times \mathrm{O}(2)$ by its diagonal central $\mathbb{Z}/2\mathbb{Z}$. It has exactly 3 order 2 characters, namely $\det, \mu$ and ϵ. Its tautological 2-dimensional representation P satisfies $\det P = \det$

and $P^* \simeq P \otimes \mu$ (with symmetric pairing); it is irreducible, non self-dual, and satisfies $P \simeq P \otimes \epsilon$.

Definition 5.2. *We denote by $\mathcal{G}$ the subgroup of all elements (g_1, g_2) in $\mathrm{O}(2)^{\pm} \times \mathrm{O}(2)^{\pm}$ such that $\det g_1 = \det g_2$ and $\mu(g_2) = \epsilon(g_1)$.*

The group $\mathcal{G}$ has two natural 2-dimensional complex representations P_1 and P_2, given by the tautological representations of the first and second factors respectively. Fix an embedding $\rho : \mathcal{G} \to \mathrm{SU}(4)$ with underlying 4-dimensional representation $\simeq P_1 \oplus P_2$. Such a ρ is unique up to conjugacy by Lemma A.1, but to fix ideas we take $P_1 = \mathbb{C}^2 \times 0$ and $P_2 = 0 \times \mathbb{C}^2$ with standard quadratic and hermitian forms on each factor (namely $(x, y) \mapsto x^2 + y^2$ and $|x|^2 + |y|^2$). We define morphisms υ and $\mathrm{d} : \mathcal{G} \to \{\pm 1\}$ by setting, for $g = (g_1, g_2) \in \mathcal{G}$,

$$\upsilon(g) = \epsilon(g_1) = \mu(g_2) \text{ and } \mathrm{d}(g) = \det(g_1) = \det(g_2). \tag{7}$$

Proposition 5.3. *View ρ as a morphism $\mathcal{G} \to N$. Then ρ and $\upsilon\rho$ are element conjugate in N, but not N-conjugate.*

Proof. By the discussion above, we have $P_1 \otimes \upsilon \simeq P_1$ and $P_2 \otimes \upsilon \simeq P_2^*$, hence

$$\rho \simeq P_1 \oplus P_2 \qquad \text{and} \qquad \upsilon\rho \simeq \upsilon \otimes \rho \simeq P_1 \oplus P_2^*. \tag{8}$$

As neither P_1 nor P_2 is self-dual, neither ρ nor its dual is isomorphic to $\upsilon\rho$, so ρ and $\upsilon\rho$ are not N-conjugate, by Remark 4.3. However, for all $g \in \mathcal{G}$ we have $\rho(g)$ or $\rho(g)^{-1}$ conjugate to $\upsilon(g)\rho(g)$ in $\mathrm{SU}(4)$. Indeed, this is trivial for g in $\ker \upsilon$. It is thus enough to show that on both $\ker \mathrm{d}$ and $\ker \upsilon\mathrm{d}$ we have $\rho^* \simeq \upsilon\rho$, and for that it is enough to prove that on those two subgroups we have $P_1 \simeq P_1^*$. But on $\ker \mathrm{d}$, this follows from the fact that we have $1 = \mathrm{d}(g) = \det(g_1)$ (P_1 is symplectic), and on $\ker \upsilon\mathrm{d}$, this follows from $1 = (\mathrm{d}\upsilon)(g) = \mu(g_1)$ (P_1 is orthogonal). ☐

We now start showing that any unacceptable morphism $r : \Gamma \to N$ with $r(\Gamma) \subset N^0$ can be explained by this example. We denote by V the tautological complex 4-dimensional representation of $N^0 = \mathrm{SU}(4)$. For a given morphism $r : \Gamma \to N$ with $r(\Gamma) \subset N^0$, we may view V as a (semi-simple) $\mathbb{C}[\Gamma]$-module.

Lemma 5.4. *Let $r : \Gamma \to \mathrm{SU}(4)$ and $\eta : \Gamma \to \{\pm 1\}$ be two morphisms. Assume ηr and r are element conjugate in N but not N-conjugate. Then the $\mathbb{C}[\Gamma]$-module V defined by r has a unique decomposition $V = A \oplus B$ such that:*

(i) $\dim A = \dim B = 2$,

(ii) *we have $\mathbb{C}[\Gamma]$-module isomorphisms $A \simeq \eta \otimes A$ and $B^* \simeq \eta \otimes B$.*

Moreover, the following properties hold:

(a) *neither A nor B is self-dual,*

(b) *the character $\det A = \det B$ has order 2, and we have $\det A \neq \eta$,*

(c) *the* $\mathbb{C}[\Gamma]$*-module* V *is multiplicity free,*

(d) A *is reducible if, and only if, we have* $A \simeq a \oplus \eta a$ *with* a *of order* 4*,*

(e) B *is reducible if, and only if, we have* $B \simeq b_1 \oplus b_2$ *with* $b_1^2 = b_2^2 = \eta$*.*

Proof. We start as in [Lar96, Lemma 2.4]. Denote respectively by V_1 (resp. V_2) the 4-dimensional representation of Γ on V defined by r (resp. ηr). We have thus

$$V_2 \simeq \eta \otimes V_1. \tag{9}$$

and $\det V_i = 1$. The V_i are semi-simple. By (U1) and Remark 4.3 we have $V_1 \oplus V_1^* \simeq V_2 \oplus V_2^*$. We may write $V_1 \simeq A \oplus B$ and $V_2 \simeq A \oplus C$ with $\mathrm{Hom}_\Gamma(B, C) = 0$. The previous relation shows $B \oplus B^* \simeq C \oplus C^*$ and then $C \simeq B^*$. In particular, we have $\det A \det B = \det A (\det B)^{-1} = 1$, so

$$c := \det A = \det B$$

is a character of Γ of order ≤ 2. Also, (U2) is equivalent to $V_1 \not\simeq V_2$ and $V_1 \not\simeq V_2^*$, *i.e.* to: neither A nor B is self-dual. This proves property (a) of the statement. In particular, both A and B have dimension ≥ 2, hence must have dimension 2. As a summary, we have shown

$$V_1 \simeq A \oplus B,\ V_2 \simeq A \oplus B^*,\ A^* \not\simeq A,\ B^* \not\simeq B,\ \dim A = \dim B = 2. \tag{10}$$

In particular, the character c must be non trivial, since any 2-dimensional representation P satisfies $P^* \otimes \det P \simeq P$. So c has order 2.

By (9) we have $A \oplus B^* \simeq \eta \otimes (A \oplus B)$. Note that if two semi-simple 2-dimensional representations of same determinant share a 1-dimensional constituent, then they are isomorphic. As a consequence, if A is not isomorphic to $\eta \otimes A$, then we have $A \simeq \eta \otimes B$ and $B^* \simeq \eta \otimes A$. But these two relations imply $A \simeq A^*$, a contradiction. We have proved that the decomposition $V = A \oplus B$ satisfies (i) and (ii).

Note that $A \simeq \eta \otimes A$ is equivalent to $A^* \simeq c\eta \otimes A$. In particular, we have $c \neq \eta$ as A is not selfdual, and this ends proving property (b). If A is reducible and contains say the character $a : \Gamma \to \mathbb{C}^\times$, it also contains $\eta a \neq a$, and we have $A \simeq a \oplus \eta a$ with $c = a^2\eta$, $a^2 \neq 1$ and $a^4 = c^2 = 1$, hence property (d). Similarly, if B is reducible and contains b, we have $B \simeq b \oplus cb^{-1}$ and the relations $B^* \simeq \eta \otimes B$ and $c \neq \eta$ imply $b^{-1} = \eta b$. This proves property (e). In any case, both A and B are multiplicity free as we have $\eta, c\eta \neq 1$. Also, there is no nonzero Γ-equivariant morphism $A \to B$, since such a morphism would be an isomorphism and would imply $B \simeq \eta \otimes B \simeq B^*$. We have proved property (c): V is multiplicity free.

It only remains to show the uniqueness statement. Assume we have a $\mathbb{C}[\Gamma]$-module decomposition $V = A' \oplus B'$ with A' and B' satisfying (i) and (ii). Consider the Γ-equivariant projection $f : A' \to B$ with kernel A. It cannot be an isomorphism, since it would imply $B \simeq \eta \otimes B$ by (ii) and then $B^* \simeq B$. If f is nonzero, then both A and B are reducible and we have $A' \simeq a \oplus b$ with a (resp. b) a constituent of A (resp. B), and in particular $a^2\eta = c$ and $b^2 = \eta$. But

$A' \simeq \eta \otimes A'$ implies $b = \eta a$, so we have $c = \eta a^2 = \eta b^2 = 1$: a contradiction. We deduce $f(A') = 0$, *i.e.* $A' \subset A$, and then $A' = A$ and $B' \simeq B$. As V is multiplicity free, this forces $B' = B$ as well. $\square$

Remark 5.5. *If both A and B are reducible, it follows from Lemma 5.4 that we have $A \simeq a \oplus ab^2$, $B \simeq b \oplus ba^2$, a, b of order 4 and $\eta = b^2$. This is thus exactly the case studied in* Example 1.

For later use we mention the following complement to Lemma 5.4.

Lemma 5.6. *In the notations of Lemma 5.4, the $\mathbb{C}[\Gamma]$-module $A \otimes B$ does not contain any character $\chi : \Gamma \to \mathbb{C}^\times$ with $\chi^2 = 1$.*

Proof. Indeed, such a χ induces a nonzero $\mathbb{C}[\Gamma]$-equivariant morphism $\chi \otimes A^* \to B$. This forces $\chi \otimes A^* \simeq B$ since both $\chi \otimes A^*$ and B have determinant $\chi^2 c^{-1} = c$, as $\chi^2 = 1$. This implies $c\chi \otimes A \simeq B$ and then $\mathrm{Sym}^2 A \simeq \mathrm{Sym}^2 B$. But $\mathrm{Sym}^2 B$ contains η by Lemma 5.9 (i), hence so does $\mathrm{Sym}^2 A$. This forces $\eta \otimes A^* \simeq A$ (again since both sides have the same determinant) and then $A^* \simeq A$: a contradiction. $\square$

The following corollary applies for instance to $\Gamma = \mathbb{Z} \times \mathrm{SU}(2)$.

Corollary 5.7. *Assume there is no surjective morphism $\Gamma \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then any morphism $\Gamma \to \mathrm{Spin}(7)$ is acceptable.*

Proof. If (r, η) is unacceptable, the assumption implies $\mathrm{Hom}(\Gamma, \{\pm 1\}) = \{1, \eta\}$. But then $\mathrm{Y}(r) = \{1\}$ by Corollary 3.8 and Theorem 1.3, so r is of type I. But this contradicts the property $\det A \neq 1, \eta$ in Lemma 5.4 $\square$

Theorem 5.8. *Assume $r, r' : \Gamma \to N$ are element conjugate but non conjugate morphisms with values in $\mathrm{SU}(4) = N^0$. Then there exists a morphism $f : \Gamma \to \mathcal{G}$ such that, up to replacing r and r' by some N-conjugate if necessary, we have:*

(i) $r = \rho \circ f$,

(ii) $r' = \eta r$ *with* $\eta := \upsilon \circ f$, *and*

(iii) η *and* $\mathrm{d} \circ f$ *are distinct, order* 2, *characters of* Γ.

Proof. By Proposition-Definition 4.5, up to conjugating r' by some element of N we may assume that we have $r' = \eta r$ for some order 2 character $\eta : \Gamma \to \{\pm 1\}$. Write $V \simeq A \oplus B$ as in Lemma 5.4 and set $c = \det A$. Apply Lemma 5.9 (i) below to both $(P, \eta) = (A, c\eta)$ and $(P, \eta) = (B, \eta)$. It endows A and B with nondegenerate Γ-equivariant symmetric pairings b_A and b_B with respective similitude factors $c\eta$ and η. We may assume that both quadratic spaces (A, b_A) and (B, b_B) are $\mathbb{C}^2$ equipped with the standard form $x^2 + y^2$. The action of Γ on A and B thus gives rise to morphisms f_A and $f_B : \Gamma \to \mathrm{GO}_2(\mathbb{C})$, with respective similitude factors

$\mu \circ f_A = c\eta$ and $\mu \circ f_B = \eta$, and $\det \circ f_A = \det \circ f_B = c$. But Γ also preserves some positive definite hermitian form inherited from V, hence $f_A(\Gamma)$ and $f_B(\Gamma)$ have a compact closure in $\mathrm{GO}_2(\mathbb{C})$, and so are included in $\mathrm{GO}(2)$, and even in $\mathrm{O}(2)^{\pm}$ as we have $\eta^2 = (c\eta)^2 = 1$. We have thus constructed a morphism $f := f_A \times f_B : \Gamma \to \mathcal{G}$ such that the two morphisms $\rho \circ f$ and r, from Γ to $\mathrm{SU}(4)$, define two isomorphic representations of Γ on $\mathbb{C}^4$. By Lemma A.1, they are conjugate in $\mathrm{SU}(4)$. So up to replacing r with a conjugate we may assume $\rho \circ f = r$. The result follows from the formulae $\upsilon \circ \rho = \mu \circ f_B = \eta$ and $\mathrm{d} \circ \rho = \det \circ f_A = c$. $\square$

In the following lemma, we denote by D_8 the dihedral group of order 8.

Lemma 5.9. *Let Γ be a group, P a 2-dimensional complex representation of Γ and $\eta : \Gamma \to \mathbb{C}^\times$ a morphism. Assume $\det P \neq \eta$ and $P^* \simeq P \otimes \eta^{-1}$.*

- (i) *There is a nondegenerate symmetric pairing on P such that Γ acts on P as orthogonal similitudes with similitude factor η, and $\mathrm{Sym}^2 P$ contains η.*
- (ii) *Assume furthermore that $\det P$ and η are two order 2 characters of Γ. Then P is self-dual if, and only if, the image of Γ in $\mathrm{GL}(P)$ is isomorphic to D_8.*

Proof. The given isomorphism $P^* \simeq P \otimes \eta^{-1}$ may be viewed a nondegenerate Γ-equivariant pairing $P \otimes P \to \eta$. As we have $\Lambda^2 P = \det P \neq \eta$ by assumption, this pairing factors through $\mathrm{Sym}^2 P$. This proves (i). We now prove assertion (ii).

Of course, the unique faithful 2-dimensional representation of D_8 is self-dual. Conversely, assume $P^* \simeq P$. We have then $P^* \simeq P \otimes \nu$ for each of the four distinct characters $\nu = 1, \eta, \det, \eta \det$, and thus $\mathrm{Sym}^2 P \simeq 1 \oplus \eta \oplus \eta \det$. Note that $(\eta, \det) : \Gamma \to \mu_2^2$ is surjective, and that the kernel of the natural morphism $\mathrm{GL}(P) \to \mathrm{GL}(\mathrm{Sym}^2 P)$ is the central μ_2. As a consequence, the image I of Γ in $\mathrm{GL}(P)$ is an extension of μ_2^2 by a central subgroup of order ≤ 2. This shows $|I| \leq 8$. Note also that P cannot contain any character ν, otherwise it would contain the three distinct characters ν, $\nu\eta$, $\nu \det$. So P is irreducible and I is nonabelian of order 8. Since the unique 2-dimensional faithful representation of the quaternion group of order 8 has determinant 1, I must be isomorphic to D_8. $\square$

We now state a converse to Theorem 5.8.

Proposition 5.10. *Let Γ be a group and $f : \Gamma \to \mathcal{G}$ a morphism such that $\mathrm{d} \circ f$ and $\eta := \upsilon \circ f$ are distinct and nontrivial. Then the N-valued morphisms $r := \rho \circ f$ and ηr are element-conjugate by Proposition 5.3. They are non conjugate if, and only if, none of the two projections $\Gamma \to \mathrm{O}(2)^{\pm}$ has an image isomorphic to D_8.*

Proof. This follows from Remark 4.3, Formula (8) and Lemma 5.9 (ii). $\square$

We finally provide an alternative study of the unacceptable morphisms of type I from the point of view of their standard representation on E. View ρ as a morphism $\mathcal{G} \to \mathrm{Spin}(7)$, and consider the associated $\mathbb{R}[\mathcal{G}]$-module E. We have a $\mathcal{G}$-stable decomposition $E = L \perp F$, with $L \simeq 1$ as an $\mathbb{R}[\mathcal{G}]$-module, and by Remark 4.9 we have a $\mathbb{C}[\mathcal{G}]$-module decomposition

$$F \otimes \mathbb{C} \simeq P_1 \otimes P_2 \oplus \mathrm{d} \oplus \mathrm{d}. \tag{11}$$

Proposition 5.11. *We have an orthogonal decomposition*

$$F = F_1 \perp F_2 \perp F_3, \quad \text{with } \dim F_1 = \dim F_2 = \dim F_3 = 2, \tag{12}$$

such that $\mathcal{G}$ coincides with the subgroup of elements $g \in N$ with determinant 1 on F, preserving the pair $\{F_1, F_2\}$ (hence F_3), and acting on F_3 as a homothety whose sign coincides with the signature of g on the 2-elements set $\{F_1, F_2\}$. Moreover, this latter signature is $\mathrm{d}(g)$, and if we assume $\mathrm{d}(g) = 1$ then we have $\upsilon(g) = \det_{F_1}(g) = \det_{F_2}(g)$.

In the decomposition above, both F_3 and the pair $\{F_1, F_2\}$ are thus unique, but of course the numberings of F_1 and F_2 are not.

Proof. Consider the subgroup $T := \mathrm{SO}(2) \times \mathrm{SO}(2)$ of $\mathcal{G}$ (a torus). By Formula (11), there is a unique triple $\{F_1, F_2, F_3\}$ of planes of F with $F = F_1 \perp F_2 \perp F_3$, such that T acts trivially on F_3 and the natural map $T \to \mathrm{SO}(F_1) \times \mathrm{SO}(F_2)$ is surjective, with kernel $\{\pm 1\}$. More precisely, if we write $P_1 \simeq x \oplus x^{-1}$ and $P_2 \simeq y \oplus y^{-1}$ as $\mathbb{C}[T]$-modules, then up to replacing x with x^{-1} we may assume $F_1 \otimes \mathbb{C} \simeq xy \oplus (xy)^{-1}$ and $F_2 \otimes \mathbb{C} \simeq xy^{-1} \oplus x^{-1}y$. By Definition 5.2, the group $\mathcal{G}$ is generated by T and the two elements $g_1 = (\sigma, i1_2)$ and $g_2 = (i1_2, \sigma)$, for any $\sigma \in \mathrm{O}(2)$ with determinant -1. Both g_1 and g_2 exchange thus F_1 and F_2. We conclude as we have $\upsilon(g_1) = -1$, $\upsilon(g_2) = 1$, $\mathrm{d}(g_1) = \mathrm{d}(g_2) = -1$, and $\det_{F_j}(g_1 g_2) = -1$ for $j = 1, 2$. $\square$

Let $\mathcal{G}_1 \subset \mathcal{G}$ be the kernel of d and consider the 4-dimensional Euclidean space

$$F' = F_1 \perp F_2.$$

By Proposition 5.11, we may identify F' with the natural direct sum $\mathbb{R}^2 \perp \mathbb{R}^2$, $\mathcal{G}/\{\pm 1\}$ with the subgroup $\underline{\mathcal{G}}$ of $\mathrm{SO}(4)$ preserving this sum, and $\mathcal{G}_1/\{\pm 1\}$ with

$$\underline{\mathcal{G}_1} := \{(g_1, g_2) \in \mathrm{O}(2) \times \mathrm{O}(2) \mid \det(g_1) = \det(g_2)\}.$$

The element $\theta = \begin{bmatrix} 0 & 1_2 \\ 1_2 & 0 \end{bmatrix}$ has determinant 1 and is in $\underline{\mathcal{G}} \smallsetminus \underline{\mathcal{G}_1}$, with $\theta^2 = 1$. For all $(g_1, g_2) \in \underline{\mathcal{G}_1}$, we have $\theta(g_1, g_2)\theta^{-1} = (g_2, g_1)$ and $\upsilon(g_1, g_2) = \det(g_1) = \det(g_2)$ (the value of υ on θ will be irrelevant).

We have proved that up to conjugacy, any type I unacceptable morphism (r, η) arises from a morphism $\Gamma \to \mathcal{G}$ in the sense of Theorem 5.8. A first converse statement was given in Proposition 5.10, in terms of the natural morphism $\Gamma \to$

$N^0 = \mathrm{SU}(4)$. Our aim now is to provide a second one, rather in terms of the properties of the $\mathbb{R}[\Gamma]$-module E. We first highlight a few examples.

Proposition 5.12. *We denote by $\Delta \simeq \mathrm{O}(2)$ the diagonal subgroup of $\mathrm{O}(2) \times \mathrm{O}(2)$ and by $\mu \simeq \{\pm 1\}^4$ the diagonal subgroup of $\mathrm{O}(4)$. We define three subgroups H_1, H_2 and H_3 of $\mathrm{SO}(4)$ contained in $\underline{\mathcal{G}}$ as follows:*

(i) *H_1 is generated by $\mu \cap \Delta$ and $\mathrm{diag}(s, -s)\theta$ with $s = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,*

(ii) *H_2 is generated by Δ and θ,*

(iii) *H_3 is generated by Δ and $t := \mathrm{diag}(1_2, -1_2)\theta$.*

For each i, the characters $\delta = \mathrm{d}_{|H_i}$ and $\eta = \upsilon_{|H_i}$ of H_i are distinct and of order 2, and we denote by r the inclusion of H_i in $\mathrm{Spin}(7)$. Then (r, η) is acceptable for $i = 1, 2$ and unacceptable for $i = 3$.

Proof. Each H_i contains elements of the form g and $g'\theta$ with $g, g' \in \Delta$ and $\upsilon(g) = -1$, hence the first assertion. We have $H_1 \simeq \mathrm{D}_8$ and $H_2 \simeq \mathrm{O}(2) \times \mathbb{Z}/2\mathbb{Z}$, and for $i = 1, 2$, an $\mathbb{R}[H_i]$-module isomorphism $F' \simeq \delta \otimes P \oplus P$ with P irreducible and $\det P \in \{\eta, \eta\delta\}$. So $\eta \in \mathrm{X}(r)$ and (r, η) is acceptable for $i = 1, 2$. On the other hand, we have $H_3 \simeq (\mathrm{O}(2) \times \mu_4)/\mathrm{diag}\, \mu_2$ and an $\mathbb{R}[H_3]$-module isomorphism $F' \simeq P \otimes Q$, with P the (inflation of the) tautological representation of the $\mathrm{O}(2)$ factor, and Q (that of) the irreducible 2-dimensional real representation of the μ_4 factor. Indeed, the element t commutes with Δ and satisfies $t^2 = \mathrm{diag}(-1_2, -1_2) \in \Delta$. So F' is irreducible, we have $\mathrm{X}(r) = \{1, \delta\}$, and (r, η) is unacceptable for $i = 3$. □

The two first examples turn out to explain all the acceptable examples. For minor reasons we first need to introduce the subgroup $M \subset N$ preserving F_1, F_2 (hence F_3) and acting trivially on F_3. We have $M/\{\pm 1\} \simeq \mathrm{O}(2) \times \mathrm{O}(2)$, $M \cap \mathcal{G} = \mathcal{G}_1$ and M normalizes $\mathcal{G}$ by fixing the character d and exchanging υ and $\upsilon\mathrm{d}$.

Proposition 5.13. *Consider a morphism $f : \Gamma \to \mathcal{G}$ such that $\eta = \upsilon \circ f$ and $\delta = \mathrm{d} \circ f$ are distinct and of order 2, and set $r = \rho \circ f$. Then (r, η) satisfies* (U1)*. It is acceptable if, and only if, up to conjugating f with some element of M, the image of $f(\Gamma)$ in $\underline{\mathcal{G}}$ is included in one of the subgroups H_1 and H_2 of Proposition 5.12.*

Proof. That (r, η) satisfies (U1) follows from Proposition 5.3, so we only have to check the last assertion. The sufficiency of the given condition follows directly from Proposition 5.12, so we assume from now on that (r, η) is acceptable, or equivalently, that we have $\eta \in \mathrm{X}(r)$. Set $\Gamma_1 = \ker \delta$.

By (12) we may find some $\mathbb{R}[\Gamma]$-submodule $U \subset F'$ with $\det_U \in \{\eta, \eta\delta\}$. Il all cases $\det_U$ coincides with η on Γ_1, and up to replacing U by its orthogonal in F' if necessary, we may also assume $1 \leq \dim U \leq 2$. Note that U is not included in F_1 or in F_2, since Γ permutes transitively $\{F_1, F_2\}$ as $\delta \neq 1$. So the natural

Γ_1-equivariant projection $U \to F_i$ has a nonzero image. As U, F_1 and F_2 have the same determinant η on Γ_1, this implies that F_1 and F_2 are isomorphic as $\mathbb{R}[\Gamma_1]$-modules (and isomorphic to $U_{|\Gamma_1}$ in the case $\dim U = 2$). By Lemma A.1, and up to replacing f by some M-conjugate, we may thus assume $f(\Gamma_1) \subset \Delta$. (We freely use the notations of Proposition 5.12).

As η is nontrivial on Γ_1, there is an element $h \in \Gamma$ with $f(h) = \mathrm{diag}(a, b)\theta$, $a, b \in \mathrm{O}(2)$ and $\det a = \det b = -1$. The element $f(h^2) = f(h)^2 = \mathrm{diag}(ab, ba)$ is in Δ, so the two reflections a and b in $\mathrm{O}(2)$ commute. This forces $b = \epsilon a$ for some sign $\epsilon = \pm 1$. If $\epsilon = 1$ then we have $f(\Gamma) \subset H_2$ and we are done. So we definitely assume $\epsilon = -1$, *i.e.* $f(\Gamma) \subset H_3$, and we write $H_3 \simeq (\mathrm{O}(2) \times \mu_4)/\mathrm{diag}\,\mu_2$ and $F' \simeq P \otimes Q$ as in the proof of Proposition 5.12.

Assume first P is irreducible as an $\mathbb{R}[\Gamma_1]$-module (necessarily absolutely irreducible, since $\det P \neq 1$). We claim that F' is an irreducible $\mathbb{R}[\Gamma]$-module. Indeed, we have $(F')_{|\Gamma_1} \simeq P \otimes Q$ with Q trivial, so the proper $\mathbb{R}[\Gamma_1]$-submodules of F' are the $P \otimes v$ with $v \in Q$ nonzero. But none of those is stable by the element $f(h) \in \Delta t$, since t has no eigenvector in Q. So we have $\det F' = 1$, $\mathrm{X}(r) = \{1, \delta\}$ and $\eta \notin \mathrm{X}(r)$, contradicting the acceptability of (r, η).

So the $\mathbb{R}[\Gamma_1]$-module P is reducible, necessarily the sum of two distinct real characters as $\det P = \eta \neq 1$. Up to replacing f by some M-conjugate, we may thus assume $f(\Gamma_1) \subset \mu \cap \Delta$. Since $f(h) = \mathrm{diag}(a, -a)\theta$ normalizes $f(\Gamma_1)$, the reflection $a \in \mathrm{O}(2)$ normalizes the diagonal subgroup $\{\pm 1\}^2$ of $\mathrm{O}(2)$, so we either have $a = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $a = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In the first case we have $f(\Gamma) \subset H_1$ and we are done. In the second case, P is reducible as $\mathbb{R}[\Gamma]$-module, say $P \simeq \alpha \oplus \beta$, and we have $F' \simeq \alpha \otimes Q \oplus \beta \otimes Q$, a sum of two irreducible $\mathbb{R}[\Gamma]$-modules with determinant 1. This shows $\mathrm{X}(r) = \{1, \delta\}$ and contradicts the acceptability of (r, η). □

We end this paragraph by discussing a few other properties of type I unacceptable morphisms.

Corollary 5.14. *Assume $r : \Gamma \to \mathrm{Spin}(7)$ is unacceptable of type* I. *We have $\mathrm{Y}(r) = \{1, \delta\}$ where* 1 *and δ have respective multiplicity* 1 *and* 2 *in E.*

Proof. This is a corollary of Proposition 4.4, Theorem 5.8 and Lemma 5.6. □

The next proposition asserts that for an unacceptable morphism r of type I, the action of $\mathrm{X}(r)$ on $\mathrm{E}(r)$ discussed in Remark 3.5 is transitive.

Proposition 5.15. *Assume $r : \Gamma \to \mathrm{Spin}(7)$ is unacceptable of type* I. *Then up to the multiplication by an element of $\mathrm{X}(r)$, there is a unique morphism $\eta : \Gamma \to \{\pm 1\}$ such that (r, η) is unacceptable.*

Proof. Let δ be as in Corollary 5.14 and set $\Gamma_1 = \ker \delta$. Choose η such that (r, η) is unacceptable, and set $\eta_1 = \eta_{|\Gamma_1}$. By Theorem 5.8 and Proposition 5.11 we have an $\mathbb{R}[\Gamma]$-module isomorphism $E \simeq 1 \oplus \delta \oplus \delta \oplus \mathrm{ind}_{\Gamma_1}^{\Gamma} U$, where U is an $\mathbb{R}[\Gamma_1]$-module of dimension 2 with $\det U = \eta_1$. If U is irreducible, then the 2-dimensional irreducible constituents of the $\mathbb{R}[\Gamma_1]$-module E are U and its outer conjugate under Γ/Γ_1, and both have determinant η_1. It follows that η_1 is uniquely defined, hence that η is unique up to multiplication by $\delta \in \mathrm{X}(r)$ in this case. So we may assume U is reducible, say $U \simeq \alpha \oplus \alpha\eta_1$ for some character $\alpha : \Gamma_1 \to \{\pm 1\}$. We have

$$\mathrm{ind}_{\Gamma_1}^{\Gamma} U \simeq V \oplus \eta \otimes V, \text{ with } V := \mathrm{ind}_{\Gamma_1}^{\Gamma} \alpha.$$

Denote by α^c the outer-conjugate of α under Γ/Γ_1. Let us assume first $\alpha^c \neq \alpha$. Then the image of Γ_1 and Γ in $\mathrm{O}(V)$ are thus respectively isomorphic to μ_2^2 and D_8. The three order 2 characters of this D_8 must be δ, $\delta \det_V$ and $\det_V$, and we have $\mathrm{X}(r) = \langle \delta, \det_V \rangle$. As r is unacceptable, η is not in $\mathrm{X}(r)$, so the group $\pi(r(\Gamma))$ is isomorphic to $\mathrm{D}_8 \times \mu_2$. By Proposition 3.3, η is an order 2 character of this group. We conclude as there is a unique non trivial character of $\pi(r(\Gamma))$ modulo $\mathrm{X}(r)$. Assume now $\alpha^c = \alpha$. Then the image of Γ_1 and Γ in $\mathrm{O}(V)$ are respectively isomorphic to μ_2 and μ_4. Indeed, that of Γ cannot be $\mu_2 \times \mu_2$, otherwise V would contain a 1-dimensional representation and contradict Corollary 5.14. So we have $\mathrm{X}(r) = \{1, \delta\}$, and $\pi(r(\Gamma)) \simeq \mu_4 \times \mu_2$ has a unique order 2 character modulo $\mathrm{X}(r)$. (This case is the one of Example 1). □

6. Type II unacceptable morphisms

Definition 6.1. *Let $r : \Gamma \to \mathrm{Spin}(7)$ be unacceptable, $\chi \in \mathrm{Y}(r)$* non trivial*, and set $\Gamma_0 = \ker \chi$ (an index 2 subgroup of Γ). We say that r is of type* II *with respect to χ if there is $\eta : \Gamma \to \{\pm 1\}$ such that both (r, η) and $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$ are unacceptable. In this situation, we also say that (r, η) is unacceptable of type* II *with respect to χ. We shall also say that r is of type* II *if it is so with respect to some χ.*

Of course, to check the unacceptability of both (r, η) and $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$, it is enough to check that (r, η) satisfies (U1) and that $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$ satisfies (U2). The following remark shows that types I and II are exclusive.

Remark 6.2. *Assume that $r : \Gamma \to \mathrm{Spin}(7)$ is unacceptable of type* I*. By Corollary 5.14, we have $\mathrm{Y}(r) = \{1, \delta\}$ with 1 (resp. δ) occuring with multiplicity 1 (resp. 2). So $r_{|\ker \delta}$ contains 1 with multiplicity 2, and r is not of type* II*.*

Notation : Up to the end of this section, we assume that Γ is a group, that $\chi : \Gamma \to \{\pm 1\}$ is a non trivial character, and we denote by Γ_0 the kernel of χ, an index 2 subgroup of Γ.

Lemma 6.3. *Let $r : \Gamma \to \mathrm{Spin}(7)$ with $\chi \in \mathrm{Y}(r)$. Let $\eta : \Gamma \to \{\pm 1\}$ be a morphism.*

(i) *The pair (r, η) satisfies* (U1) *if, and only if, $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$ satisfies* (U1).

(ii) *If* $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$ *is unacceptable, then* $r_{|\Gamma_0}$ *is of type* I.

Proof. The *only if* part of (i) is obvious. For the *if* part, assume $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$ satisfies (U1). We have to show that for $\gamma \in \Gamma \smallsetminus \Gamma_0$, $r(\gamma)$ and $\eta(\gamma)r(\gamma)$ are conjugate in $\mathrm{Spin}(7)$, or equivalently (by Proposition 3.12), that $\eta(\gamma)$ is an eigenvalue of γ in E. We conclude as both 1 and $\chi(\gamma) = -1$ are eigenvalues of γ on E, as $\dim E$ is odd and $\chi \in \mathrm{Y}(r)$. Assertion (ii) is clear since we have $1 = \chi_{|\Gamma_0} \in \mathrm{Y}(r_{|\Gamma_0})$. □

Recall the set $\mathrm{E}(r)$ defined in Remark 3.5.

Corollary 6.4. *Let* $r : \Gamma \to \mathrm{Spin}(7)$ *be a morphism such that* $\chi \in \mathrm{Y}(r)$*. Then* r *is unacceptable of type* II *with respect to* χ *if, and only if,* $r_{|\Gamma_0}$ *is unacceptable of type* I *and at least one element of* $\mathrm{E}(r_{|\Gamma_0})$ *extends to an order* 2 *character of* Γ.

Proof. Set $r_0 = r_{|\Gamma_0}$. If r is unacceptable of type II with respect to χ, then by definition there is $\eta \in \mathrm{E}(r)$ such that $\eta_{|\Gamma_0} \in \mathrm{E}(r_0)$ (in particular, $\eta_{|\Gamma_0} \neq 1$). Also, r_0 is of type I by Corollary 6.3 (ii). Conversely, assume r_0 is unacceptable of type I and choose η_0 in $\mathrm{E}(r_0)$ extending to $\eta : \Gamma \to \{\pm 1\}$. Then (r, η) satisfies (U1) by Corollary 6.3 (i), and (r_0, η_0) is unacceptable: r is of type II with respect to χ. □

This corollary reduces the classification of type II unacceptable morphisms to that of type I ones, done in the previous section, together with some extension problem. We shall not say more about this problem here, and are happy to leave this task to a motivated reader. We shall content ourselves below with one example of unacceptable type II morphism, and with two criteria for their inexistence.

Example 6.5. *Let* $\vartheta \in N \smallsetminus N^0$ *be an order* 2 *element with* $\vartheta g \vartheta^{-1} = \overline{g}$ *for all* $g \in \mathrm{SU}(4)$*. For all* $(g_1, g_2) \in \mathcal{G} \subset \mathrm{O}(2)^{\pm} \times \mathrm{O}(2)^{\pm}$ *we have* $\vartheta(g_1, g_2)\vartheta^{-1} = (\pm g_1, \pm g_2)$*. Since the* 4 *characters of* $\mathrm{O}(2)^{\pm}$ *are trivial on* -1_2*, the character* υ *of* $\mathcal{G}$ *extends to* $\mathcal{G}' = \mathcal{G} \rtimes \langle \vartheta \rangle$*, and with order* 2 *as* $\vartheta^2 = 1$*. By Proposition 5.3 and Corollary 6.4, the inclusion of* $\mathcal{G}'$ *in* $\mathrm{Spin}(7)$ *is unacceptable of type* II *with respect to* $\kappa_{|\mathcal{G}'}$*.*

Proposition 6.6. *If* $r : \Gamma \to \mathrm{Spin}(7)$ *is unacceptable of type* II*, then the group* Γ *has a quotient isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^3$ *or to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$*.*

Proof. Choose χ such that r is of type II with respect to χ. Set $\Gamma_0 = \ker \chi$ and $r_0 = r_{|\Gamma_0}$. Choose $\eta \in \mathrm{E}(r)$ such $(r_0, \eta_{|\Gamma_0})$ is unacceptable. By Corollary 5.14, there is a unique non trivial character δ_0 in $\mathrm{Y}(r_0)$. But the 7-dimensional representation $\pi \circ r_0$ of Γ_0 extends to Γ, so it is isomorphic to its outer conjugate under $\Gamma/\Gamma_0 \simeq \mathbb{Z}/2\mathbb{Z}$, hence so is δ_0. It follows that δ_0 extends to a character $\delta : \Gamma \to \mu_4$. Consider the morphism $f : \Gamma \to \{\pm 1\}^2 \times \mu_4$, given by $f = (\eta, \chi, \delta)$. As η and δ_0 are distinct of order 2 over Γ_0, and $\chi \neq 1$, the five characters $1, \chi, \eta, \delta, \eta\delta$ of Γ are distinct. We have thus $f(\Gamma_0) \simeq (\mathbb{Z}/2\mathbb{Z})^2$, $|f(\Gamma)| = 2|f(\Gamma_0)| = 8$, $f(\Gamma) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ if δ has order 4, and $f(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^3$ otherwise. □

Definition 6.7. *A group morphism $r : \Gamma \to G$ is called discrete if the centralizer of* $\operatorname{Im} r$ *in G is finite.*

In the case G is $\operatorname{Spin}(n)$, it is equivalent to ask that $\pi \circ r : \Gamma \to \operatorname{SO}(n)$ is discrete, by arguments given in the proof of Proposition 3.6. Type I unacceptable morphisms are not discrete by Formula (11). On the other hand, the inclusion $\mathcal{G}' \to \operatorname{Spin}(7)$ in Example 6.5 is discrete. Indeed, its centralizer must be included in N, hence coincides with $\mu_2 \times \mu_2$.

Proposition 6.8. *If $r : \Gamma \to \operatorname{Spin}(7)$ is discrete and unacceptable of type* II, *then the group Γ has a quotient isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.*

Proof. Define $\chi, \eta, r_0, \delta_0, \delta$ and f as the proof of Proposition 6.6. By Corollary 5.14, the isotypic component U of δ_0 in the $\mathbb{R}[\Gamma_0]$-module E has dimension 2. This plane $U \subset E$ is Γ-stable as we have $\mathrm{Y}(r_0) = \{\delta_0\}$. As r is discrete, the centralizer of the image of Γ in $\mathrm{O}(U)$ is finite, hence this image is not included in $\mathrm{SO}(U)$ (infinite abelian). If we write $\Gamma = \Gamma_0 \coprod z\Gamma_0$, this forces z to act on U as a reflection. But then we have $U \simeq \alpha \oplus \beta$ for some characters $\alpha, \beta : \Gamma \to \{\pm 1\}$, and both α and β extend δ_0. So we have $\delta \in \{\alpha, \kappa\alpha\}$, $\delta(\Gamma) = \mu_2$, and by the proof of Proposition 6.6, $f(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^3$. □

7. Type III unacceptable morphisms

In this section, we still assume that Γ is a group, that $\chi : \Gamma \to \{\pm 1\}$ is a non trivial character, and we denote by Γ_0 the kernel of χ, an index 2 subgroup of Γ.

Proposition 7.1. *Let $r : \Gamma \to \operatorname{Spin}(7)$ be a morphism with $\chi \in \mathrm{Y}(r)$. If $\eta : \Gamma \to \{\pm 1\}$ is an order* 2 *character, the following are equivalent:*

(i) *(r, η) is unacceptable and $(r_{|\Gamma_0}, \eta_{|\Gamma_0|})$ is acceptable,*
(ii) *we have $\eta \notin \mathrm{X}(r)$ but $\eta_{|\Gamma_0} \in \mathrm{X}(r_{|\Gamma_0})$.*

Proof. By Lemma 6.3 (i), the assumption $\eta_{|\Gamma_0} \in \mathrm{X}(r_{|\Gamma_0})$ implies that (r, η) satisfies (U1). The equivalence between (i) and (ii) follows then from Definition 3.4. □

Observe that if (r, η) is unacceptable of type I, and if χ is the unique non trivial element in $\mathrm{Y}(r)$ (Corollary 5.14), then $(r_{|\Gamma_0}, \eta_{|\Gamma_0})$ is acceptable (otherwise it would have type I and contain 3 times the trivial character). This is why we exclude this case in the following definition.

Definition 7.2. *A morphism $r : \Gamma \to \operatorname{Spin}(7)$ is called* unacceptable of type III *with respect to χ if we have $1 \notin \mathrm{Y}(r)$ and if there exists an order two character η of Γ satisfying the equivalent properties (i) and (ii) in Proposition 7.1. For such an η we also say that (r, η) is (unacceptable) of type* III *with respect to χ.*

By definition, any unacceptable $\mathrm{Spin}(7)$-valued morphism r is either of type I, or of type II or III for some element in $\mathrm{Y}(r)$. Let us analyze in more details condition (ii) in Proposition 7.1.

Definition-Proposition 7.3. *Assume r is unacceptable of type* III *with respect to χ and write $E \simeq F \oplus \chi$ as an $\mathbb{R}[\Gamma]$-module. There is an order 2 character η of Γ with $\eta \notin \mathrm{X}(r)$, and an $\mathbb{R}[\Gamma_0]$-submodule $V_0 \subset F$ (not necessarily irreducible) with determinant $\eta_{|\Gamma_0}$, such that one of the following holds:*

(a) *either* $\dim V_0 = 2$ *and there is an $\mathbb{R}[\Gamma]$-module Q, with* $\dim Q = 2$ *and* $\det Q = \chi$*, satisfying $F \simeq Q \oplus \mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0$,*

(b) *or* $\dim V_0 = 3$ *and we have an $\mathbb{R}[\Gamma]$-module isomorphism $F \simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0$.*

If we are in case (a) *(resp.* (b)*), we say that (r, η) is unacceptable* of type IIIa *(resp.* IIIb) *with respect to χ.*

Proof. By the Definition 7.2, there is an order 2 character η' of Γ and an $\mathbb{R}[\Gamma_0]$-submodule $U_0 \subset F$ with $\det U_0 = \eta'_{|\Gamma_0}$ and $\eta' \notin \mathrm{X}(r)$. Note that U_0 is not Γ-stable, otherwise the character $\det U_0 \in \mathrm{X}(r)$ would coincide with η' on Γ_0, which forces $\det U_0 = \eta'$ or $\chi\eta'$, hence $\eta' \in \mathrm{X}(r)$. The biggest Γ-stable subspace in U_0 is $U := U_0 \cap zU_0$, where z is any element in $\Gamma \smallsetminus \Gamma_0$. We have thus an $\mathbb{R}[\Gamma_0]$-module decomposition $U_0 = U \perp V_0$, with $V_0 \neq 0$, and $\det V_0 = (\eta' \det U)_{|\Gamma_0}$. The character $\eta := \eta' \det U$ is not in the group $\mathrm{X}(r)$ (see also Remark 3.5), and satisfies $\det V_0 = \eta_{|\Gamma_0}$. We also have $V_0 \cap zV_0 = \{0\}$ by construction, and thus an $\mathbb{R}[\Gamma]$-module embedding $\mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0 \hookrightarrow F$, as well as $1 \leq \dim V_0 \leq 3$.

Assume first $\dim V_0$ is odd and set $W_0 = \eta_{|\Gamma_0} \otimes V_0$. We have $\det W_0 = 1$ and an isomorphism $\mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0 \simeq \eta \otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma} W_0$. In the case $\dim W_0 = 1$ we must have $W_0 \simeq 1$, hence $\mathrm{Ind}_{\Gamma_0}^{\Gamma} W_0 \simeq 1 \oplus \chi$. But this shows $\eta \in \mathrm{Y}(r) \subset \mathrm{X}(r)$, a contradiction. This proves $\dim V_0 = 3$ and we are in case (b).

Assume now $\dim V_0 = 2$. Define Q as the orthogonal of $P = \mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0$ in F. By Lemma A.2 (ii), we have $\det P = \chi^2 t = t$ where $t : \Gamma \to \{\pm 1\}$ is the transfer of $\det V_0 = \eta_{|\Gamma_0}$ to Γ. As we have $\eta^2 = 1$ and η is a character of Γ, we have $t = 1$ by Lemma A.2 (i). This shows $\det P = 1$, hence $\det Q = \det F = \chi$. $\square$

Example 3

Fix a decomposition $F = P \perp P' \perp Q$ with P, P' and Q each of dimension 2. Define $\underline{\mathcal{H}} \subset \mathrm{O}(F)$ as the subgroup of isometries that:

– preserve the pair $\{P, P'\}$ and Q,

– have determinant 1 on $P \oplus P'$, and

– whose signature on $\{P, P'\}$ coincides with their determinant on Q.

Define $\underline{\mathcal{H}}_0$ as the subgroup of $\underline{\mathcal{H}}$ with trivial determinant on Q. We clearly have

$$\underline{\mathcal{H}}_0 \simeq (\mathrm{O}(2) \times \mathrm{O}(2))^{\det_1 = \det_2} \times \mathrm{SO}(2)$$

and using this identification $\underline{\mathcal{H}}$ may be identified with the semidirect product of $\mathbb{Z}/2\mathbb{Z}$ by $\underline{\mathcal{H}}_0$ with respect to the involution $(g_1, g_2, g_3) \mapsto (g_2, g_1, g_3^{-1})$. Set $\alpha = \det Q$ and denote by ε any of the two characters of $\underline{\mathcal{H}}$ that coincide with $\det P = \det P'$ on $\underline{\mathcal{H}}_0$ (the other one being $\alpha\varepsilon$). Define $\mathcal{H} \subset N \subset \mathrm{Spin}(7)$ as the inverse image of $\underline{\mathcal{H}}$. The characters α and ε may be viewed as characters of $\mathcal{H}$ by inflation, and we set $\mathcal{H}_0 = \ker \alpha$. Consider the element $n \in N \smallsetminus N^0$ acting trivially on P' and Q, and as a reflection on P. Observe that n normalizes $\mathcal{H}$ by preserving $\mathcal{H}_0$ (hence the character α) and exchanges ε and $\alpha\varepsilon$, as we have $\alpha(\theta n \theta^{-1} n^{-1}) = -1$. We have $F \simeq Q \oplus \mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} P$ as $\mathbb{R}[\mathcal{H}]$-modules. We denote by $\rho : \mathcal{H} \to \mathrm{Spin}(7)$ the natural inclusion.

Theorem 7.4. *The pair (ρ, ε) is unacceptable of type* IIIa *with respect to α. Conversely, if we have $r : \Gamma \to \mathrm{Spin}(7)$ and $\eta : \Gamma \to \{\pm 1\}$ with (r, η) unacceptable of type* IIIa *with respect to χ, then up to replacing r with a conjugate there is a morphism $f : \Gamma \to \mathcal{H}$ with $\rho \circ f = r$, $\chi = \alpha \circ f$ and $\eta = \varepsilon \circ f$.*

Proof. The characters α and ε of $\mathcal{H}$ are distinct and non trivial. Set $\rho_0 = \rho_{|\mathcal{H}_0}$ and $\varepsilon_0 = \varepsilon_{|\mathcal{H}_0}$. By Proposition 3.6, the morphisms ρ_0 and $\varepsilon_0\rho_0$ are Spin(7)-conjugate, since $\mathrm{X}(\rho_0)$ contains $\varepsilon_0 = \det P$. That (ρ, ε) satisfies (U1) follows then from Lemma 6.3 (i). The irreducible sub-representations of the $\mathbb{R}[\mathcal{H}]$-module F are clearly Q and $\mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} P$, whose determinant are respectively α and 1. We have thus $\mathrm{X}(\rho) = \{1, \alpha\}$ by Proposition 3.6, $\mathrm{Y}(\rho) = \{\alpha\}$, and (ρ, ε) satisfies (U2) as well since $\varepsilon \notin \mathrm{X}(\rho)$: it is unacceptable, of type IIIa with respect to α by construction.

Suppose conversely that (r, η) is unacceptable of type IIIa with respect to χ. Choose $z \in \Gamma \smallsetminus \Gamma_0$. By Lemmas A.4 (ii) and A.3, there is an $\mathbb{R}[\Gamma_0]$-submodule $V_0 \subset F$ of dimension 2 and determinant $\eta_{|\Gamma_0}$ with $F = V_0 \perp zV_0 \perp Q'$, and Q' is Γ-stable with determinant χ. Choose $g \in \mathrm{Spin}(7)$ with $g(V_0) = P$, $g(zV_0) = P'$ and $g(Q') = Q$, we have $g\, r(\Gamma)\, g^{-1} \subset \rho(\mathcal{H})$. Up to replacing r with its g conjugate, the morphism $f = \rho^{-1} \circ r$ satisfies $\rho \circ f = r$, $\chi = \alpha \circ f$ and $\eta_{|\Gamma_0} = \varepsilon \circ f_{|\Gamma_0}$. The last condition implies $\varepsilon \circ f = \eta$ or $\eta\chi$. So either f, or the conjugate of f by the element n defined above, has all the required properties. ☐

Remark 7.5. *Of course, if we have a morphism $f : \Gamma \to \mathcal{H}$, and if we set $r = \rho \circ f$, $\chi = \alpha \circ f$ and $\eta = \varepsilon \circ f$, then (r, η) satisfies* (U1) *but not automatically* (U2)*, as we may have $\eta \in \mathrm{X}(r)$ even if $\varepsilon \notin \mathrm{X}(\rho)$. We leave to the reader a study of the condition* (U2) *in the spirit of that made in Proposition 5.13.*

Remark 7.6. (*An alternative description of $\mathcal{H}$*) By definition, we have

$$\mathcal{H} \subset \mathrm{Spin}(A) \cdot \mathrm{Spin}(B), \text{ with } A = L \perp Q \text{ and } B = P \perp P',$$

and thus $\mathrm{Spin}(A) \simeq \mathrm{SU}(2)$ and $\mathrm{Spin}(B) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$. The neutral component $\mathcal{H}^0$ of $\mathcal{H}$ is a maximal torus of $\mathrm{Spin}(A) \cdot \mathrm{Spin}(B)$. Let T be the diagonal torus of $\mathrm{SU}(2)$, C the normalizer of T in $\mathrm{SU}(2)$ and $s : C \to \{\pm 1\}$ the order 2 character with kernel T. There is a morphism $\xi : \mathrm{SU}(2)^3 \to \mathrm{Spin}(7)$ with $\xi(\mathrm{SU}(2) \times 1) = \mathrm{Spin}(A)$, $\xi(1 \times \mathrm{SU}(2)^2) = \mathrm{Spin}(B)$ and $\xi(T^3) = \mathcal{H}^0$. We have $\ker \xi = \langle(-1_2, -1_2, -1_2)\rangle \simeq \mathbb{Z}/2\mathbb{Z}$. The group $\mathcal{H}$ has index 2 in $\xi(C^3)$ and a simple computation shows

$$\mathcal{H} = \xi(H) \quad \text{with} \quad H = \{(c_1, c_2, c_3) \in C^3 \mid s(c_1)s(c_2)s(c_3) = 1\}. \tag{13}$$

Moreover, if $s_i : H \to \{\pm 1\}$ denotes the character $(c_1, c_2, c_3) \mapsto s(c_i)$, we have $\alpha \circ \xi = s_1$ and $\varepsilon \circ \xi = s_2$ or s_3.

Example 4

Consider an orthogonal decomposition $F = T \perp T'$ with $\dim T = \dim T' = 3$. Define $\underline{\mathcal{I}} \subset \mathrm{O}(F)$ as the subgroup of isometries that:

– preserve the pair $\{T, T'\}$, and

– whose signature on $\{T, T'\}$ coincides with their determinant on F.

Its neutral component is $\underline{\mathcal{I}}^0 = \mathrm{SO}(T) \times \mathrm{SO}(T') \simeq \mathrm{SO}(3) \times \mathrm{SO}(3)$ and we have

$$\underline{\mathcal{I}}/\underline{\mathcal{I}}^0 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{14}$$

Indeed, $\underline{\mathcal{I}}$ is generated by any order 2 element in $\mathrm{O}(F)$ exchanging T and T' (those elements have determinant -1) and by its index 2 subgroup

$$\underline{\mathcal{I}}_0 = \{(g_1, g_2) \in \mathrm{O}(T) \times \mathrm{O}(T') \mid \det g_1 = \det g_2\} \simeq (\mathrm{O}(3) \times \mathrm{O}(3))^{\det_1 = \det_2}$$

Set $\alpha = \det F$, so that we also have $\underline{\mathcal{I}}_0 = \ker \alpha$. Denote by ε any of the two characters of $\underline{\mathcal{I}}$ that coincide with $\det T = \det T'$ on $\underline{\mathcal{I}}_0$ (the other one being then $\alpha\varepsilon$); we have $\varepsilon^2 = 1$ by (14). Define $\mathcal{I} \subset N \subset \mathrm{Spin}(7)$ as the inverse image of $\underline{\mathcal{I}}$. The characters α and ε may be viewed as characters of $\mathcal{I}$ by inflation, and we set $\mathcal{I}_0 = \ker \alpha$. We also have $F \simeq \mathrm{Ind}_{\mathcal{I}_0}^{\mathcal{I}} T$ as $\mathbb{R}[\mathcal{I}]$-modules (irreducible). Denote by $\rho : \mathcal{I} \to \mathrm{Spin}(7)$ the natural inclusion. The proof of the following theorem is similar to that of Theorem 7.4.

Theorem 7.7. *The pair (ρ, ε) is unacceptable of type* IIIb *with respect to α. Conversely, if we have $r : \Gamma \to \mathrm{Spin}(7)$ and $\eta : \Gamma \to \{\pm 1\}$ with (r, η) unacceptable of type* IIIb *with respect to χ, then up to replacing r with a conjugate there is a morphism $f : \Gamma \to \mathcal{I}$ with $r = \rho \circ f$, $\chi = \alpha \circ f$ and $\eta = \varepsilon \circ f$.*

We end this section with several remarks. First, an analogue of Remark 7.5 also applies in this context. Moreover:

Remark 7.8. *The pair (ρ, ε) in Theorem 7.7 is an unacceptable morphism with biggest possible image, of dimension 6. Using that theorem and the well-known classification of closed subgroups of* $\mathrm{SO}(3)$*, we easily find examples of unacceptable*

$\mathrm{Spin}(7)$-valued morphisms r of type IIIb *such that $\pi \circ r(\Gamma)$ is an extension of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by $H \times H$ with $H \simeq \mathrm{A}_4$, S_4 or A_5.*

We leave as an exercise to the reader to verify that the following alternative description of $\mathcal{I}$ holds.

Remark 7.9. (*An alternative description of $\mathcal{I}$*) There are closed normal subgroups μ and O of $\mathcal{I}$, with $\mu \simeq \mu_4$ and $\mathrm{O} \simeq \mathrm{O}(4)$, as well as a decomposition

$$\mathcal{I} = \mu \cdot \mathrm{O} \text{ with } \mu \cap \mathrm{O} = \{\pm 1\}, \tag{15}$$

such that if i is a generator of μ, then for all $g \in \mathrm{O}$ we have

$$g\, i\, g^{-1} = \det(g)\, i.$$

We also have $\alpha_{|\mu} = 1$, $\alpha_{|\mathrm{O}} = \det$ and $\varepsilon_{|\mu} \neq 1$.

Remark 7.10. (*Morphisms of both types* IIIa *and* IIIb) We mention that there is an example of a triple (r, η, χ) such that (r, η) is unacceptable of both types IIIa and IIIb with respect to χ. Furthermore, this example is unique in a natural sense. We omit the details, but simply say that in this example, we have $\pi(r(\Gamma)) \simeq \mathrm{D}_8 \times \mu_2$ and the $\mathbb{R}[\Gamma]$-module F is isomorphic to $P \,\oplus\, \eta \otimes P \,\oplus\, \det P \,\oplus\, \chi \det P$, with P irreducible and $\dim P = 2$.

8. A few examples and properties in the Weil group case

In this section, p is a prime, F denotes a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, and q is the cardinality of the residue field of F (a p-th power). We denote by W_F the Weil group of F (see [Tat79]), a locally compact topological group, and we consider the unacceptable continuous group morphisms $\mathrm{W}_F \to \mathrm{Spin}(7)$. As already mentioned in the introduction, these morphisms are of interest for the approach toward a local Langlands correspondence for the group $\mathrm{PGSp}_6(F)$ discussed in [GS22]. We start with a simple lemma.

Lemma 8.1. *Assume Γ is a topological group and $r : \Gamma \to \mathrm{Spin}(n)$ is continuous. Then the elements of $\mathrm{X}(r)$ and of $\mathrm{E}(r)$ are continuous characters of Γ.*

Proof. Recall the morphism $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. The assertion about $\mathrm{X}(r)$ is clear as π is continuous. The one about η follows from the fact that there is an open neighborhood U of $1 \in \mathrm{Spin}(n)$ such that for all $g \in U$, $-g$ is not conjugate to g. Indeed, there is a neighborhood U of 1 satisfying $\mathrm{id}_E + \pi(U) \subset \mathrm{GL}(E)$ and we conclude by Lemma 3.10 (or Remark 3.11 for n even). $\square$

That being said, we only consider *continuous* morphisms or characters from now on, without further mention. As is well-known, we have

$$\mathrm{Hom}(\mathrm{W}_F, \{\pm 1\}) \simeq F^\times/F^{\times,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{\delta_F} \text{ with } \delta_F = \begin{cases} 2 & \text{if } p \text{ is odd,} \\ 2+[F:\mathbb{Q}_2] & \text{for } p=2. \end{cases} \tag{16}$$

In particular, we always have $\delta_F \geq 2$, a necessary condition for the existence of unacceptable morphisms by Corollary 5.7.[4] If E is a finite extension of F, we will denote by $\mathrm{N}_{E/F} : E^\times \to F^\times$ the norm morphism and by $\mathrm{S}^1(E/F)$ its kernel, a compact subgroup of $E^\times$.

Lemma 8.2. *Let E be a quadratic extension of F and $\epsilon : F^\times \to \mathbb{C}^\times$ a character. For any integer $n \geq 1$, there is a character $E^\times \to \mathbb{C}^\times$ whose restriction to $F^\times$ is ϵ and whose restriction to $\mathrm{S}^1(E/F) \subset E^\times$ has order divisible by p^n.*

Proof. Set $S = \mathrm{S}^1(E/F)$. The subgroup $F^\times \cdot S$ of $E^\times$ is open, of finite index, and we have $F^\times \cap S = \{\pm 1\}$. We may write $S = S_{\mathrm{tor}} \cdot S_f$ with $S_{\mathrm{tor}} \subset S$ the finite torsion subgroup and $S_f \simeq \mathbb{Z}_p^d$ with $d = [F : \mathbb{Q}_p]$. We extend first ϵ to a character ϵ' of $F^\times \cdot S_{\mathrm{tor}}$. As we have $(F^\times \cdot S_{\mathrm{tor}}) \cap S_f = \{1\}$, we may extend ϵ' to a character ϵ'' of $F^\times \cdot S$ so that $\epsilon''_{|S_f}$ has order p^n. Any extension of ϵ'' to $E^\times$ does the trick. □

Recall the *reciprocity isomorphism* $\mathrm{rec}_F : F^\times \xrightarrow{\sim} \mathrm{W}_F^{\mathrm{ab}}$ from local class field theory [TAT79, §1.1]. For any quadratic extension E/F we denote by $\mathrm{sgn}_{E/F} : \mathrm{W}_F \to \{\pm 1\}$ the order 2 character with kernel W_E. The kernel of $\mathrm{sgn}_{E/F} \circ \mathrm{rec}_F$ is the index 2 subgroup $\mathrm{N}_{E/F}(E^\times)$ of $F^\times$. If $c : E^\times \to \mathbb{C}^\times$ is a character, we consider the induced representation $\mathrm{I}(c) := \mathrm{Ind}_{\mathrm{W}_E}^{\mathrm{W}_F} c \circ \mathrm{rec}_E^{-1}$. For general reasons we have

$$\mathrm{I}(c) \simeq \mathrm{I}(c)^* \otimes \det \mathrm{I}(c) \quad \text{and} \quad \mathrm{I}(c) \simeq \mathrm{I}(c) \otimes \mathrm{sgn}_{E/F}. \tag{17}$$

By [TAT79, §1.1 (W3)] the transfer of $c \circ \mathrm{rec}_E^{-1}$ to W_F is $c_{|F^\times} \circ \mathrm{rec}_F^{-1}$, so we have

$$\det \mathrm{I}(c) \circ \mathrm{rec}_F = \mathrm{sgn}_{E/F} \circ \mathrm{rec}_F \cdot c_{|F^\times} \tag{18}$$

by Lemma A.2 (ii).

Proposition 8.3. *There exist continuous morphisms $r : \mathrm{W}_F \to \mathrm{Spin}(7)$ which are unacceptable of type* I *and of arbitrary large finite image.*

Proof. By (16) we may choose two different quadratic extensions E and E' of F. Set $s = \mathrm{sgn}_{E/F}$ and $s' = \mathrm{sgn}_{E'/F}$, two order 2 characters of W_F. The character $s'' := ss'$ is the character $\mathrm{sgn}_{E''/F}$ of the third quadratic extension E'' of F in the compositum $E \cdot E'$. By Lemma 8.2, we may choose a character $c : E^\times \to \mathbb{C}^\times$ with $c_{|F^\times} = s' \circ \mathrm{rec}_F$ and of arbitrary large finite order over $\mathrm{S}^1(E/F)$. Such a character has finite image as s' has this property and $F^\times \cdot \mathrm{S}^1(E/F)$ has finite index in $E^\times$.

[4] For the Archimedean local fields $F = \mathbb{R}$ or $F = \mathbb{C}$, this same corollary also shows that any continuous morphism $\mathrm{W}_F \to \mathrm{Spin}(7)$ is acceptable.

By (18) we have $\det \mathrm{I}(c) = ss' = s''$. By (17) we deduce $\mathrm{I}(c)^* \simeq \mathrm{I}(c) \otimes s'$. Using $s' \neq \det \mathrm{I}(c)$, Lemma 5.9 (i) shows that $\mathrm{I}(c)$ defines a morphism $r : \mathrm{W}_F \to \mathrm{O}(2)^{\pm}$ with $\mu \circ r = s'$, $\det \circ r = s''$ and thus $\epsilon \circ r = s$. The group $r(\mathrm{W}_F)$ is not isomorphic to D_8 as long as we choose c to have order > 2. By exchanging the roles of E and E' we may also find a morphism $r' : \mathrm{W}_F \to \mathrm{O}(2)^{\pm}$ with $\mu \circ r = s$, $\det \circ r = s''$, $\epsilon \circ r = s'$ and $r'(\mathrm{W}_F) \not\simeq \mathrm{D}_8$. The pair $\rho := (r, r')$ so defined is a morphism $\rho : \mathrm{W}_F \to \mathcal{G}$ with $\nu \circ \rho = s$ and $\mathrm{d} \circ \rho = s''$. By Proposition 5.10, the pair (ρ, s) is unacceptable of type I. As the order of c is arbitrary large, so is the cardinality of the finite group $\rho(\mathrm{W}_F)$. □

As a consequence, there are always plenty of unacceptable morphisms $\mathrm{W}_F \to \mathrm{Spin}(7)$. An interesting question, from the point of view of the Langlands classification, is the existence of *discrete* unacceptable morphisms $\mathrm{W}_F \to \mathrm{Spin}(7)$ (see Definition 6.7).

Proposition 8.4. (a) *For any group* Γ*, there is no type* I *discrete unacceptable morphism* $\Gamma \to \mathrm{Spin}(7)$*.*

(b) *When* p *is odd, there is no type* II *discrete unacceptable morphism* $\mathrm{W}_F \to \mathrm{Spin}(7)$

Proof. (a) If $r : \Gamma \to \mathrm{Spin}(7)$ unacceptable of type I, the $\mathbb{R}[\Gamma]$-module E contains some order 2 character with multiplicity 2 by Corollary 5.14, so the centralizer of $r(\Gamma)$ in $\mathrm{Spin}(7)$ contains some $\mathrm{Spin}(2) \simeq \mathrm{S}^1$ (infinite).

(b) For p odd, there is no surjective morphism $F^\times \to (\mathbb{Z}/2\mathbb{Z})^3$ by (16), so the second assertion follows from Proposition 6.8. □

On the other hand, there are always plenty of unacceptable discrete morphisms $\mathrm{W}_F \to \mathrm{Spin}(7)$ of type III.

Proposition 8.5. *There exist discrete continuous morphisms* $r : \mathrm{W}_F \to \mathrm{Spin}(7)$ *which are unacceptable of type* IIIa *and of arbitrary large finite image.*

Proof. Let K be a Galois extension of F with $\mathrm{Gal}(K/F) \simeq (\mathbb{Z}/2\mathbb{Z})^2$, and let F_1, F_2 and F_3 be the three quadratic extensions of F inside K. For each $1 \leq i \leq 3$ we may choose by Lemma 8.2 a character $c_i : F_i^\times \to \mathbb{C}^\times$ with $(c_i)_{|F^\times} = \mathrm{sgn}_{F_i/F} \circ \mathrm{rec}_F$ and whose restriction to $\mathrm{S}^1(F_i/F)$ has an arbitrarily high order > 4. As $F^\times \cdot \mathrm{S}^1(F_i/F)$ has finite index in $F_i^\times$, the image of c_i is finite. The representation $\mathrm{I}(c_i)$ of W_F has determinant 1 by Formula (18), so each r_i defines a morphism $r_i : \mathrm{W}_F \to \mathrm{SU}(2)$.

We now use the description of the group $\mathcal{H}$ given in Remark 7.6. We borrow the notations T, C, H, s, s_i, ξ of this remark. Up to conjugating r_i we may assume $r_i(\mathrm{W}_{F_i}) \subset T$, which forces $r_i(\mathrm{W}_F) \subset C$ as we have $c_i^2 \neq 1$. The composition $\mathrm{W}_F \xrightarrow{r_i} C \xrightarrow{s} \{\pm 1\}$ is $\mathrm{sgn}_{F_i/F}$ by construction, and we have the identity

$$\mathrm{sgn}_{F_1/F}\, \mathrm{sgn}_{F_2/F}\, \mathrm{sgn}_{F_3/F} \;=\; 1,$$

so that $r := (r_1, r_2, r_3)$ defines a morphism $r : \mathrm{W}_F \to H$. We now consider $\xi \circ r : \mathrm{W}_F \to \mathcal{H}$. We have $\varepsilon \circ \xi \circ r = \mathrm{sgn}_{F_2/F}$ and $\alpha \circ \xi \circ r = \mathrm{sgn}_{F_1/F}$. Then $(\xi \circ r, \mathrm{sgn}_{F_2/F})$ satisfies (U1) by the first assertion of Theorem 7.4, and if we can show it satisfies (U2), then it will be of type IIIa with respect to $\mathrm{sgn}_{F_1/F}$. So it only remains to show that (U2) holds, *i.e.* $\mathrm{sgn}_{F_2/F} \notin \mathrm{X}(\xi \circ r)$.

The $\mathbb{R}[H]$-module E has an (absolutely) irreducible decomposition of the form $E = \mathrm{s}_1 \oplus Q \oplus S$ with $\dim Q = 2$, $\det Q = s_1$, $\dim S = 4$ and $\det S = 1$ (we use the letter S here instead of F since the later already denotes the local field). As we have $\mathrm{sgn}_{F_1/F} \neq \mathrm{sgn}_{F_2/F}$ it is enough to show that S and Q are absolutely irreducible as $\mathbb{R}[\mathrm{W}_F]$-modules. This is clear for Q as we have $\det Q \neq 1$ and the image of W_F in $\mathrm{O}(Q)$ has order > 4 by assumption on c_1. So we now deal with S. For each $1 \leq i \leq 3$ define a character $a_i : K^\times \to \mathbb{C}^\times$ by $a_i = c_i \circ \mathrm{N}_{K/F_i}$. By the basic properties of the reciprocity morphisms, we have $\mathrm{I}(c_i)_{|\mathrm{W}_K} \simeq a_i \oplus a_i^{-1}$, and then a $\mathbb{C}[\mathrm{W}_K]$-module isomorphism

$$S \otimes \mathbb{C} \simeq a_2 a_3 \oplus (a_2 a_3)^{-1} \oplus a_2^{-1} a_3 \oplus a_2 a_3^{-1}.$$

As each of s_1, s_2 and s_3 is nontrivial over W_F, the four characters $a_2^{\pm 1} a_3^{\pm 1}$ are conjugate under W_F. It is thus enough to show that they are distinct, or equivalently, that a_2^2, a_3^2 and $(a_2 a_3)^2$ are all nontrivial over $K^\times$. Denote by $\sigma_i \in \mathrm{Gal}(K/F)$ the order 2 element fixing pointwise F_i. For $x \in K^\times$ we have $a_i(x) = c_i(x \sigma_i(x))$ by definition, hence $a_{i|F_i^\times} = c_i^2$, which has order > 2 by assumption on c_i. Last but not least, for $x \in \mathrm{S}^1(F_2/F) \subset F_2^\times \subset K^\times$ we have just seen $a_2(x) = c_2(x)^2$, but we also have $\tau_3(x) = x^{-1}$, hence $a_3(x) = c_3(x x^{-1}) = 1$, and thus the identity

$$(a_2 a_3)_{|\mathrm{S}^1(F_2/F)} = (c_2^2)_{|\mathrm{S}^1(F_2/F)}.$$

We conclude as the character on the right has order > 2 by assumption. □

From the point of view of Langlands' theory of endoscopy, a more specific class of discrete morphisms $r : \mathrm{W}_F \to \mathrm{Spin}(n)$ is of interest, namely those such that the centralizer of $r(\mathrm{W}_F)$ in $\mathrm{Spin}(n)$ is the center of $\mathrm{Spin}(n)$, hence *as small as possible.* We call *stable* such a morphism, a meaningful terminology from the point of view of endoscopy. It is easy to see that the discrete type III examples of Proposition 8.5 are not stable (use Lemma 8.7 (i) below). For p odd, even more is happily true:

Proposition 8.6. *For p odd, there is no discrete, stable and unacceptable morphism $r : \mathrm{W}_F \to \mathrm{Spin}(7)$.*

Proof. Set $\Gamma = \mathrm{W}_F$ and let $r : \Gamma \to \mathrm{Spin}(7)$ be discrete, stable and unacceptable. By (16) and Lemma 8.7 (ii) below, the $\mathbb{R}[\Gamma]$-module E has at most 2 irreducible summands. An immediate inspection of E in types I, II and III (Formula (11), Corollary (5.14) and Proposition-Definition (7.3)) shows that the only possibility is that r is of type IIIb. More precisely, there are 2 characters $\chi, \eta : \Gamma \to \{\pm 1\}$ as well as real 3-dimensional, necessarily irreducible, representation V_0 of the kernel Γ_0 of χ, such that we have an $\mathbb{R}[\Gamma]$-module decomposition

$$E \simeq \chi \oplus \mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0.$$

So V_0 gives rise to an irreducible representation $\Gamma_0 \to \mathrm{O}(3)$, and after twisting it by $\det V_0$, to an irreducible representation $\Gamma_0 \to \mathrm{SO}(3)$. But we know since Klein that the finite irreducible subgroups of $\mathrm{SO}(3)$ are isomorphic to A_4, S_4 and A_5. But it is well-known that none of these groups can be the Galois group of a finite Galois extension of p-adic fields with p odd [Wei74, §13]. □

Lemma 8.7. *Assume $r : \Gamma \to \mathrm{Spin}(n)$ is discrete and stable with n odd.*

(i) *There is no $\mathbb{R}[\Gamma]$-submodule $\{0\} \subsetneq V \subsetneq E$ with $\det V = 1$.*

(ii) *If r is furthermore unacceptable, then either E has ≤ 2 irreducible summands, or there is a surjective group morphism $\Gamma \to (\mathbb{Z}/2\mathbb{Z})^3$.*

Proof. To prove assertion (i), assume we have a nontrivial $\mathbb{R}[\Gamma]$-stable decomposition $E = A \perp B$. Then we have $\det A = \det B$, and if this character is trivial the group $r(\Gamma)$ falls inside the subgroup $\mathrm{Spin}(A) \cdot \mathrm{Spin}(B)$ of $\mathrm{Spin}(A, B)$, whose center strictly contains $\{\pm 1\}$. For assertion (ii), assume we have a Γ-stable decomposition $E = E_1 \oplus E_2 \oplus E_3$ with E_i nonzero for each i. By assertion (i), the three characters $\det E_i$ are nontrivial, distinct, and of course in $\mathrm{X}(r)$. But by the unacceptability of r there is another order 2 character η of Γ, with $\eta \notin \mathrm{X}(r)$. □

The reader aware of Langlands' parameterizations knows that we have to consider more generally continuous morphisms $\mathrm{W}_F \times \mathrm{SU}(2) \to \mathrm{Spin}(7)$. We did not emphasize this extra $\mathrm{SU}(2)$ earlier because of the following proposition.

Proposition 8.8. *Let W be any group and assume $r : W \times \mathrm{SU}(2) \to \mathrm{Spin}(7)$ is a morphism whose restriction to $\mathrm{SU}(2)$ is continuous and nontrivial. Then r is acceptable.*

Proof. Set $\Gamma = W \times \mathrm{SU}(2)$ and assume r is unacceptable by contradiction. There are no non trivial continuous morphisms from $\mathrm{SU}(2)$ to $\mathcal{G}$ or $\mathcal{H}$, since those two groups have an abelian neutral component. By Theorems 5.8 and 7.4, the morphism r is neither of type I, nor of type IIIa. So there are characters $\chi, \eta : \Gamma \to \{\pm 1\}$ such that (r, η) is of type II or IIIb with respect to χ. A character of Γ is necessarily trivial over $1 \times \mathrm{SU}(2)$. So the kernel Γ_0 of χ has the form

$$\Gamma_0 = W_0 \times \mathrm{SU}(2)$$

for some index 2 subgroup W_0 of W. By Lemma 6.3 (ii), if $r_{|\Gamma_0}$ is unacceptable then it is of type I, which contradicts the previous paragraph applied to $r_{|W_0 \times \mathrm{SU}(2)}$, so (r, η) is of type IIIb. By Definition-Proposition 7.3, we have

$$E \simeq \chi \oplus \mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0$$

for some $\mathbb{R}[\Gamma_0]$-module V_0 with $\dim V_0 = 3$ and $\det V_0 = \eta_{|\Gamma_0}$. If $1 \times \mathrm{SU}(2)$ acts trivially on V_0, then it acts trivially as well on E, a contradiction. But then it must act absolutely irreducibly on V_0, as we have $\dim V_0 = 3$. As $W_0 \times 1$ centralizes $1 \times \mathrm{SU}(2)$, it must act by multiplication by a real character on V_0, necessarily equal to $\eta_{|\Gamma_0}$ by taking the determinant. So we have $V_0 \simeq \eta_{|W_0} \boxtimes U$, and then

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0 \simeq (\mathrm{Ind}_{W_0}^{W} \eta_{|W_0}) \boxtimes U \simeq \eta_{|W} \boxtimes U \oplus (\chi\eta)_{|W} \boxtimes U.$$

The factor $\eta_{|W} \boxtimes U$ has determinant η, contradicting the unacceptability of r. $\square$

9. The GSpin(n) variant

In this section, n is any integer ≥ 1 and we explain how the previous results can be applied to study the unacceptable $\mathrm{GSpin}(n)$-valued morphisms. Recall that $\mathrm{GSpin}(n)$ is the compact subroup of $\mathrm{Cl}(E)^{\times}$ generated by its unit scalar subgroup $Z \simeq \mathrm{U}(1)$ and $\mathrm{Spin}(n)$ (see Sect. 2). So Z is central in $\mathrm{GSpin}(n)$ and we have

$$\mathrm{GSpin}(n) = Z \cdot \mathrm{Spin}(n) \text{ and } Z \cap \mathrm{Spin}(n) = \{\pm 1\}. \tag{19}$$

The morphism $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ defined *loc. cit.* extends to a morphism $\mathrm{GSpin}(n) \to \mathrm{SO}(n)$ with kernel Z and still denoted by π. We fix a morphism $r : \Gamma \to \mathrm{GSpin}(n)$ and set

$$\Gamma(r) = \{(\gamma, \sigma) \in \Gamma \times \mathrm{Spin}(n) \mid \pi(r(\gamma)) = \pi(\sigma)\}.$$

The first projection $\Gamma(r) \to \Gamma$ is surjective with kernel $\{(1, \pm 1)\}$; any morphism f with source Γ can be thus inflated to a morphism $\widetilde{f}$ with source $\Gamma(r)$ using this surjection. We also have two natural morphisms $r_S : \Gamma(r) \to \mathrm{Spin}(n)$ and $r_Z : \Gamma(r) \to Z$, defined by $r_S(\gamma, \sigma) = \sigma$ and $r_Z(\gamma, \sigma) = r(\gamma)\sigma^{-1}$, and satisfying

$$\widetilde{r}(g) = r_Z(g)\, r_S(g), \ \forall g \in \Gamma(r). \tag{20}$$

We denote by $\mathrm{G}(r)$ the set of morphisms $\Gamma \to \mathrm{GSpin}(n)$ which are element conjugate to r, and by $\mathrm{S}(r)$ the set of morphisms $\Gamma(r) \to \mathrm{Spin}(n)$ which are element conjugate to r_S. Of course, we have $r \in \mathrm{G}(r)$ and $r_S \in \mathrm{S}(r)$. The group $\mathrm{GSpin}(n)$ naturally acts on $\mathrm{G}(r)$ and $\mathrm{S}(r)$ by conjugation.

Proposition 9.1. *Let $r : \Gamma \to \mathrm{GSpin}(n)$ be a fixed morphism. There is a natural, $\mathrm{GSpin}(n)$-equivariant, bijection $b : \mathrm{G}(r) \xrightarrow{\sim} \mathrm{S}(r)$ satisfying $b(r) = r_S$. In particular, r is unacceptable if, and only if, r_S is unacceptable.*

Since we have $\widetilde{\pi \circ r} = \pi \circ r_S$ by Formula (20), most of what we have done for $\mathrm{Spin}(n)$-valued morphisms will apply to $\mathrm{GSpin}(n)$-valued ones by this proposition. We refer to the proof of Proposition 10.2 for a concrete example.

Proof. We follow a construction in the proof of [Lar94, Prop.1.4]. For $r' \in \mathrm{G}(r)$ and $g = (\gamma, \sigma) \in \Gamma(r)$, we set $b(r')(g) := r_Z(g)^{-1} r'(\gamma) \in \mathrm{GSpin}(n)$. This element is $\mathrm{Spin}(n)$-conjugate to $r_S(g) = r_Z(g)^{-1}r(\gamma)$ by Formulas (20) and (19). As $b(r')$ is a group morphism $\Gamma(r) \to \mathrm{GSpin}(r)$, we have $b(r') \in \mathrm{S}(r)$ and the identity

$$(21) \qquad \widetilde{r'}(g) = r_Z(g)\, b(r')(g), \ \ \forall g \in \Gamma(r).$$

Conversely, for any $f \in \mathrm{S}(r)$ and $g \in \Gamma(r)$ the element $r_Z(g)f(g) \in \mathrm{GSpin}(n)$ is conjugate to $\widetilde{r}(g) = r_Z(g)r_S(g)$ by assumption. In particular, it is trivial on $(1,-1)$, and so there is a unique $r' \in \mathrm{G}(r)$ with $b(r') = f$. We clearly have $b(gr'g^{-1}) = gb(r')g^{-1}$ for all $r' \in \mathrm{G}(r)$ and $g \in \mathrm{GSpin}(n)$. The last assertion follows since $\mathrm{GSpin}(n)$-conjugacy and $\mathrm{Spin}(n)$-conjugacy coincide in $\mathrm{Spin}(n)$ by Formula (19). □

10. An application in the non compact case

In this last section, we fix an *algebraically closed field k that embeds into $\mathbb{C}$*. Our aim is to give some statement about morphisms[5] $\Gamma \to \mathrm{GSpin}_n(k)$ in the spirit of [KS, §4 & 5] and that follows from our results. This forces us to discuss an analogue of the acceptability condition for morphisms to linear algebraic k-groups that is useful in practice but which slightly differs from the case of morphisms to compact groups. In the proof of Proposition 10.2 below, and in Proposition 10.1, we will see how to pass from a setting to the other.

Let G be a linear algebraic k-group, Γ an arbitrary group and $r : \Gamma \to G$ a group morphism.[6] We denote by $\mathrm{Zar}(r)$ the Zariski closure of $r(\Gamma)$ in G. Recall that r is called *semi-simple* if $\mathrm{Zar}(r)$ is reductive. [7] Two elements g, g' of G are called *ss-conjugate* if the semi-simple parts in their Jordan decompositions are conjugate, or equivalently, if g and g' have the same trace in any algebraic k-linear representation of G. This equivalence relation on G is thus Zariski-closed in $G \times G$. Two morphisms $r, r' : \Gamma \to G$ are called *element ss-conjugate* if, for each $\gamma \in \Gamma$, $r(\gamma)$ and $r'(\gamma)$ are *ss-conjugate*. Finally, we say that a morphism $r : \Gamma \to G$ is *ss-acceptable* if, for each $r' : \Gamma \to G$ element ss-conjugate to r, then r' is actually G-conjugate to r. The following proposition slightly strengthens [Lar94, Prop. 1.7].

Proposition 10.1. *Let G_1 and G_2 be complex reductive linear algebraic groups, and let K_1 and K_2 be maximal compact subgroups of G_1 and G_2 respectively. Assume $r, r' : G_1 \to G_2$ are two algebraic morphisms with*[8] *$r(K_1) \subset K_2$ and $r'(K_1) \subset K_2$, and consider the two morphisms $r_{|K_1}, r'_{|K_1} : K_1 \to K_2$. Then $r_{|K_1}$ and $r'_{|K_1}$ are element conjugate (resp. K_2-conjugate) if, and only if, r and r' are element ss-conjugate (resp. G_2-conjugate).*

[5] In the applications to Galois representations, such as those in [KS], GSpin-valued morphisms are much more common than Spin-valued morphisms.

[6] We identify a linear algebraic k-group with its group of k-points, as in [Hum98] for instance.

[7] We do not assume that a reductive group is connected.

[8] We can always achieve this property by conjugating r and r' in G_2.

Proof. Consider the Cartan (or *polar*) decomposition $G_2 = K_2P$ of the linear reductive Lie group G_2 with respect to its maximal compact subgroup K_2. Recall that P is a subset of G_2 stable by K_2-conjugacy such that the multiplication $K_2 \times P \to G_2$ is bijective. Assume $gxg^{-1} = y$ with $x, y \in K_2$ and $g \in G_2$, and write $g = pk$ with $k \in K_2$ and $p \in P$. The uniqueness of Cartan decomposition and $kPk^{-1} = P$ show $kxk^{-1} = y$ (and $py = yp$). As any element of K_2 is semi-simple, this shows that $r_{|K_1}$ and $r'_{|K_1}$ are element conjugate (resp. K_2-conjugate) if r and r' are element ss-conjugate (resp. G_2-conjugate). The converse follows from the Zariski density of K_1 in G_1, as "element ss-conjugacy" is a Zariski closed relation in $G_2 \times G_2$. □

Consider the quadratic space $E = k^n$ with quadratic form $x_1^2 + \cdots + x_n^2$. As in §2, we have associated reductive linear algebraic k-groups $\mathrm{SO}(E)$ and $\mathrm{Spin}_n(k) \subset \mathrm{GSpin}_n(k) \subset \mathrm{Cl}(E)^\times$, and a natural surjective morphism $\pi : \mathrm{GSpin}_n(k) \to \mathrm{SO}(E)$.

Proposition 10.2. *Let $r : \Gamma \to \mathrm{GSpin}_n(k)$ be a semi-simple morphism with $n \leq 7$. In the case $n = 7$, we assume that one of the two following conditions hold:*

(i) *The $k[\Gamma]$-module E does not contain any character $c : \Gamma \to k^\times$ with $c^2 = 1$.*

(ii) *The multiplicity of the weight 0 in the $k[\Gamma]$-module E is ≤ 2 and $r(\Gamma)$ contains a non trivial unipotent element.*

Then r is ss-acceptable.

The first part of condition (ii) means that for some (hence any) maximal torus T of $\mathrm{Zar}(r)$, the invariants of T in E have dimension ≤ 2.

Proof. Set $G = \mathrm{GSpin}_n(k)$. Assume $r' : \Gamma \to G$ is element ss-conjugate to r. In order to show that r' is G-conjugate to r, we may assume $k = \mathbb{C}$ by the Nullstellensatz, since k embeds in $\mathbb{C}$ by assumption. In the style of Remark 1.5, up to replacing Γ by the Zariski closure of its image in $r \times r' : \Gamma \to G \times G$, and r and r' by the two projections, we may assume that Γ is a complex reductive linear algebraic group, and that r and r' are injective algebraic morphisms.

The group $\mathrm{GSpin}(n)$ is maximal compact in G. Choose K a maximal compact subgroup of Γ. Up to replacing r and r' by some G-conjugate if necessary, we may assume we have $r(K), r'(K) \subset \mathrm{GSpin}(n)$, and consider $r_{|K}$ and $r'_{|K} : K \to \mathrm{GSpin}(n)$ as in Proposition 10.1. By this proposition, $r_{|K}$ and $r'_{|K}$ are element conjugate, and they are $\mathrm{GSpin}(n)$-conjugate if, and only if, r and r' are G-conjugate. The acceptability of $\mathrm{Spin}(n)$ for $n \leq 6$, hence that of $\mathrm{GSpin}(n)$ by Proposition 9.1, concludes the proof for $n \leq 6$.

We may thus assume $n = 7$ and that the morphism $f := r_{|K}$ is unacceptable. We apply to this $f : K \to \mathrm{GSpin}(7)$ the considerations of §9 and use the notations $K(f)$ and f_S *loc. cit.*, with $\Gamma = K$ and $r = f$. By Proposition 9.1, the morphism $f_S : K(f) \to \mathrm{Spin}(7)$ is unacceptable as well, and $\pi \circ f_S$ factors through $K(f) \to K$ and coincides then with $\pi \circ r_{|K}$. By Theorem 1.3, there is a line L in the Euclidean space $E_{\mathbb{R}} = \mathbb{R}^7$ (a real structure of the complex quadratic space E) on which K acts by a character $K \to \{\pm 1\}$. As K is Zariski dense in Γ, it follows that Γ also acts on $L \otimes \mathbb{C} \subset E$ by such a character, contradicting assumption (i). So we may assume that (ii) holds.

Fix T a maximal torus in K. If f_S is of type I, II or IIIb, it follows from Theorem 5.8, Corollary 6.4 and Theorem 7.7 respectively that the invariants of T in $E_{\mathbb{R}}$ have dimension ≥ 3 (consider the action of a maximal torus of $\mathcal{G}$ and $\mathcal{I}$ on $E_{\mathbb{R}}$). As T is Zariski dense in a maximal torus of Γ, and as r induces an isomorphism $\Gamma \simeq \mathrm{Z}(r)$, this contradicts the first assertion in assumption (ii).

So we may assume f_S is of type IIIa. In this case, it follows from Theorem 7.7 that $\pi(r(K))^0$ is a torus, since $\mathcal{H}^0$ is, so we have $T = K^0$ and $\mathrm{Z}(r)^0$ is a complex torus. But then any element of $\mathrm{Z}(r)$, hence of $r(\Gamma)$, is semi-simple, in contradiction with the second assertion in assumption (ii). $\square$

Example 10.3. *Assume $k = \overline{\mathbb{Q}_\ell}$ is an algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, Γ is the absolute Galois group of $\mathbb{Q}$, and r is a geometric $\mathrm{GSpin}_n(\overline{\mathbb{Q}_\ell})$-valued representation in the sense of Fontaine and Mazur. The first part of assumption (ii) holds for instance if the multiplicity of the Hodge-Tate weight 0 of $\pi \circ r$ is < 3. Under the assumptions of Proposition 10.2, it follows that the collection of conjugacy classes in $\mathrm{GSpin}_n(\overline{\mathbb{Q}_\ell})$ of the semi-simplified Frobenius elements of r determine r up to conjugacy. In the case $n = 7$, this improves* [KS, Prop. 5.2].

Appendix A.

We gather in this appendix a few lemmas that we used. The first is the following folklore variant of the acceptability of $\mathrm{O}(n)$ and $\mathrm{U}(n)$.

Lemma A.1. *Let G be either $\mathrm{U}(n)$ or $\mathrm{O}(n)$ for $n \geq 1$, $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ its tautological representation, and $r, r' : \Gamma \to G$ two group morphisms. Then r and r' are conjugate in G if, and only if, the representations $\rho \circ r$ and $\rho \circ r'$ of Γ are isomorphic. Moreover, the same result holds if G is $\mathrm{SU}(n)$, or $\mathrm{SO}(n)$ with n odd.*

Proof. Two elements $g, g' \in G$ are conjugate in G if, and only if, $\rho(g)$ and $\rho(g')$ have the same characteristic polynomial. The non trivial implication of the statement is then equivalent to the acceptability of G (proved *e.g* in [Lar94]). $\square$

The next lemma is about the transfer morphism to an index 2 subgroup.

Lemma A.2. *Let Γ be a group, $\chi : \Gamma \to \{\pm 1\}$ an order 2 character, $\Gamma_0 \subset \Gamma$ the kernel of χ, c a character of Γ_0 and t the transfer of c to Γ. Then:*

(i) *For all $\gamma \in \Gamma_0$ and $z \in \Gamma \smallsetminus \Gamma_0$, we have $t(z) = c(z^2)$ and $t(\gamma) = c(\gamma z^{-1} \gamma z)$.*

(ii) *If U is a finite dimensional representation of Γ_0 with determinant c, then $\det \mathrm{Ind}_{\Gamma_0}^{\Gamma} U = \chi^{\dim U} t$.*

Proof. Part (i) is straightforward and part (ii) is due to Gallagher [Gal65]. $\square$

The second is about a notion of *orthogonal induction.*

Lemma A.3. *Let V be an Euclidean space, Γ a group, $\rho : \Gamma \to \mathrm{O}(V)$ a representation, $\Gamma_0 \subset \Gamma$ an index 2 subgroup and $z \in \Gamma \smallsetminus \Gamma_0$. Assume that there is a Γ_0-stable subspace $V_0 \subset V$ such that:*

(i) *V_0 is a direct sum of absolutely irreducible representations of Γ_0.*

(ii) *$V = V_0 \oplus zV_0$, or equivalently, the natural morphism $\mathrm{Ind}_{\Gamma_0}^{\Gamma} V_0 \to V$ is an isomorphism.*

Then there is a Γ_0-stable subspace $U_0 \subset V$ which is isomorphic to V_0 as Γ_0-module, and satisfying $V = U_0 \perp zU_0$.

Proof. Consider first the case where V_0 is an absolutely irreducible $\mathbb{R}[\Gamma_0]$-module. We have the Γ_0-stable decompositions $V = V_0 \oplus zV_0$ and $V = V_0 \perp V_0^{\perp}$. If zV_0 is not isomorphic to V_0, the orthogonal projection $zV_0 \to V_0$ is zero, so we have $zV_0 = V_0^{\perp}$ and we are done. Otherwise, the $\mathbb{R}[\Gamma_0]$-module V is isotypical. As we have $\mathrm{End}_{\mathbb{R}[\Gamma_0]}(V_0) = \mathbb{R}$ by assumption, and by the theory of isotypic components, we may assume that V is the tensor product of V_0 and of some Euclidean plane $P \simeq \mathbb{R}^2$, and that the action of Γ_0 on $V = V_0 \otimes P$ is the given one on the first factor, and trivial on the second.

The centralizer of $\rho(\Gamma_0)$ in $\mathrm{O}(V)$ is $1 \otimes \mathrm{O}(P)$, and that of $1 \otimes \mathrm{O}(P)$ is $\mathrm{O}(V_0) \otimes 1$. The element $\rho(z)$ acts on V by normalizing $\rho(\Gamma_0)$, hence by normalizing $1 \otimes \mathrm{O}(P)$ as well. As each automorphism of $\mathrm{O}(P)$ is inner, we may thus write $\rho(z) = \gamma \otimes \delta$ for some $\gamma \in \mathrm{O}(V_0)$ and $\delta \in \mathrm{O}(P)$. As $z^2 \in \Gamma_0$, we have

$$\delta^2 \in (\mathrm{O}(V_0) \otimes 1) \cap (1 \otimes \mathrm{O}(P)) = \{\pm \mathrm{id}_V\}.$$

The proper Γ_0-stable subspaces of V are the $V_0 \otimes v$ for $v \in P$ nonzero. By assumption (ii), there is $v \in P$ such that $\delta(v) \notin \mathbb{R}v$, *i.e.* δ is not a homothety. It follows that either δ is an orthogonal symmetry (case $\delta^2 = 1$), or a rotation of angle $\pi/2$ (case $\delta^2 = -1$). In both cases there is a nonzero $v_0 \in P$ such that v_0 and $\delta(v_0)$ are orthogonal. The $\mathbb{R}[\Gamma_0]$-module $U_0 = V_0 \otimes v_0$ does the trick.

Consider now the general case. Let A be an irreducible $\mathbb{R}[\Gamma_0]$-submodule of V_0. We have $zA \cap A = \{0\}$ by (ii). By applying the first paragraph to $V_1 = A \oplus zA$, we may find a Γ_0-stable $A' \subset V_1$ isomorphic to A and with $V_1 = A' \perp zA'$. Write $V = V_1 \perp V_2$; both V_i are Γ-stable. Let B be an $\mathbb{R}[\Gamma_0]$-module such that $V_0 \simeq A \oplus B$. We must have $V_2 \simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma} B$ by semi-simplicity. By induction on $\dim V$, we may write $V_2 = B' \perp zB'$ with B' a Γ_0-stable subspace of V_2 isomorphic to B as Γ_0-module. The subspace $U_0 = A' \perp B'$ concludes the proof. $\square$

We also used the more specific:

Lemma A.4. *Assume we are in the situation of Proposition 7.3. Then:*

(i) *the* $\mathbb{R}[\Gamma_0]$*-module* F *does not contain* 1 *nor* $\eta_{|\Gamma_0}$*,*

(ii) *the* $\mathbb{R}[\Gamma_0]$*-module* V_0 *is a direct sum of absolutely irreducible representations.*

Proof. Assume that the trivial representation 1_0 of Γ_0 appears in F. Recall that the trivial representation 1 of Γ does not appear in F by definition in types II or III. If 1_0 appears in V_0 (or equivalently, in its outer conjugate by Γ/Γ_0), then $\mathrm{Ind}_{\Gamma_0}^{\Gamma} 1_0 \simeq 1 \oplus \chi$ embeds in F, a contradiction. So we are in type IIIa and 1_0 appears in $Q_{|\Gamma_0}$. But $\det Q$ is 1 on Γ_0, so we have $Q_{|\Gamma_0} \simeq 1_0 \oplus 1_0$ and again $Q \simeq 1 \oplus \chi$. For similar reasons, $\eta_{|\Gamma_0}$ does not occur in F: we have

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma} \eta_{|\Gamma_0} \simeq \eta \otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma} 1 \simeq \eta \oplus \eta\chi$$

and neither η nor $\eta\chi$ appears in F as (r, η) is unacceptable. This proves (i). If S is an irreducible $\mathbb{R}[\Gamma_0]$-submodule of V_0, we have $1 \leq \dim S \leq 3$. If S is not absolutely irreducible, we necessarily have $\dim S = 2$ and $\mathrm{End}_{\mathbb{R}[\Gamma_0]} S = \mathbb{C}$. But in this case we have $\det S = 1$. As $\det V_0 = \eta_{|\Gamma_0} \neq 1$, we have $\dim V_0 = 3$ and so $V_0 \simeq S \oplus \eta_{|\Gamma_0}$, in contradiction with (i). $\square$

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