TRIALITY AND FUNCTORIALITY

GAETAN CHENEVIER AND WEE TECK GAN

Abstract. We use the triality automorphism of simple algebraic groups of type D_4 to prove some new instances of global Langlands functorial lifting. In particular, we prove the (weak) spin lifting from GSp_6 to GL_8 and the tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GSp}_4$ to GL_8. As an arithmetic application, we establish the expected properties of the spinor L-function attached to an arbitrary Siegel modular cusp form for $\mathrm{Sp}_6(\mathbb{Z})$ generating a holomorphic discrete series.

1. Introduction

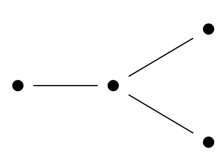
It is a truth universally acknowledged, that a single mathematical object in possession of a good many symmetries compared to its peers, must be in want of our attention.[1]

Taking this principle to heart and putting aside our own *Pride and Prejudice*, we consider in this paper the series D_n of simple Lie groups or Lie algebras, whose Dynkin diagram is given by:

with n equal to the number of vertices. Hence, one may be looking at the split linear algebraic groups SO_{2n} (over some field F) or its Lie algebra $\mathfrak{so}_{2n}$. By Lie theory, the outer automorphism group $\mathrm{Out}(D_n)$ of $\mathfrak{so}_{2n}$ is equal to the group of symmetries of its Dynkin diagram. Thus one finds that

$$\mathrm{Out}(D_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } n \neq 4; \\ \text{the symmetric group } S_3, & \text{if } n = 4, \end{cases}$$

so that the Lie algebra of type D_4 has more symmetries than the generic Lie algebra of type D_n, as is evident from its Dynkin diagram:



2010 *Mathematics Subject Classification.* Primary 11F70, Secondary 22E50.

[1] a tribute to Jane Austen, on the occasion of her 250th anniversaire

In particular, it has an outer automorphism θ of order 3 and this phenomenon is aptly named *triality*. The purpose of this paper is to investigate some implications of the existence of this extra symmetry in the Langlands program.

Let us be more precise. If one considers the special orthogonal group SO_{2n}, then its outer automorphism group is indeed $\mathbb{Z}/2\mathbb{Z}$ for all n, with an outer automorphism realised by an element in $\mathrm{O}_{2n} \setminus \mathrm{SO}_{2n}$. This is because the triality automorphism θ of $\mathfrak{so}_8$ cannot be realised on SO_8, but only on the simply-connected or adjoint form of the group, i.e. the group Spin_8 or PGSO_8 respectively. The center Z_{Spin_8} of Spin_8 is the finite group scheme $\mu_2 \times \mu_2$ which has 3 nontrivial subgroups of order 2. The triality automorphism preserves the center, permuting the 3 subgroups of order 2, so that it descends to an automorphism of the adjoint group PGSO_8. This implies that one has a commutative diagram of isogenies:

$$\begin{tikzcd} & \mathrm{Spin}_8 \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathrm{SO}_8 \arrow[dr] & \mathrm{SO}_8 \arrow[d] & \mathrm{SO}_8 \arrow[dl] \\ & \mathrm{PGSO}_8 & \end{tikzcd}$$

In particular, there are three non-conjugate maps

$$\rho_j : \mathrm{Spin}_8 \twoheadrightarrow \mathrm{SO}_8,$$

whose kernels are the three central μ_2-subgroups in Spin_8. These three maps are cyclically permuted by the triality automorphism, and together give an embedding

$$\rho = \prod_j \rho_j : \mathrm{Spin}_8 \hookrightarrow \mathrm{SO}_8^3.$$

In Section 2, following [KMRT], we describe a construction of the group Spin_8 as a subgroup of SO_8^3, equipped with an outer automorphism of order 3 which is simply the restriction of the cyclic permutation on SO_8^3. The construction in §2 thus gives a realisation of the three 8-dimensional irreducible fundamental representations of Spin_8. Likewise, there are 3 non-conjugate maps

$$f_j : \mathrm{SO}_8 \longrightarrow \mathrm{PGSO}_8$$

which are cyclically permuted by the triality automorphism. The existence of the 3 different maps f_j (or ρ_j) is a manifestation of the principle of triality.

In this paper, we shall consider some implications of the triality automorphism in the Langlands program. In particular, we shall exploit triality to construct some new instances of Langlands functoriality.

1.1. **Langlands functoriality.** We begin with a brief recollection of the notion of Langlands functoriality. Suppose that $\pi = \otimes_v \pi_v$ is an irreducible automorphic representation of a split reductive group H over a number field k. For v outside a finite set S of places of k, the local representation π_v of $H(k_v)$ is unramified and gives rise to a semisimple conjugacy class $c(\pi_v)$ in the Langlands dual group $H^\vee$ via the Satake isomorphism. Thus, π gives rise to a collection of semisimple conjugacy classes

$$c(\pi) := \{c(\pi_v) : v \notin S\}$$

which encodes the Hecke eigenvalues of π for the spherical Hecke algebras at almost all places of k. We consider two such collections of semisimple classes in $H^\vee$ to be equivalent if they agree at all but finitely many places; in other words, we are allowed to enlarge the finite set S if necessary. We shall call (the equivalence class of) $c(\pi)$ the *Hecke-Satake family* of π.

Now suppose that G is another split reductive group and we have an algebraic group homomorphism

$$\iota : H^\vee \longrightarrow G^\vee.$$

Then, by composition with ι, a Hecke-Satake family $c(\pi)$ gives rise to a family of semisimple conjugacy classes

$$\iota(c(\pi)) := \{\iota(c(\pi_v)) : v \notin S\}$$

in $G^\vee$. One may ask if there exists an irreducible automorphic representation σ of G whose Hecke-Satake family agrees with $\iota(c(\pi))$ for almost all v; we shall express this as:

$$\iota(c(\pi)) = c(\sigma).$$

The Langlands functoriality conjecture asserts that the answer is affirmative, in which case one says that the automorphic representation σ is a *weak functorial lifting* of π.

As an example, consider the map of dual groups

$$f_j^\vee : \mathrm{Spin}_8(\mathbb{C}) \longrightarrow \mathrm{SO}_8(\mathbb{C})$$

which is dual to the map

$$f_j : \mathrm{SO}_8 \longrightarrow \mathrm{PGSO}_8.$$

Then the weak functorial lifting associated to $f_j^\vee$ exists and is simply given by any irreducible constituent of the pullback of an automorphic representation by f_j (see Proposition 3.2).

As another example, suppose that H is a symplectic or orthogonal group, so that its Langlands dual group $H^\vee$ is also a classical group. Then $H^\vee$ has a standard representation

$$\mathrm{std} : H^\vee \to \mathrm{GL}_N(\mathbb{C}),$$

which should induce a weak functorial lifting of automorphic representations from H to GL_N. The existence of this weak functorial lifting is highly nontrivial and has been shown by Cogdell-Kim-Piatetski-Shapiro-Shahidi [CKPSS] (for generic automorphic representations) and Cai-Friedberg-Kaplan [CFK] (for all automorphic representations) using Converse Theorems, and independently by Arthur [A] (for all automorphic representations) using the stable twisted trace formula. The most pertinent case for us is $H = \mathrm{SO}_8$ with its standard representation $\mathrm{std} : \mathrm{SO}_8(\mathbb{C}) \longrightarrow \mathrm{GL}_8(\mathbb{C})$.

1.2. **Triality and Langlands functoriality.** We shall see that the principle of triality can be exploited to obtain some interesting results towards Langlands functoriality. The basic idea is the following. Suppose one has a morphism of dual groups

$$H^\vee \xrightarrow{\ \iota\ } \mathrm{PGSO}_8^\vee = \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{\ f_1^\vee\ } \mathrm{SO}_8(\mathbb{C})$$

and we are able to construct the weak functorial lifting for ι and hence for $f_1^\vee \circ \iota$ (since the weak functorial lifting for $f_1^\vee$ is given by pullback of automorphic forms, as we have explained in the previous subsection). Now one may compose ι with the triality automorphism θ and obtain another map

$$H^{\vee} \xrightarrow{\iota} \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{\theta} \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{f_1^{\vee}} \mathrm{SO}_8(\mathbb{C}).$$

This composite map may give a drastically different instance of Langlands functoriality from H to SO_8 since $\theta \circ \iota$ and ι may not be conjugate in $\mathrm{Spin}_8(\mathbb{C})$. The map $f_1^{\vee} \circ \theta \circ \iota$ can be more simply described as

$$H^{\vee} \xrightarrow{\iota} \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{f_2^{\vee}} \mathrm{SO}_8(\mathbb{C}).$$

The existence of the triality automorphism essentially gives one this new functorial lifting for $f_2^{\vee} \circ \iota$ with minimal effort. This idea has already been observed and exploited in the monograph [CL] of the first author and Lannes and we push it further in the present paper. It has also played a critical role in the proof of the local Langlands conjecture for G_2 given in [GS2].

By using this simple idea, we prove the following instances of Langlands functorial lifting:

Theorem 1.1. *(i) Consider the map of dual groups*

$$\mathrm{spin} : \mathrm{Spin}_7(\mathbb{C}) \longrightarrow \mathrm{GL}_8(\mathbb{C})$$

given by the Spin representation. Then for any cuspidal representation π of PGSp_6 *whose restriction to* Sp_6 *has a generic (or tempered) A-parameter, the corresponding weak Spin lifting of π to* GL_8 *exists.*

(ii) The Rankin-Selberg lifting of automorphic representations of symplectic type *from* $\mathrm{GL}_4 \times \mathrm{GL}_2$ *to* GL_8*, corresponding to the following maps of dual groups:*

$$\mathrm{Sp}_4(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\boxtimes} \mathrm{SO}_8(\mathbb{C}) \xrightarrow{\mathrm{std}} \mathrm{GL}_8(\mathbb{C})$$

exists.

We refer the reader to Theorem 3.8, Theorem 3.10 and Theorem 5.3 for more details about the Spin lifting and Rankin-Selberg lifting of the theorem. In §6, we remove the hypothesis of "trivial central character" in the above theorem (see Theorems 6.3 and 6.9). In particular, we demonstrate the Spin lifting from GSp_6 to GL_8 induced by $\mathrm{GSpin}_7(\mathbb{C}) \to \mathrm{GL}_8(\mathbb{C})$. We also give an application to a lifting from selfdual cuspidal automorphic representations of PGL_7 to SL_8. In §7, we refine Theorems 3.8 and 3.10 further in the setting of automorphic representations of PGSp_6 over $\mathbb{Q}$ generated by holomorphic Siegel modular forms for the full Siegel modular group $\mathrm{Sp}_6(\mathbb{Z})$. In this arithmetic situation, we show that these theorems also hold if we do not impose the genericity (of A-parameters) assumption. These improvements are possible by arguments using Galois representations. As an example, we prove the following result (see Theorem 7.15):

Corollary 1.2. *Assume π is a cuspidal automorphic representation of* PGSp_6 *over* $\mathbb{Q}$ *generated by a Siegel modular cusp form for* $\mathrm{Sp}_6(\mathbb{Z})$ *with weights $k_1 \geq k_2 \geq k_3 \geq 4$. Then:*

(i) $\mathrm{L}(s, \pi, \mathrm{spin})$ *has a meromorphic continuation to all of* $\mathbb{C}$*, with at most a simple pole at $s = 0$ and 1, and no other poles. It satisfies* $\mathrm{L}(s, \pi, \mathrm{spin}) = \mathrm{L}(1 - s, \pi, \mathrm{spin})$.

(iii) *Moreover,* $\mathrm{L}(s, \pi, \mathrm{spin})$ *has a pole at $s = 1$ if, and only if, π is of type* G_2*, in which case* $\mathrm{L}(s, \pi, \mathrm{spin}) = \zeta(s) \cdot \mathrm{L}(s, \pi, \mathrm{std})$.

A weaker statement had been proved by Pollack in [Po1, Thm. 1.2], assuming that the associated Siegel modular form has a nonzero Fourier coefficient at the maximal order of a definite quaternion algebra. For PGSp_6 over a general number field, note that Theorem 1.1 shows that the partial spinor L-function of any cuspidal π with a generic standard A-parameter is a product of Godement-Jacquet L-functions, providing a rather different approach to the constructions in [BG, Vo].

Finally, we remark that Theorem 1.1 depends on results of Arthur [A], B. Xu [X1, X2] and Moeglin-Waldspurger [MW1, MW2] as stated, through the use of the stable twisted trace formula and the notions of A-parameters and A-packets. As such, they are currently subject to the hypothesis of the twisted weighted fundamental lemma. However, one can obtain slightly weaker versions of these results unconditionally. For example, in Theorem 1.1(i), if we had assumed that π is globally generic, there will be no need to invoke [A]. But since the various hypotheses present in the aforementioned references should soon be fully verified (see for example [AGIKMS]), we prefer not to split hairs on such issues, so as to preserve our own *Sense and Sensibility*.

Acknowledgments: The authors thank Bin Xu for several helpful conversations about his work. The first author thanks Olivier Taïbi for useful discussions. This project was initiated when the authors participated in the Oberwolfach workshop "Harmonic Analysis and Trace Formula" during May 22-26, 2017; we thank the organizers Werner Mueller, Sug Woo Shin, Birgit Speh and Nicholas Templier for their invitation, and the Mathematisches Forschungsinstitut Oberwolfach for the excellent working conditions. The main work for this project was completed when the second author visited the IHES during May 15-July 12, 2017; the second author thanks the IHES for its support, as well as providing excellent academic atmosphere and living conditions. The first author is partially supported by the C.N.R.S. and the project ANR-19-CE40-0015-02 (COLOSS). The second author is supported by a Tan Chin Tuan Centennial Professorship at NUS.

2. **Triality**

In this section, we fix our notations for the algebraic groups used in the article and describe some background material on the triality automorphism θ of D_4. As we mentioned in the introduction, the automorphism θ can only be realised on the simply-connected group Spin_8 or the adjoint group PGSO_8. In the process, we shall give a construction of Spin_8 that makes triality especially transparent, following the treatment of [KMRT].

2.1. **Spin groups and representations.** We begin by reviewing the classical construction of Spin groups and their spin representations. Let F be an arbitrary field with $\mathrm{char}(F) \neq 2$. Recall that for any non-degenerate quadratic space (V, Q) over F, we have the naturally associated special orthogonal and proper similitude groups $\mathrm{SO}(V) \subset \mathrm{GSO}(V)$. When we want to emphasize the quadratic form Q, we also write $\mathrm{SO}(V, Q)$ for $\mathrm{SO}(V)$ for example. Via the theory of Clifford algebras, one may construct the associated Spin and general Spin groups $\mathrm{Spin}(V) \subset \mathrm{GSpin}(V)$: these are classically defined as subgroups of the invertible elements of the even Clifford algebra $\mathrm{Cl}^0(V)$ of V. One has a natural projection

$$\rho : \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V)$$

often called the *standard* morphism. The resulting representation of $\mathrm{GSpin}(V)$ on V is its *standard representation*

$$\mathrm{std} : \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{GL}(V).$$

The restrictions to $\mathrm{Spin}(V) \subset \mathrm{GSpin}(V)$ of ρ and std will still be denoted by ρ and std.

In the case V is split with $\dim V = r$, we shall often denote the groups introduced above by $\mathrm{SO}_r \subset \mathrm{GSO}_r$ and $\mathrm{Spin}_r \subset \mathrm{GSpin}_r$. Assume V is split, or more generally that V has trivial discriminant and Clifford invariant (see [KMRT, §35A]). If $\dim V = 2n+1$ is odd, then $\mathrm{Cl}^0(V)$ has a spinor module S_V, unique up to isomorphism, which defines a spin representation

$$\mathrm{spin} : \mathrm{GSpin}(V) \longrightarrow \mathrm{GL}(S_V)$$

of dimension 2^n. On the other hand, if $\dim V = 2n$ is even, then any choice of spinor modules $S_{V,\pm}$ for $\mathrm{Cl}^0(V)$ defines two half-spin representations

$$\mathrm{spin}_\pm : \mathrm{GSpin}(V) \to \mathrm{GL}(S_{V,\pm}),$$

each being of dimension 2^{n-1} over F. These half-spin representations are known to be orthogonal when $2n \equiv 0 \bmod 8$. Of course, we shall also denote by spin or $\mathrm{spin}^\pm$ the restriction of these representations to $\mathrm{Spin}(V)$.

Suppose that $V = V_1 \oplus V_2$ is an orthogonal decomposition of V into the sum of two non-degenerate quadratic spaces. Then one has a natural commutative diagram

$$\begin{tikzcd}
\mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2) \arrow[r, "\iota"] \arrow[d, "\rho_1 \times \rho_2"'] & \mathrm{GSpin}(V) \arrow[d, "\rho"] \\
\mathrm{SO}(V_1) \times \mathrm{SO}(V_2) \arrow[r, "\iota_\flat"] & \mathrm{SO}(V)
\end{tikzcd}$$

so that

$$\mathrm{std}_V \circ \iota \simeq \mathrm{std}_{V_1} \oplus \mathrm{std}_{V_2}$$

as representations of $\mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2)$. On the other hand, one may consider the pullback of the spin representations under ι. For this, we have:

- when $\dim V$ is even but each $\dim V_i$ is odd:

$$S_{V,\pm} \circ \iota \simeq S_{V_1} \boxtimes S_{V_2}$$

  as representations of $\mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2)$;
- when $\dim V$ and $\dim V_i$ are all even:

$$S_{V,+} \circ \iota \simeq S_{V_1,+} \boxtimes S_{V_2,+} \oplus S_{V_1,-} \boxtimes S_{V_2,-}$$

  and

$$S_{V,-} \circ \iota \simeq S_{V_1,+} \boxtimes S_{V_2,-} \oplus S_{V_1,-} \boxtimes S_{V_2,+}$$

  up to relabelling the half-spin modules;
- when $\dim V$ and $\dim V_1$ are odd:

$$S_V \simeq S_{V_1} \boxtimes S_{V_2,+} \oplus S_{V_1} \boxtimes S_{V_2,-}.$$

We now specialize to the case when $\dim V = 8$. Observe that in this case, the standard and half-spin representations are all orthogonal of dimension 8. As we mentioned in the introduction, these three inequivalent irreducible representations of $\mathrm{Spin}(V)$ are permuted transitively by the triality (outer) automorphism. However, the above classical construction does not readily reveal the triality automorphism. This is not surprising since the construction applies to quadratic spaces of any rank. In the rest of this section, we shall explain an alternative construction which applies only when $\dim V = 8$ and which visibly exhibits the triality automorphism. This alternative construction makes use of more structures on V than that of a quadratic space, namely the structure of an octonion algebra.

2.2. **Octonion algebras.** Let $\mathbb{O}$ be an octonion algebra over F. Thus, $\mathbb{O}$ is an 8-dimensional (non-commutative and non-associative) composition algebra with unit, see [KMRT, §33]. In particular, it is equipped with an F-linear involution $x \mapsto \bar{x}$ such that $N(x) = x \cdot \bar{x}$ is the quadratic form permitting composition:

$$N(x \cdot y) = N(x) \cdot N(y) \quad \text{ for } x, y \in \mathbb{O}.$$

We denote by b_N the symmetric bilinear form associated to N and 1 the unit of $\mathbb{O}$. The automorphism group of $\mathbb{O}$ is an exceptional simple algebraic group over F of type G_2; it is a subgroup of $\mathrm{SO}(\mathbb{O}, N)$ that we denote by $\mathrm{G}_2^{\mathbb{O}}$. It is split over F if $\mathbb{O}$ is, in which case we often simply denote $\mathrm{G}_2^{\mathbb{O}}$ by G_2. The action of $\mathrm{G}_2^{\mathbb{O}}$ on $\mathbb{O}$ fixes 1, and hence its orthogonal complement with respect to b_N, which is the 7-dimensional standard representation (note $\mathrm{char}(F) \neq 2$).

2.3. **Symmetric Composition Algebras.** For the purpose of constructing the Spin group with its triality automorphism, it will be more convenient to work with a modified multiplication law on $\mathbb{O}$. We set

$$x * y := \bar{x} \cdot \bar{y} \quad \text{for } x, y \in \mathbb{O}.$$

Then $(\mathbb{O}, *, N)$ is called a *para-octonion* algebra and satisfies the axioms of a *symmetric composition algebra* [KMRT, Pg. 463-464]: it is a non-unital composition algebra with respect to N which satisfies:

$$b_N(x * y, z) = b_N(x, y * z) \quad \text{ for all } x, y, z \in \mathbb{O}.$$

However, automorphisms of $\mathbb{O}$ give rise to automorphisms of $(\mathbb{O}, *)$ and vice versa, so that $\mathrm{Aut}(\mathbb{O}, *) = \mathrm{G}_2^{\mathbb{O}}$ [KMRT, Cor. 34.6].

2.4. **The Spin group in dimension** 8. Assume $(V, *, Q)$ is a symmetric composition algebra of dimension 8. It follows from [KMRT, §35A] that the even Clifford algebra of (V, Q) is naturally isomorphic to $\mathrm{End}(V) \times \mathrm{End}(V)$. In particular, (V, Q) has trivial discriminant and Clifford invariant, and we may take $S_{V,\pm} = V$. Better, let us define

$$\mathrm{Spin}(V, *, Q) = \{(g_1, g_2, g_3) \in \mathrm{SO}(V, Q)^3 : g_1(x * y) = g_2(x) * g_3(y) \text{ for } x, y \in V\}.$$

Then $\mathrm{Spin}(V, *, Q)$ is an algebraic group over F isomorphic to $\mathrm{Spin}(V, Q)$ by [KMRT, §35C]. A charm of this rather simple definition is that not only does it not involve anymore the Clifford algebra, but also that it renders triality entirely natural and transparent. Indeed, by construction the group $\mathrm{Spin}(V, *, Q)$ is equipped with the following extra structures:

- the three projections to $\mathrm{SO}(V,Q)$ gives 3 homomorphisms
$$\rho_j : \mathrm{Spin}(V,*,Q) \longrightarrow \mathrm{SO}(V,Q),$$
with $\mathrm{Ker}(\rho_j) \simeq \mu_2 \subset Z_{\mathrm{Spin}(V,*,Q)}$.
- the cyclic permutation of the three coordinates in $\mathrm{SO}(V,Q)^3$ preserves the subgroup $\mathrm{Spin}(V,*,Q)$ and thus defines an order 3 automorphism
$$\theta : \mathrm{Spin}(V,*,Q) \longrightarrow \mathrm{Spin}(V,*,Q).$$
This is a triality automorphism which permutes the 3 maps ρ_j cyclically. The fixed group of θ is, by definition, the subgroup $\mathrm{Aut}(V,*,Q) \subset \mathrm{SO}(V,Q)$.
- Because the automorphism θ necessarily preserves the center $Z_{\mathrm{Spin}(V,*,Q)}$, it descends to an automorphism of the adjoint group $\mathrm{PGSO}(V,Q)$, still denoted by θ. The projection from the three $\mathrm{SO}(V,Q)$'s thus gives 3 maps
$$f_j : \mathrm{SO}(V,Q) \longrightarrow \mathrm{PGSO}(V,Q),$$
which are permuted by θ.

The maps f_j and ρ_j alluded to in the introduction are those defined above. By [KMRT, Prop. 35.7], we may identify $\mathrm{Spin}(V,*,Q)$ with $\mathrm{Spin}(V,Q)$ in such a way that we have $\rho_1 = \rho$, and that ρ_2 and ρ_3 induce the two half-spin representations of $\mathrm{Spin}(V,Q)$. This (breaking of symmetry!) being done, [KMRT, Prop. 35.1] shows that the two isogenies ρ_2 and ρ_3 naturally extend to $\mathrm{GSpin}(V,Q) \supset \mathrm{Spin}(V,Q)$ and give rise to two isogenies
$$\widetilde{\rho_j} : \mathrm{GSpin}(V,Q) \longrightarrow \mathrm{GSO}(V,Q), \ \ j = 2,3$$
inducing as well the two half-spin representations of $\mathrm{GSpin}(V,Q)$ on $S_{V,\pm} = V$.

3. Spin Lifting for PGSp_6

We shall now investigate the consequences of triality for Langlands functoriality. In particular, in this section, we shall establish the so-called Spin lifting from PGSp_6 to GL_8.

Let k be a number field with ring of adèles $\mathbb{A}$. For a reductive group G over k, set $[G] = G(k)\backslash G(\mathbb{A})$ and let $\mathcal{A}(G)$ denote the space of automorphic forms for G. The subspace of cusp forms is denoted by $\mathcal{A}_{cusp}(G)$.

3.1. **Simple functorial lifting.** We have introduced the notion of weak Langlands functorial lifting in §1.1. For the sake of convenient reference, we document an instance of functorial lifting already alluded to in the introduction and which we will appeal to repeatedly later on.

With G and H split connected reductive algebraic groups over k, Z a commutative algebraic group of multiplicative type, and S a torus over k, we shall consider an exact sequence of k-groups of the form:

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{\phi} H \longrightarrow S \longrightarrow 1. \tag{3.1}$$

In particular, Z is isomorphic to a central subgroup of G, and S is a co-central quotient of H (and is necessarily split). Here are some examples we have in mind:

- $G = H$, Z and S are trivial, and ϕ is an automorphism of G (not necessarily inner);

- Z is a finite central subgroup and S is trivial, so that ϕ is an isogeny;
- Z is trivial and $S = \mathbb{G}_m$.

The above exact sequence induces a corresponding map of Langlands dual groups:

$$\phi^\vee : H^\vee \longrightarrow G^\vee.$$

The map $\phi^\vee$ gives rise to a simple instance of weak Langlands functoriality from H to G, as the following folklore proposition documents.

Proposition 3.2. *Let $\phi : G \longrightarrow H$ be as in* (3.1).

(i) *Assume that v is a finite place of k and π_v is an unramified representation of $H(k_v)$. Then any constituent σ_v of $\pi_v|_{G(k_v)}$ is unramified, and we have $c(\sigma_v) = \phi^\vee(c(\pi_v))$.*

(ii) *If π is an automorphic representation of H, and if σ is any irreducible automorphic constituent of $\pi|_{G(\mathbb{A})}$, then σ is a weak functorial lifting of π with respect to $\phi^\vee$.*

Before we give the proof, recall that if I is a split connected reductive group over a non-Archimedean local field F (of characteristic 0 in what follows), and if τ is an irreducible smooth representation of $I(F)$, we say that τ is *unramified* if it has nonzero invariant vectors under some compact open subgroup $K \subset I(F)$ which is *hyperspecial* in the sense of Tits [Ti, §1.10]. When we want to specify K, as we shall do in the proof below, we rather say that τ is *K-unramified.* Hyperspecial (compact) subgroups exist as I is split [Ti, §1.10.2]. We shall use below their following properties, in which $\mathcal{O}$ denotes the valuation ring of F :

(HSa) the hyperspecial subgroups of $I(F)$ are exactly the subgroups of the form $f(\mathcal{I}(\mathcal{O}))$ where $(\mathcal{I},\ f : \mathcal{I} \times_{\mathcal{O}} F \xrightarrow{\sim} I)$ is a reductive $\mathcal{O}$-model of I [Ti, §3.8.1],

(HSb) the natural action $(\varphi, K) \mapsto \varphi(K)$ of $\mathrm{Aut}_F(I)$ on the subgroups of $I(F)$ preserves the hyperspecial subgroups, and $I_{\mathrm{ad}}(F) \subset \mathrm{Aut}_F(I)$ permutes them transitively [Ti, §2.5].

For example, if I is a split torus, then its unique maximal compact subgroup is its unique hyperspecial subgroup by (HSa). The following lemma is presumably classical, but we provide a proof for lack of reference.

Lemma 3.3. *Assume $\phi : G \to H$ is a morphism of split connected reductive groups over the non-Archimedean local field F belonging to an exact sequence as in* (3.1). *Then for any hyperspecial subgroup K of $H(F)$, there is a hyperspecial subgroup K' of $G(F)$ with $\phi(K') \subset K$.*

Proof. Observe first that it is enough to find some hyperspecial subgroups $A \subset G(F)$ and $B \subset H(F)$ such that $\phi(A) \subset B$. Indeed, let $K \subset H(F)$ be hyperspecial. By (HSb) we may choose $h \in H_{\mathrm{ad}}(F)$ with $K = hBh^{-1}$. But ϕ induces by (3.1) an isomorphism $G_{\mathrm{ad}} \simeq H_{\mathrm{ad}}$, so there is $g \in G_{\mathrm{ad}}(F)$ with $\phi(g) = h$, and $K' := gAg^{-1} \subset G(F)$ works.

As a second and independent observation, we claim that we may reduce the lemma to the special case where ϕ is surjective (hence S trivial). Indeed, let us denote by C the neutral component of the center of H, so that C is a split torus. Then the morphism $\phi' : C \times G \to H, (c, g) \mapsto c\phi(g)$ is surjective with central kernel. Now any hyperspecial compact subgroup of $C(F) \times G(F)$ has the form $A' \times A$ with A' and A hyperspecial subgroups in $C(F)$ and $G(F)$ respectively, *e.g.* by (HSb). Hence, if the desired result is known for the surjective morphism

ϕ', we can find a hyperspecial subgroup B of $H(F)$ such that $\phi(A) \subset \phi'(A' \times A) \subset B$, this proving the desired result for ϕ.

It remains to prove the lemma when ϕ is surjective. In this case, ϕ factors as

$$G \xrightarrow{f} G/Z \xrightarrow{g} H,$$

with f the canonical map and g an isomorphism. As the lemma is obvious for isomorphisms, we may and do assume $\phi = f$. Choose $\mathcal{G}$ a reductive (necessarily split) $\mathcal{O}$-model of G, let $\mathcal{T}$ be a maximal $\mathcal{O}$-split torus in $\mathcal{G}$ and set $T = \mathcal{T} \times_{\mathcal{O}} F$. The closed subgroup $Z \subset T$ certainly has an $\mathcal{O}$-model $\mathcal{Z} \subset \mathcal{T}$ (with same character group as Z). By [D, Prop. 4.3.1 (i)], the quotient group scheme $\mathcal{G}/\mathcal{Z}$ is reductive over $\mathcal{O}$, and $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{Z}$ is a model of ϕ over $\mathcal{O}$. By (HSa), the subgroups $A := \mathcal{G}(\mathcal{O})$ and $B := (\mathcal{G}/\mathcal{Z})(\mathcal{O})$ are hyperspecial in $G(F)$ and $H(F) = (G/Z)(F)$ and satisfy $\phi(A) \subset B$, and we are done by the first observation. $\square$

Proof. (Of Proposition 3.2) Part (i) trivially implies (ii), so we focus on (i). We first prove the first assertion of (i). Let K be a hyperspecial subgroup of $H(k_v)$ such that π_v is K-unramified. By Lemma 3.3, there is a hyperspecial subgroup K' of $G(k_v)$ such that $\phi(K') \subset K$. As k_v is a local field, the normal subgroup $\phi(G(k_v)) \subset H(k)$ generates, together with the center of $H(k_v)$, a finite index subgroup of $H(k_v)$, by Tate's finiteness of Galois cohomology. It follows that $\pi_v|_{G(k_v)}$ is a finite direct sum of irreducible representations which are permuted transitively by $H(k_v)$. These constituents are even $G_{\mathrm{ad}}(k_v)$-conjugate, as ϕ induces an isomorphism $G_{\mathrm{ad}} \simeq H_{\mathrm{ad}}$. As $\phi(K') \subset K$, some constituent of $\pi_v|_{G(k_v)}$ is K'-unramified, so any constituent of $\pi_v|_{G(k_v)}$ is unramified with respect to a suitable $G_{\mathrm{ad}}(k_v)$-conjugate of K'.

We now prove the second assertion of (i). Fix a constituent σ of $\pi_v|_{G(k_v)}$. Choose a Borel pair (B, T) defined over k in H and set $B' = \phi^{-1}(B)$ and $T' = \phi^{-1}(T)$ in G. Then π_v is an irreducible constituent of the (normalized) principal series $\mathrm{Ind}_{B(k_v)}^{H(k_v)}\chi$ for some unramified character $\chi : T(k_v) \to \mathbb{C}^\times$. Recall that T and T' being split, we have the decomposition $H(k_v) = T(k_v)\phi(G(k_v))$, by a standard Galois cohomology argument using Hilbert 90. This shows that the map $f \mapsto f \circ \varphi$ induces an injective $G(k_v)$-equivariant map

$$\left(\mathrm{Ind}_{B(k_v)}^{H(k_v)} \chi \right)|_{G(k_v)} \longrightarrow \mathrm{Ind}_{B'(k_v)}^{G(k_v)} \chi',$$

with $\chi' = \chi \circ \phi$, an unramified character of $T'(k_v)$, and thus σ is a constituent of the right-hand side. This shows that $c(\sigma) = \phi^\vee(c(\pi_v))$ by the properties of the Satake isomorphism. $\square$

3.2. **Spin lifting.** Consider the split group PGSp_{2n} whose dual group is $\mathrm{Spin}_{2n+1}(\mathbb{C})$. Recall from §2.1 that one has the standard morphism

$$\rho_{\mathbb{C}} : \mathrm{Spin}_{2n+1}(\mathbb{C}) \longrightarrow \mathrm{SO}_{2n+1}(\mathbb{C})$$

and by Proposition 3.2, the associated functorial lifting is simply the restriction of automorphic forms

$$\mathcal{A}(\mathrm{PGSp}_{2n}) \xrightarrow{\text{rest.}} \mathcal{A}(\mathrm{Sp}_{2n}).$$

On the other hand, by §2.1 again, one has the faithful Spin representation

$$\mathrm{spin} : \mathrm{Spin}_{2n+1}(\mathbb{C}) \longrightarrow \mathrm{GL}_{2^n}(\mathbb{C}).$$

Accordingly, there should be an associated Langlands functorial lifting

$$\mathcal{A}(\mathrm{PGSp}_{2n}) \longrightarrow \mathcal{A}(\mathrm{GL}_{2^n})$$

from automorphic forms of GSp_{2n} to GL_{2^n}; we call this the *Spin lifting* of PGSp_{2n}.

3.3. **The case $n = 3$.** We shall now specialize to the case $n = 3$. In this case, one can describe the spin representation of $\mathrm{Spin}_7(\mathbb{C})$ using triality. The point is that there are 3 conjugacy classes of embeddings

$$\mathrm{Spin}_7(\mathbb{C}) \longrightarrow \mathrm{Spin}_8(\mathbb{C})$$

permuted by the triality automorphism of $\mathrm{Spin}_8(\mathbb{C})$. These 3 conjugacy classes of embeddings are characterized by their intersection with the center $Z_{\mathrm{Spin}_8} \simeq \mu_2 \times \mu_2$ of $\mathrm{Spin}_8(\mathbb{C})$. One may fix such an embedding ι so that the standard representation of $\mathrm{Spin}_7(\mathbb{C})$ is compatible with the map $f_1^\vee$ (dual to the standard $f_1 : \mathrm{SO}_8 \longrightarrow \mathrm{PGSO}_8$), in the sense that one has a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}_7(\mathbb{C}) & \xrightarrow{\iota} & \mathrm{Spin}_8(\mathbb{C}) \\ {\scriptstyle\rho_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle f_1^\vee} \\ \mathrm{SO}_7(\mathbb{C}) & \longrightarrow & \mathrm{SO}_8(\mathbb{C}) \end{array} \tag{3.4}$$

We may designate the map $f_1^\vee$ as the standard morphism. On the other hand, recall from the discussion in §2.4 that there are two other isogenies $f_2, f_3 : \mathrm{SO}_8 \longrightarrow \mathrm{PGSO}_8$ which factor as $f_2 = \theta \circ f_1$ and $f_3 = \theta^2 \circ f_1$, with θ a triality automorphism of PGSO_8. These induce dual isogenies

$$f_2^\vee : \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{\theta^\vee} \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{f_1^\vee} \mathrm{SO}_8(\mathbb{C}),$$

and likewise $f_3^\vee$, with $\theta^\vee$ a triality automorphism of $\mathrm{Spin}_8(\mathbb{C})$. We may designate $f_2^\vee$ and $f_3^\vee$ as the half-spin representations of $\mathrm{Spin}_8(\mathbb{C})$. By our discussion in §2.1, the restriction of these half-spin representations, via $\iota : \mathrm{Spin}_7(\mathbb{C}) \to \mathrm{Spin}_8(\mathbb{C})$, give rise to the spin representation of $\mathrm{Spin}_7(\mathbb{C})$. In other words, the spin representation of $\mathrm{Spin}_7(\mathbb{C})$ is given by the composite map

$$\mathrm{Spin}_7(\mathbb{C}) \xrightarrow{\iota} \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{f_2^\vee} \mathrm{SO}_8(\mathbb{C}) \xrightarrow{\mathrm{std}} \mathrm{GL}_8(\mathbb{C}).$$

This then suggests a construction of the spin lifting which is summarised by the following sequence of liftings:

$$\mathcal{A}(\mathrm{PGSp}_6) \xrightarrow{\iota_*} \mathcal{A}(\mathrm{PGSO}_8) \xrightarrow{f_2^*} \mathcal{A}(\mathrm{SO}_8) \xrightarrow{\text{[A, CKPSS]}} \mathcal{A}(\mathrm{GL}_8)$$

As we explain next, the functorial lifting ι_* can be constructed by the similitude theta correspondence.

3.4. **Theta correspondence.** For our discussion of theta correspondence, we consider the dual reductive pair $\mathrm{Sp}_{2n} \times \mathrm{O}_{2m}$ of symplectic and orthogonal groups associated to a skew-symmetric and quadratic space of dimension $2n$ and $2m$ respectively. For simplicity, assume that the quadratic space underlying O_{2m} has trivial discriminant. In our applications later on in the paper, we will assume that $m > n$.

Given a nontrivial additive character ψ of $k\backslash\mathbb{A}$, the dual pair $\mathrm{Sp}_{2n} \times \mathrm{O}_{2m}$ is equipped with a Weil representation Ω_ψ. One has an automorphic realization

$$\theta : \Omega_\psi \longrightarrow C^\infty([\mathrm{Sp}_{2n} \times \mathrm{O}_{2m}])$$

given by the formation of theta series. The global theta lifting of isometry groups is an equivariant map

$$\Theta : \Omega_\psi \otimes \overline{\mathcal{A}_{cusp}(\mathrm{Sp}_{2n})} \longrightarrow \mathcal{A}(\mathrm{O}_{2m})$$

defined by

$$\Theta(\phi, f)(h) = \int_{[\mathrm{Sp}_{2n}]} \theta(\phi)(gh) \cdot \overline{f(g)}\, dg, \quad \text{for } \phi \in \Omega_\psi \text{ and } f \in \mathcal{A}_{cusp}(\mathrm{Sp}_{2n}).$$

It is known that this theory of theta correspondence can be extended to the setting of the similitude dual pair $\mathrm{GSp}_{2n} \times \mathrm{GO}_{2m}$. More precisely, the Weil representation has a natural extension to the group

$$(\mathrm{GSp}_{2n} \times \mathrm{GO}_{2m})^{\mathrm{sim}} = \{(g,h) \in \mathrm{GSp}_{2n} \times \mathrm{GO}_{2m} : \mathrm{sim}(g) \cdot \mathrm{sim}(h) = 1\}.$$

Observe that this group sits in the short exact sequences:

$$1 \longrightarrow \mathrm{Sp}_{2n} \longrightarrow (\mathrm{GSp}_{2n} \times \mathrm{GO}_{2m})^{\mathrm{sim}} \xrightarrow{p} \mathrm{GO}_{2m} \longrightarrow 1.$$

$$1 \longrightarrow \mathrm{O}_{2m} \longrightarrow (\mathrm{GSp}_{2n} \times \mathrm{GO}_{2m})^{\mathrm{sim}} \xrightarrow{q} \mathrm{GSp}_{2n} \longrightarrow 1$$

where p and q are the natural projections on the two factors. Hence, for $\phi \in \Omega_\psi$ and a cusp form $f \in \mathcal{A}_{cusp}([\mathrm{GSp}_{2n}])$ with a fixed central character, one can define an automorphic form $\theta(\phi, f)$ on GO_{2m} by:

$$\Theta(\phi, f)(h) = \int_{[\mathrm{Sp}_{2n}]} \theta(\phi)(t_h g, h) \cdot \overline{f(g)}\, dg,$$

for any $t_h \in \mathrm{GSp}_{2n}(\mathbb{A})$ such that $(t_h, h) \in (\mathrm{GSp}_{2n}(\mathbb{A}) \times \mathrm{GO}_{2m}(\mathbb{A}))^{\mathrm{sim}}$. Note that the above definition is independent of the choice of t_h. It can be more elegantly expressed as:

$$\Theta(\phi, f) = p_! \left(\theta(\phi) \cdot q^*(\overline{f})\right).$$

The automorphic form $\Theta(\phi, f)$ has the same central character as f, on identifying the centers of GO_{2m} and GSp_{2n} with $\mathbb{G}_m$ via their action on the underlying quadratic and skew-symmetric spaces.

To summarize, one has a commutative diagram of global theta liftings, with the vertical arrows given by pullback and restriction of automorphic forms:

$$\begin{array}{ccc}
\Omega_\psi \otimes \overline{\mathcal{A}_{cusp}(\mathrm{PGSp}_{2n})} & \xrightarrow{\Theta} & \mathcal{A}(\mathrm{PGO}_{2m}) \\
\downarrow & & \downarrow \\
\Omega_\psi \otimes \overline{\mathcal{A}_{cusp}(\mathrm{GSp}_{2n})} & \xrightarrow{\Theta} & \mathcal{A}(\mathrm{GO}_{2m}) \\
\downarrow & & \downarrow \\
\Omega_\psi \otimes \overline{\mathcal{A}_{cusp}(\mathrm{Sp}_{2n})} & \xrightarrow{\Theta} & \mathcal{A}(\mathrm{O}_{2m})
\end{array}$$

If $\pi \in \mathcal{A}_{cusp}(\mathrm{GSp}_{2n})$ or $\mathcal{A}_{cusp}(\mathrm{Sp}_{2n})$ is a cuspidal representation, then its global theta lift to GO_{2m} or O_{2m} is the subrepresentation

$$\Theta(\pi) = \langle \Theta(\phi, f) : \phi \in \Omega_\psi,\ f \in \pi \rangle \subset \mathcal{A}(\mathrm{GO}_{2m}) \quad \text{or} \quad \mathcal{A}(\mathrm{O}_{2m}).$$

For our purpose of constructing the spin lifting, we shall consider the case $m = n+1$. In this case, one knows that if $\pi \in \mathcal{A}_{cusp}(\mathrm{GSp}_{2n})$ is a cuspidal representation which is globally generic, then $\Theta(\pi)$ on GO_{2n+2} is globally generic and thus is nonzero [GRS1].

There is also an analogous theory of local (isometry and similitude) theta correspondence, for which we refer the reader to [Ro] and [GT1, GT2]. Another result we need is the local theta correspondence of unramified representations. More precisely, we have:

Proposition 3.5. *Assume that* $\mathrm{Sp}_{2n} \times \mathrm{O}_{2n+2}$ *is an unramified dual pair over a non-Archimedean local field* F *of characteristic* 0. *Let* π *be an unramified irreducible representation of* $\mathrm{GSp}_{2n}(F)$ *and consider its local (small) theta lift* $\theta(\pi)$ *on* $\mathrm{GO}_{2n+2}(F)$. *One has:*

(i) $\theta(\pi)$ *is nonzero, irreducible and unramified;*
(ii) $\theta(\pi)$ *has the same central character as* π*;*
(iii) $\theta(\pi)$ *remains irreducible when restricted to* $\mathrm{GSO}_{2n+2}(F)$.

Hence, the local theta correspondence gives a map

$$\theta : \mathrm{Irr}_{ur,\chi}(\mathrm{GSp}_{2n}) \longrightarrow \mathrm{Irr}_{ur,\chi}(\mathrm{GSO}_{2n+2})$$

where $\mathrm{Irr}_{ur,\chi}$ *denotes the set of irreducible unramified representations with central character* χ. *At the level of Satake parameters, this map is given by the top arrow* ι *in the following natural diagram of dual groups:*

$$\begin{array}{ccc} \mathrm{GSpin}_{2n+1}(\mathbb{C}) & \xrightarrow{\iota} & \mathrm{GSpin}_{2n+2}(\mathbb{C}) \\ {\scriptstyle\rho'}\big\downarrow & & \big\downarrow{\scriptstyle\rho'} \\ \mathrm{SO}_{2n+1}(\mathbb{C}) & \xrightarrow{\iota^\flat} & \mathrm{SO}_{2n+2}(\mathbb{C}). \end{array} \tag{3.6}$$

So as not to disrupt our discussion, the proof of this proposition is given in Appendix A at the end of the paper. We should also mention that the global and local (similitude) theta correspondences are compatible, in the following sense [G, Prop. 3.1]:

Proposition 3.7. *Suppose* π *is a cuspidal automorphic representation of* $\mathrm{GSp}_{2n}(\mathbb{A})$ *such that its theta lift* $\Theta(\pi)$ *to* $\mathrm{GO}_{2m}(\mathbb{A})$ *is nonzero and contained in the space of square-integrable automorphic forms (with fixed central character). Then* $\Theta(\pi)$ *is irreducible and for all places* v,

$$\Theta(\pi)_v \simeq \theta(\pi_v)$$

where the RHS is the local theta lift of π_v. *More generally (i.e. without assuming that* $\Theta(\pi)$ *is contained in the space of square-integrable automorphic forms), for any irreducible subquotient* σ *of* $\Theta(\pi)$,

$$\sigma_v \simeq \theta(\pi_v) \quad \text{for almost all } v.$$

3.5. **Spin lifting for $n = 3$.** We now specialise to the case $n = 3$. We shall prove the following theorem which extends [CL, Thm. 2.3(ii)]:

Theorem 3.8. *Let π be a cuspidal representation of PGSp_6 whose restriction to Sp_6 possesses a generic A-parameter. Then the (weak) Spin lifting of π to GL_8 exists.*

Let us first establish a lemma which may be of independent interest.

Lemma 3.9. *Let π be a cuspidal representation of GSp_{2n} whose restriction to Sp_{2n} possesses a generic A-parameter.*

(i) The cuspidal representation π is nearly equivalent to a globally generic cuspidal representation π' of GSp_{2n}.

(ii) Fix a place v of k and π' as in (i). If π_v is unramified, then $\pi'_v \simeq \pi_v$.

Proof. (i) Consider
$$\pi|_{\mathrm{Sp}_{2n}} := \{f|_{\mathrm{Sp}_{2n}(\mathbb{A})} : f \in \pi\} \subset \mathcal{A}_{cusp}(\mathrm{Sp}_{2n}).$$
By hypothesis, this submodule of $\mathcal{A}_{cusp}(\mathrm{Sp}_{2n})$ has a generic A-parameter $\Psi_\flat$ with an associated submodule $\mathcal{A}_{\Psi_\flat} \subset \mathcal{A}_{cusp}(\mathrm{Sp}_{2n})$ (the global A-packet), so that
$$\pi|_{\mathrm{Sp}_{2n}} \subset \mathcal{A}_{\Psi_\flat} \subset \mathcal{A}_{cusp}(\mathrm{Sp}_{2n}).$$
As shown by Ginzburg-Rallis-Soudry via automorphic descent [GRS2], the A-packet $\mathcal{A}_{\Psi_\flat}$ contains a globally generic cuspidal representation $\pi'_\flat$. In particular, the irreducible summands of $\pi|_{\mathrm{Sp}_{2n}}$ are nearly equivalent to $\pi'_\flat$. In [X2], Bin Xu has constructed a global A-packet
$$\tilde{\mathcal{A}}_{\Psi_\flat} \subset \mathcal{A}_{cusp}(\mathrm{GSp}_{2n}),$$
such that
$$\tilde{\mathcal{A}}_{\Psi_\flat}|_{\mathrm{Sp}_{2n}} = \mathcal{A}_{\Psi_\flat}.$$
Moreover, the global A-packet $\tilde{\mathcal{A}}_{\Psi_\flat}$ is unique up to twisting by automorphic quadratic characters. Up to replacing $\tilde{\mathcal{A}}_{\Psi_\flat}$ by an automorphic quadratic twist, we may thus assume that $\pi \subset \tilde{\mathcal{A}}_{\Psi_\flat}$. Now there is some irreducible summand $\pi' \subset \tilde{\mathcal{A}}_{\Psi_\flat}$ such that
$$\pi'_\flat \subset \pi'|_{\mathrm{Sp}_{2n}}.$$
Then π is nearly equivalent to π', which is globally generic (since $\pi'_\flat$ is).

(ii) The representations π and π' both belong to the generic global A-packet $\tilde{\mathcal{A}}_{\Psi_\flat}$ introduced above. By [X2], the members of $\tilde{\mathcal{A}}_{\Psi_\flat}$ (that Xu also calls a global L-packet) are constructed as a restricted tensor product, over all places w of k, of the members of local L-packets $\tilde{\mathcal{A}}_{\Psi_{\flat,w}}$ defined by B. Xu in [X1]. By [X1, Thm. 4.6] (see also the discussion *loc. cit.* after Prop. 3.10), for each finite place w, the irreducible constituents of the restriction of the members of $\tilde{\mathcal{A}}_{\Psi_{\flat,w}}$ form a local L-packet $\Pi^\flat_w$ of $\mathrm{Sp}_{2n}(k_w)$ (as defined in [A]). Moreover, the restriction to $\mathrm{Sp}_{2n}(k_w)$ also induces a bijection between $\tilde{\mathcal{A}}_{\Psi_{\flat,w}}$ and the $\mathrm{GSp}_{2n}(k_w)$-orbits in $\Pi^\flat_w$, by [X1, Prop. 4.4 (2)].

Assume π_w is unramified. An argument given in the proof of Proposition 3.2 (i) shows that the restriction of π_w to $\mathrm{Sp}_{2n}(k_w)$ is a finite direct sum of unramified representations which are permuted transitively by $\mathrm{GSp}_{2n}(k_w)$. But by the unramified case of the local Langlands correspondence for $\mathrm{Sp}_{2n}(k_w)$ in [A], the set of these constituents is the full L-packet $\Pi_w^\flat$. By the properties of $\tilde{\mathcal{A}}_{\Psi_\flat,w}$ recalled above, this shows $\tilde{\mathcal{A}}_{\Psi_\flat,w} = \{\pi_w\}$. $\square$

We can now prove Theorem 3.8.

Proof. (of Theorem 3.8) By Lemma 3.9 (i), π is nearly equivalent to a globally generic cuspidal representation π' of PGSp_6. By [GRS1], the global theta lift $\Theta(\pi')$ is a nonzero globally generic (not necessarily cuspidal) automorphic representation of PGSO_8. In any case, by the commutative diagram (3.6) and Proposition 3.7, the Hecke-Satake family of $\Theta(\pi')$ is the image of $c(\pi') = c(\pi)$ under the natural map $\iota : \mathrm{Spin}_7(\mathbb{C}) \to \mathrm{Spin}_8(\mathbb{C})$, and the pullback of $\Theta(\pi')$ to SO_8 via the natural map $f_1 : \mathrm{SO}_8 \to \mathrm{PGSO}_8$ has A-parameter

$$\Psi' = \Psi \boxplus \mathbf{1},$$

where Ψ is the A-parameter of $\pi|_{\mathrm{Sp}_6}$. On the other hand, the discussion in §3.2 and §3.3, together with Proposition 3.2 (ii), show that using the pullback of $\Theta(\pi')$ via $f_2 : \mathrm{SO}_8 \to \mathrm{PGSO}_8$, the map

$$\pi \mapsto \text{the constituents of } f_2^*(\Theta(\pi'))$$

exhibits the Spin lifting from PGSp_6 to SO_8. Composing this with the lifting from SO_8 to GL_8 due to [CKPSS] or [A], we obtain the desired Spin lifting from PGSp_6 to GL_8. $\square$

In fact, by being more careful with the above proof, one has the following slight strengthening of Theorem 3.8, which says that the weak Spin lifting provided by Theorem 3.8 is *strong* at unramified places and Archimedean places.

Theorem 3.10. *Let $\pi \subset \mathcal{A}_{cusp}(\mathrm{PGSp}_6)$ be as in Theorem 3.8 and let σ be its automorphic weak spin lift constructed on GL_8 therein.*

(i) For each finite place v of k,

$$\pi_v \textit{ unramified} \Longrightarrow \sigma_v \textit{ unramified}$$

with $c(\sigma_v) = \mathrm{spin}(c(\pi_v))$.

(ii) Let v be an Archimedean place of v and let $c(\pi_v)$ be the infinitesimal character of π_v, regarded as a semisimple element in $\mathfrak{spin}_7 = \mathrm{Lie}(\mathrm{PGSp}_6^\vee) \otimes_{k_v} \mathbb{C}$. Then the infinitesimal character of σ_v is given by

$$c(\sigma_v) = \mathrm{dspin}(c(\pi_v)) \in \mathfrak{gl}_8.$$

(iii) Assume that the A-parameter of $\pi|_{\mathrm{Sp}_6}$ does not contain the trivial representation $\mathbf{1}$. Then the automorphic representation σ of GL_8 is an isobaric sum

$$\sigma = \sigma_1 \boxplus \sigma_2 \boxplus \cdots \boxplus \sigma_k$$

where each σ_i is a self-dual cuspidal representation of some GL_{n_i} which is of orthogonal type (i.e. $L^S(s, \sigma_i, \mathrm{Sym}^2)$ has a pole at $s = 1$) and $\sigma_i \neq \sigma_j$ if $i \neq j$.

Proof. (i) Consider the globally generic π' provided by Lemma 3.9(i) and used in the proof of Theorem 3.8, so that π and π' both belong to the global A-packet $\tilde{\mathcal{A}}_{\Psi_\flat}$. By Lemma 3.9(ii), $\pi'_v \simeq \pi_v$ is unramified and generic. It follows by Proposition 3.5 and Proposition 3.7 that the global theta lift $\Theta(\pi')$ is unramified at the place v with

$$c(\Theta(\pi')_v) = c(\theta(\pi_v)) = \iota(c(\pi_v)),$$

where $\iota : \mathrm{Spin}_7(\mathbb{C}) \hookrightarrow \mathrm{Spin}_8(\mathbb{C})$ is as in (3.4). In the proof of Theorem 3.8, the weak lift σ on GL_8 is obtained by considering $f_2^*(\Theta(\pi'))$ on SO_8 followed by the Arthur transfer std_* from SO_8 to GL_8. Both of these respect unramified representations (or rather unramified L-packets) with the expected effect on their Satake parameters: for f_2^* this follows from Proposition 3.2 (i), and for std_* from [A]. Thus we deduce that σ_v is unramified with

$$c(\sigma_v) = \mathrm{std}(f_2^\vee(\iota(c(\pi_v)))) = \mathrm{spin}(c(\pi_v)),$$

as desired.

(ii) For an Archimedean place v, note that the elements in the local L-packet $\tilde{\mathcal{A}}_{\Psi_{\flat,v}}$ all have the same infinitesimal character. In particular, $c(\pi_v) = c(\pi'_v)$. We also know that $c(\theta(\pi'_v)) = d\iota(c(\pi'_v))$ by the correspondence of infinitesimal character under local theta correspondence (see [Pr, Li]). Since infinitesimal characters behave in the expected way under pulling back by an isogeny and the Arthur transfer from SO_8 to GL_8 (by [A]), we deduce the desired statement in (ii).

(iii) If the A-parameter of $\pi|_{\mathrm{Sp}_6}$ does not contain the trivial representation $\mathbf{1}$, then the nonzero global theta lift $\Theta(\pi')$ in the proof of Theorem 3.8 is a globally generic cuspidal representation of PGSO_8 (by [GRS1]). Under pullback by f_2, $f_2^*(\Theta(\pi'))$ contains a globally generic cuspidal representation of SO_8 and hence has a generic discrete A-parameter, i.e. its transfer to GL_8 is a multiplicity-free isobaric sum of self-dual cuspidal representations of orthogonal type (by [A] or [CKPSS, GRS2]). □

Remark 3.11. If the A-parameter Ψ of $\pi|_{\mathrm{Sp}_6}$ contains the trivial representation $\mathbf{1}$, say $\Psi = \mathbf{1} \boxplus \Psi'$, then $\Theta(\pi')$ is globally generic but not cuspidal (by [GRS1]). Indeed, under pullback by f_1, $f_1^*(\Theta(\pi'))$ has A-parameter $\mathbf{1} \boxplus \Psi = \mathbf{1} \boxplus \mathbf{1} \boxplus \Psi'$, which contains the trivial representation $\mathbf{1}$ two times and hence is not a discrete A-parameter. In this case, the pullback $f_2^*(\Theta(\pi'))$ under f_2 has A-parameter of the form $\tau \boxplus \tau^\vee$ for a cuspidal representation τ of GL_4. Since we do not need this case later on, we will skip the details here.

In §7, we will consider the case of cuspidal representations of PGSp_6 associated to holomorphic Siegel cusp forms of level 1. In that case, we shall give a full determination of the possible shapes of the A-parameter of the Spin lift of π.

4. **Variants**

We may exploit the above combination of similitude theta correspondence and triality in a couple of other situations, providing cheap constructions of interesting nontempered discrete automorphic representations of PGSO_8. More precisely, we may consider the global theta lifting associated to the tower:

$$\begin{tikzcd} \mathrm{PGSp}_6 \arrow[dr, dash] & \\ \mathrm{PGSp}_4 \arrow[r, dash] & \mathrm{PGSO}_8 \\ \mathrm{PGSp}_2 \arrow[ur, dash] & \end{tikzcd}$$

We have considered the theta lifting from PGSp_6 to PGSO_8 in the previous section. Let us consider the theta lifting from the two smaller symplectic similitude groups in this section.

4.1. **Theta lifting.** The isometry theta lifting from Sp_2 to SO_8 takes a cuspidal representation π of Sp_2 to the near equivalence class on SO_8 given by the nontempered A-parameter

$$\Psi_\pi \boxplus S_5,$$

where Ψ_π is the L-parameter of π, which is a self-dual cuspidal representation of GL_3, and S_k will denote the k-dimensional irreducible representation of the Arthur SL_2. Similarly, the isometry theta correspondence from Sp_4 to SO_8 takes a cuspidal representation π of Sp_4 to the near equivalence class given by the A-parameter

$$\Psi_\pi \boxplus S_3,$$

where now Ψ_π is a self-dual automorphic representation of GL_5 (the A-parameter of π).

In both these cases, we see that the Hecke-Satake family of $\Theta(\pi)$ (or equivalently its A-parameter) is valued in $\mathrm{SO}_3(\mathbb{C}) \times \mathrm{SO}_5(\mathbb{C}) \subset \mathrm{SO}_8(\mathbb{C})$. If one considers the similitude theta lifting from PGSp_2 or PGSp_4 to PGSO_8, then as we show in Appendix A (Proposition 8.8), the corresponding functoriality is given by the top row of the commutative diagram:

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{C}) \times \mathrm{Sp}_4(\mathbb{C}) \arrow[r, equal] & \mathrm{Spin}_3(\mathbb{C}) \times \mathrm{Spin}_5(\mathbb{C}) \arrow[r, "j"] \arrow[d] & \mathrm{Spin}_8(\mathbb{C}) \arrow[d, "f_1^\vee"] \\
 & \mathrm{SO}_3(\mathbb{C}) \times \mathrm{SO}_5(\mathbb{C}) \arrow[r] & \mathrm{SO}_8(\mathbb{C}).
\end{tikzcd}$$

More precisely,

- if π is a cuspidal representation of PGSp_2 with Hecke-Satake family
$$c(\pi) = \{c(\pi_v)\}_{v \notin S} \subset \mathrm{Spin}_3(\mathbb{C}),$$
and $c(\mathrm{triv}) \subset \mathrm{Spin}_5(\mathbb{C})$ is the Hecke-Satake family for the trivial representation of PGSp_4, then the Hecke-Satake family for the theta lift of π to PGSO_8 is
$$\{j(c(\pi_v), c(\mathrm{triv}_v))\}_{v \notin S} \subset \mathrm{Spin}_8(\mathbb{C}).$$
- if π is a cuspidal representation of PGSp_4 with Hecke-Satake family $c(\pi) \subset \mathrm{Spin}_5(\mathbb{C})$ and $c(\mathrm{triv}) \subset \mathrm{Spin}_3(\mathbb{C})$ is the Hecke-Satake family for the trivial representation of PGSp_2, then the Hecke-Satake family of the theta lift of π to PGSO_8 is
$$\{j(c(\mathrm{triv}_v), c(\pi_v))\}_{v \notin S} \subset \mathrm{Spin}_8(\mathbb{C}).$$

In other words, the Hecke-Satake family of the similitude theta lift is contained in the subgroup $j(\mathrm{Spin}_3(\mathbb{C}) \times \mathrm{Spin}_5(\mathbb{C})) \subset \mathrm{Spin}_8(\mathbb{C})$.

Further, as we are working with the split PGSO_8, the global theta lift $\Theta(\pi)$ of a cuspidal representation π of PGSp_2 or PGSp_4 is nonzero, because one is in the so-called stable range. Moreover, $\Theta(\pi)$ is an irreducible square-integrable automorphic representation (typically not cuspidal). For the above statements, see [G, Prop. 3.2] and the references therein. Hence, in these cases, we do not need to impose further conditions (such as the restriction of π to the isometry groups having a generic A-parameter) to ensure the nonvanishing of the global theta lifting.

4.2. **Application of triality.** Now observe that

$$f_2^\vee \circ j = \mathrm{spin}_3 \boxtimes \mathrm{spin}_5$$

where the RHS denotes the Spin representations of $\mathrm{Spin}_3(\mathbb{C})$ and $\mathrm{Spin}_5(\mathbb{C})$ respectively. These are nothing but the standard representations of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{Sp}_4(\mathbb{C})$ respectively. By this discussion (and Proposition 8.8), we deduce the following result, which extends [CL, Thm. 2.2 and Thm. 2.3(i)]:

Theorem 4.1. *(i) If π is a cuspidal representation of $\mathrm{PGL}_2 \simeq \mathrm{PGSp}_2$ and $\Theta(\pi)$ is its global theta lift to the split PGSO_8 (which is nonzero), then $f_2^*(\Theta(\pi))$ is a square integrable automorphic representation with nontempered A-parameter*

$$\Psi_\pi \boxtimes S_4$$

where $\Psi_\pi = \pi$ denotes the A-parameter of π. Moreover, if π_v is unramified, then $[f_2^*(\Theta(\pi))]_v$ is also unramified, whose L-parameter (or Satake parameter) is the L-parameter associated to $\Psi_\pi \boxtimes S_4$.

(ii) If π is a cuspidal representation of PGSp_4 and $\Theta(\pi)$ is its global theta lift to the split PGSO_8 (which is nonzero), then $f_2^*(\Theta(\pi))$ is a square integrable automorphic representation with nontempered A-parameter

$$\Psi_\pi \boxtimes S_2,$$

where Ψ_π denotes the A-parameter of π viewed as a representation of $\mathrm{PGSp}_4 \simeq \mathrm{SO}_5$. Moreover, if π_v is unramified, then $[f_2^*(\Theta(\pi))]_v$ is also unramified, whose L-parameter (or Satake parameter) is the L-parameter associated to $\Psi_\pi \boxtimes S_2$.

4.3. **Ikeda's lifting.** In the context of Theorem 4.1 (i), we may further consider the global Theta lift Π of $f_2^*(\Theta(\pi))$ to Sp_8. If nonzero, it provides a rather cheap construction, of some automorphic representation of Sp_8 with standard A-parameter $(\Psi_\pi \boxtimes S_4) \boxplus \mathbf{1}$ (not relying on [A] nor on [CKPSS]).

Better, in the construction of $\Theta(\pi)$, let us replace the split GSO_8 by $\mathrm{GSO}(V)$, where V is an octonion F-algebra whose set Σ of (necessarily real) non-split places is nonempty. By [R2], $\Theta(\pi)$ is nonzero if, and only if, π_v is a holomorphic discrete series of weight ≥ 4 for all $v \in \Sigma$. The triality automorphism (hence f_2) still exists on $\mathrm{PGSO}(V)$. Assuming again that the theta lift Π of $f_2^*(\Theta(\pi))$ to Sp_8 is nonzero, then Π_v is a holomorphic discrete series for all $v \in V$, providing an alternative construction of the liftings in [I] and [IY] in the special case of Sp_8 (under slightly different assumptions).

This idea was used in [CL, §5.4] to give an elementary proof that the Schottky form on $\mathrm{Sp}_8(\mathbb{Z})$ is an Ikeda lift of the discriminant cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. For a general π, the non-vanishing of Π may be addressed using, for example, [GT3, Cor. 7.9(c)].

5. **Rankin-Selberg Lifting** $\mathrm{PGSp}_4 \times \mathrm{PGL}_2 \to \mathrm{SO}_8$

The examples considered in the previous section all arise through an application of the triality automorphism to a nontempered A-parameter factoring through the map

$$\begin{array}{ccc} \mathrm{Spin}_3(\mathbb{C}) \times \mathrm{Spin}_5(\mathbb{C}) & \xrightarrow{j} & \mathrm{Spin}_8(\mathbb{C}) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{SO}_3(\mathbb{C}) \times \mathrm{SO}_5(\mathbb{C}) & \xrightarrow{j^\flat} & \mathrm{SO}_8(\mathbb{C}). \end{array}$$

The map j of dual groups lies over the corresponding map $j^\flat$ which gives the twisted endoscopic transfer $\mathrm{Sp}_2 \times \mathrm{Sp}_4 \longrightarrow \mathrm{SO}_8$ (associated to an outer automorphism of order 2) established in Arthur's work [A]. In [X1, X2], Xu has constructed this endoscopic transfer at the level of similitude groups associated to j. We summarise his results in this particular case:

Theorem 5.1. *Let π be a cuspidal representation of $\mathrm{PGL}_2 = \mathrm{PGSp}_2$ and let σ be a cuspidal representation of PGSp_4 with generic A-parameter. Then the endoscopic lifting of (π, σ) associated to j exists.*

Let Π be a cuspidal representation of PGSO_8 which is a weak lifting of (π, σ) via j. Consider now the pullback of Π to SO_8 via f_2, followed by the transfer std_* to GL_8. Then we have:

Corollary 5.2. *On GL_8, the representation $\mathrm{std}_*(f_2^*(\Pi))$ is a weak lift of $\pi \boxtimes \sigma$ relative to the tensor product map*

$$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{Sp}_4(\mathbb{C}) \xrightarrow{\boxtimes} \mathrm{SO}_8(\mathbb{C}) \xrightarrow{\mathrm{std}} \mathrm{GL}_8(\mathbb{C}).$$

We may restate this as:

Theorem 5.3. *If π is a self-dual cuspidal representation of GL_2 and σ is a self-dual cuspidal representation of GL_4 of symplectic type, then the Rankin-Selberg lifting $\pi \boxtimes \sigma$ exists as an automorphic representation of GL_8.*

Proof. We examine the two cases:

- if π is of orthogonal type, then π is obtained by automorphic induction from a Hecke character χ of a quadratic field extension E of k, say
$$\pi = \mathrm{AI}_{E/k}(\chi).$$
One may consider the automorphic representation
$$\Pi = \mathrm{AI}_{E/k}(\mathrm{BC}_{E/k}(\sigma) \otimes \chi),$$
where $\mathrm{BC}_{E/k}$ denotes the base change lifting with respect to E/k. Then Π is the Rankin-Selberg lift of $\pi \boxtimes \sigma$. Observe that no conditions need to be imposed on σ here.
- the main case is when π and σ are both of symplectic type; this is precisely the case treated by Corollary 5.2 above.

$\square$

6. **Spin Lifting for** GSp_6

In § 3, we have shown the Spin lifting from PGSp_6 to GL_8 via:

(a) similitude theta lifting to PGSO_8,
(b) followed by an application of the triality automorphism, before pulling back to (the standard) SO_8 (or in one step, by pulling back via the nonstandard f_2), and
(c) transferring from SO_8 to GL_8 via [CKPSS] or [A],

at least for cuspidal representations of PGSp_6 with generic A-parameters when restricted to Sp_6. In this section, we first explain how to remove the trivial central character hypothesis and extend Theorem 3.8 to construct a spin lifting for cuspidal representations of GSp_6 (with generic A-parameters when restricted to Sp_6). This will be done in § 6.1, using a similar strategy as above. The first step is the same as in (a) above: using the global similitude theta lifting, we may lift a (globally generic) cuspidal representation of GSp_6 to an automorphic representation of GSO_8. For step (b), we need to replace f_2 with the natural isogeny

$$\tilde{\rho}_2 : \mathrm{GSpin}_8 \longrightarrow \mathrm{GSO}_8 \tag{6.1}$$

introduced in §2.4. For the last step (c), we rather rely on the transfer from GSpin_8 to GL_8 via the results of Asgari-Shahidi [AS].

The established spin lifting from GSp_6 to GL_8 also has interesting consequences to a certain lifting from PGL_7 to SL_8, discussed in the end of § 6.1. In the next two subsections § 6.2 and § 6.3, we explore two other applications of the ideas above: first to a Rankin-Selberg (tensor-product) lifting from $\mathrm{GL}_2 \times \mathrm{GSp}_4$ to GL_8 generalizing Theorem 1.1 (ii), and second to the study of the behavior of the endoscopic lifting $\mathrm{GSO}_4 \times \mathrm{GSO}_4 \longrightarrow \mathrm{GSO}_8$ *after applying triality.* The work of Bin Xu [X1, X2] plays some role here.

The isogeny (6.1) naturally induces $f_2 : \mathrm{SO}_8 \to \mathrm{PGSO}_8$ (see the diagram (6.4)), hence is related to triality since f_2 is; however, it is not the composition of a triality automorphism and a standard morphism contrary to f_2, since neither GSO_8 nor GSpin_8 do have a triality automorphism. We finally explain in § 6.4 a way to restore an order 3 symmetry in this picture by introducing in a certain larger group ${}^{\circledast}\mathrm{Spin}_8$, with derived subgroup Spin_8 and center $\simeq \mathbb{G}_m^3$, over which a triality θ exists. We will show that $\tilde{\rho}_2$ naturally factors as

$$\mathrm{GSpin}_8 \longrightarrow {}^{\circledast}\mathrm{Spin}_8 \xrightarrow{\theta} {}^{\circledast}\mathrm{Spin}_8 \longrightarrow \mathrm{GSO}_8, \tag{6.2}$$

with "standard" first and last maps. As a consequence, the pullback of automorphic forms via $\tilde{\rho}_2$ used to perform the Spin lifting for GSp_6 decomposes accordingly as a sequence of three pullbacks. Finally, we explain that ${}^{\circledast}\mathrm{Spin}_8$ actually appears as a Levi subgroup in the exceptional similitude group GE_6, with θ induced by an inner automorphism of GE_6.

6.1. **The Spin lifting.**

The Langlands dual group of GSp_6 is $\mathrm{GSpin}_7(\mathbb{C})$ and one has the Spin representation (see § 2.1)

$$\mathrm{spin} : \mathrm{GSpin}_7(\mathbb{C}) \longrightarrow \mathrm{GL}_8(\mathbb{C}).$$

The corresponding weak lifting from PGSp_6 to GL_8 is the *spin lifting* for GSp_6.

Theorem 6.3. *Let π be a cuspidal representation of* GSp_6 *whose restriction to* Sp_6 *has a generic A-parameter. Then the weak spin lifting of π exists on* GL_8.

By the discussion in § 2.4, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{GSpin}_8 & \xrightarrow{\tilde{\rho}_2} & \mathrm{GSO}_8 \\ \downarrow & & \downarrow \\ \mathrm{SO}_8 & \xrightarrow{f_2} & \mathrm{PGSO}_8 \end{array} \tag{6.4}$$

where the first vertical arrow is the standard morphism ρ and the second is the canonical projection. The Langlands dual of this diagram, incorporating both the natural embedding $\iota : \mathrm{GSpin}_7(\mathbb{C}) \to \mathrm{GSpin}_8(\mathbb{C})$ (defined in §2.1, and extending the ι in (3.4)) and the standard representations, is thus the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{GL}_8(\mathbb{C}) & \xleftarrow{\mathrm{std}} & \mathrm{GSO}_8(\mathbb{C}) & \xleftarrow{\tilde{\rho}_2^\vee} & \mathrm{GSpin}_8(\mathbb{C}) & \xleftarrow{\iota} & \mathrm{GSpin}_7(\mathbb{C}) \\ \| & & \uparrow & & \uparrow & & \uparrow \\ \mathrm{GL}_8(\mathbb{C}) & \xleftarrow{\mathrm{std}} & \mathrm{SO}_8(\mathbb{C}) & \xleftarrow{f_2^\vee} & \mathrm{Spin}_8(\mathbb{C}) & \xleftarrow{\iota} & \mathrm{Spin}_7(\mathbb{C}), \end{array} \tag{6.5}$$

with canonical inclusions as vertical arrows.

Lemma 6.6. *The composite map in the first row of the diagram* (6.5) *is the spin representation of* $\mathrm{GSpin}_7(\mathbb{C})$.

Proof. Let $h : \mathrm{GSpin}_7(\mathbb{C}) \to \mathrm{GL}_8(\mathbb{C})$ be the composite map of the statement. By commutativity of (6.5), and the discussion in §3.3, we know that $h|_{\mathrm{Spin}_7(\mathbb{C})}$ is the spin representation of $\mathrm{Spin}_7(\mathbb{C})$. It only remains to show that h, or equivalently $\tilde{\rho}_2^\vee$, induces the identity map $t \mapsto t$ on the natural central (or cocentral) $\mathbb{G}_m$ of its source and its target. But this follows as $\tilde{\rho}_2 : \mathrm{GSpin}_8 \to \mathrm{GSO}_8$ itself has this property, since it induces a half-spin representation of GSpin_8. □

Proof. (of Theorem 6.3) By Lemma 6.6, the Spin lifting from GSp_6 to GL_8 has been broken down into a composite of three functorial liftings appearing in the first row in (6.5), namely those induced by the three dual maps ι, $\tilde{\rho}_2^\vee$ and std. Starting from a cuspidal representation π of GSp_6 whose restriction to Sp_6 has generic A-parameter and whose Hecke-Satake family $c(\pi)$ is contained in $\mathrm{GSpin}_7(\mathbb{C})$, these three weak functorial liftings can be obtained as follows:

- (*lifting ι*) As we saw in the proof of Theorem 3.8, after replacing π by a globally generic cuspidal representation in the same global A-packet (constructed by B. Xu), the global similitude theta lifting provides an explicit construction of the weak functorial lifting associated to ι, giving rise to an automorphic representation $\Theta(\pi)$ on GSO_8 with Hecke-Satake family
$$c(\Theta(\pi)) \ = \ \iota(c(\pi)) \subset \mathrm{GSpin}_8(\mathbb{C}).$$
- (*lifting $\tilde{\rho}_2^\vee$*) It follows by Proposition 3.2 that the functoriality for $\tilde{\rho}_2^\vee$ is simply given by the pullback of automorphic forms via the isogeny $\tilde{\rho}_2$: any automorphic constituent

Π of the restriction of $\Theta(\pi)$ to GSpin_8 via $\tilde{\rho}_2$ has Hecke-Satake family

$$c(\Pi) \;=\; \tilde{\rho}_2^\vee\,(c(\Theta(\pi))) \;=\; \tilde{\rho}_2^\vee\,(\iota(c(\pi))) \;\subset \mathrm{GSO}_8(\mathbb{C}).$$

- (*lifting* std) Finally, the last functoriality from GSpin_8 to GL_8 induced by std has been shown by Asgari-Shahidi [AS], allowing us to produce an automorphic representation Π' of GL_8 with Hecke-Satake family
$$c(\Pi') \;=\; \mathrm{std}(c(\Pi)) \;=\; \mathrm{std}\left(\tilde{\rho}_2^\vee\,(\iota(c(\pi)))\right) \;\subset \mathrm{GL}_8(\mathbb{C}).$$

This completes the proof of the theorem. □

Remark 6.7. *If one knows that the Asgari-Shahidi lift from* GSpin *groups to* GL *is strong at unramified places, then one has a similar strengthening of the theorem as in Theorem 3.10 (by the same proof).*

Let us end this subsection with an application to a lifting from PGL_7 to SL_8. Observe that the spin representation of $\mathrm{GSpin}_7(\mathbb{C})$ induces a morphism

$$\overline{\mathrm{spin}} : \mathrm{SO}_7(\mathbb{C}) \longrightarrow \mathrm{PGL}_8(\mathbb{C}).$$

On the other hand, if π is a selfdual cuspidal automorphic representation of PGL_7, then π is necessarily orthogonal. In particular, if π_v is unramified then $c(\pi_v)$ is the image under std : $\mathrm{SO}_7(\mathbb{C}) \to \mathrm{GL}_7(\mathbb{C})$ of a unique semisimple conjugacy class $c'(\pi_v)$ in $\mathrm{SO}_7(\mathbb{C})$. Setting $c'(\pi) = \{c'(\pi_v) : v \notin S\}$, it makes sense to ask for the existence of an automorphic representation Π of SL_8 satisfying $c(\Pi) = \overline{\mathrm{spin}}(c'(\pi))$. We call such a Π a (weak) $\overline{\mathrm{spin}}$ *lifting* of π.

Theorem 6.8. *If π is a selfdual cuspidal automorphic representation of* PGL_7*, then there exists a* $\overline{\mathrm{spin}}$ *lifting of π.*

Proof. By [GRS2], there exists a globally generic cuspidal automorphic representation σ of Sp_6 such that $\mathrm{std}(c(\sigma)) = c(\pi)$, or equivalently, such that $c(\sigma) = c'(\pi)$. Let $\widetilde{\sigma}$ be a (necessarily globally generic) cuspidal automorphic representation of GSp_6 such that $\widetilde{\sigma}|_{\mathrm{Sp}_6(\mathbb{A})}$ contains σ; the existence of such a $\widetilde{\sigma}$ follows for instance from [X1] Lemma 5.3. The image of $c(\widetilde{\sigma})$ under $\rho : \mathrm{GSpin}_7(\mathbb{C}) \to \mathrm{SO}_7(\mathbb{C})$ is then $c(\pi')$ by Proposition 3.2. Let Π_0 be spin lifting of $\widetilde{\sigma}$ given by Theorem 6.3; it satisfies $c(\Pi_0) = \mathrm{spin}(c(\widetilde{\sigma}))$. Any automorphic constituent Π of $\Pi_0|_{\mathrm{SL}_8}$ has thus the required property, by Proposition 3.2 again. □

6.2. **Rankin-Selberg lifting.** Recall that in Corollary 5.2, we have produced the weak functorial lifting from $\mathrm{PGSp}_4 \times \mathrm{PGL}_2$ to GL_8 relative to the map

$$\mathrm{Sp}_4(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\boxtimes} \mathrm{SO}_8(\mathbb{C}) \longrightarrow \mathrm{GL}_8(\mathbb{C})$$

of dual groups. In the same vein, we can extend this result by removing the hypothesis of trivial central characters. More precisely, we have:

Theorem 6.9. *Suppose that π is a cuspidal representation of* $\mathrm{GSp}_2 = \mathrm{GL}_2$ *and σ a cuspidal representation of* GSp_4*, with Hecke-Satake family*

$$c(\pi) \subset \mathrm{GSp}_2^\vee = \mathrm{GSpin}_3(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \quad \textit{and} \quad c(\sigma) \subset \mathrm{GSpin}_5(\mathbb{C}) = \mathrm{GSp}_4(\mathbb{C}).$$

Then there is an automorphic representation $\pi \boxtimes \sigma$ of GL_8 *with Hecke-Satake family*

$$c(\pi \otimes \sigma) = \{c(\pi_v) \boxtimes c(\sigma_v)\}_{v \notin S}.$$

To prove this, note that from § 2.1 one has a natural map

$$\iota : \mathrm{GSpin}_3(\mathbb{C}) \times \mathrm{GSpin}_5(\mathbb{C}) \longrightarrow \mathrm{GSpin}_8(\mathbb{C}).$$

If we replace the map $\mathrm{GSpin}_7(\mathbb{C}) \longrightarrow \mathrm{GSpin}_8(\mathbb{C})$ in (6.5) by this map, the composite

$$\mathrm{GSpin}_3(\mathbb{C}) \times \mathrm{GSpin}_5(\mathbb{C}) \xrightarrow{\ \iota\ } \mathrm{GSpin}_8(\mathbb{C}) \xrightarrow{\tilde{\rho}_2^\vee} \mathrm{GSO}_8(\mathbb{C}) \xrightarrow{\mathrm{std}} \mathrm{GL}_8(\mathbb{C})$$

induces the Rankin-Selberg lifting $\boxtimes$.

To establish this Rankin-Selberg lifting, it thus suffices to establish the weak functorial lifting for the three maps above. The second and third steps have been described in the previous subsection. The first (induced by ι) is an endoscopic lifting for similitude groups, from $\mathrm{GSp}_2 \times \mathrm{GSp}_4$ to GSO_8. This has been proven by Bin Xu [X1, X2]. With this, the proof of the theorem is complete.

6.3. **Another functorial lifting.** The Rankin-Selberg lifting proved in Theorem 6.9, based on the endoscopic transfer from $\mathrm{GSp}_4 \times \mathrm{GL}_2$ followed by an application of triality, is a physical manifestation of the equality

$$3 + 5 = 4 \times 2.$$

In this subsection, we consider another instance of this construction starting from the endoscopic transfer induced by the natural morphism

$$\mathrm{GSpin}_4(\mathbb{C}) \times \mathrm{GSpin}_4(\mathbb{C}) \longrightarrow \mathrm{GSpin}_8(\mathbb{C}).$$

The associated endoscopic transfer from $\mathrm{GSO}_4 \times \mathrm{GSO}_4$ to GSO_8 has largely been shown in [X2, Thm. 1.2], under a hypothesis [X2, Defn. 4.4] on cuspidal automorphic representations. As we shall explain later on, the unavailability of this endoscopic lifting in full generality will not unduly bother us below. Hence, for the subsequent discussion, the reader may assume for simplicity that this endoscopic lifting is available. The question we shall consider is what happens when one applies the triality automorphism to the resulting lift.

Let us begin by setting up some notations, for bookkeeping purposes. One has the following concrete realization of the groups GSO_4 and GSpin_4:

$$\mathrm{GSO}_4 = (\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathbb{G}_m^\nabla \quad \text{and} \quad \mathrm{GSpin}_4 = \mathrm{GL}_2 \times_{\det} \mathrm{GL}_2$$

where $\mathbb{G}_m^\nabla = \{(t, t^{-1}) : t \in \mathbb{G}_m\}$ and the suberscript $_{\det}$ refers to the subgroup of those elements (g_1, g_2) satisfying $\det(g_1) = \det(g_2)$. Given the large number of GL_2's here, we will annotate the various GL_2's with card suits, so as to help the reader and ourselves to distinguish between them. In particular, we set:

$$\mathrm{GSO}_4^{\heartsuit\diamondsuit} = (\mathrm{GL}_2^\heartsuit \times \mathrm{GL}_2^\diamondsuit)/\mathbb{G}_m^\nabla \quad \text{and} \quad \mathrm{GSpin}_4^{\heartsuit\diamondsuit} = \mathrm{GL}_2^\heartsuit \times_{\det} \mathrm{GL}_2^\diamondsuit.$$

Hence we have

$$(\mathrm{GSO}_4^{\heartsuit\diamondsuit})^\vee = \mathrm{GSpin}_4^{\heartsuit\diamondsuit}(\mathbb{C}) \quad \text{and} \quad (\mathrm{GSpin}_4^{\heartsuit\diamondsuit})^\vee = \mathrm{GSO}_4^{\heartsuit\diamondsuit}(\mathbb{C}).$$

Given the above concrete realizations of GSO_4 and GSpin_4, we can describe their (L-packets of) cuspidal representations as follows. Given cuspidal representations $\pi_\heartsuit$ of $\mathrm{GL}_2^\heartsuit$ and $\pi_\diamondsuit$ of $\mathrm{GL}_2^\diamondsuit$, the restriction of $\pi_\heartsuit \otimes \pi_\diamondsuit$ to $\mathrm{GSpin}_4^{\heartsuit\diamondsuit}$ gives an L-packet $[\pi_\heartsuit \otimes \pi_\diamondsuit]$. Note that

$$[\pi_\heartsuit \otimes \pi_\diamondsuit] = [\pi_\heartsuit \cdot \chi \otimes \pi_\diamondsuit \cdot \chi^{-1}] \quad \text{for any Hecke character } \chi.$$

On the other hand, assume now that the central characters of $\pi_\heartsuit$ and $\pi_\diamondsuit$ are equal. Then $\pi_\heartsuit \otimes \pi_\diamondsuit$ defines a cuspidal representation of $\mathrm{GSO}_4^{\heartsuit\diamondsuit}$.

Again, using the above concrete realizations of GSpin_4 and GSO_4, the reader can convince herself that there is no isogeny $\mathrm{GSpin}_4 \longrightarrow \mathrm{GSO}_4$. However, we note the following curious lemma:

Lemma 6.10. *The map*

$$\mathrm{GL}_2^\heartsuit \times \mathrm{GL}_2^\diamondsuit \times \mathrm{GL}_2^\spadesuit \times \mathrm{GL}_2^\clubsuit \longrightarrow \mathrm{GL}_2^\heartsuit \times \mathrm{GL}_2^\spadesuit \times \mathrm{GL}_2^\diamondsuit \times \mathrm{GL}_2^\clubsuit$$

given by exchanging the second and third entries gives rise to an isogeny

$$f : (\mathrm{GSpin}_4^{\heartsuit\diamondsuit} \times \mathrm{GSpin}_4^{\spadesuit\clubsuit})/\mathbb{G}_m^\nabla \longrightarrow \mathrm{GSO}_4^{\heartsuit\spadesuit} \times_{\mathrm{sim}} \mathrm{GSO}_4^{\diamondsuit\clubsuit}$$

The kernel of this isogeny is the subgroup

$$\mu_2^{\heartsuit\spadesuit} = \{[(\epsilon, 1), (\epsilon, 1)] : \epsilon \in \mu_2\} \subset (\mu_2^\heartsuit \times \mu_2^\diamondsuit \times \mu_2^\spadesuit \times \mu_2^\clubsuit)/\mathbb{G}_m^\nabla.$$

The map f fits into the following diagram of morphisms of Langlands dual groups:

$$\begin{tikzcd}
\mathrm{GSO}_4^{\heartsuit\spadesuit}(\mathbb{C}) \times_{\mathrm{sim}} \mathrm{GSO}_4^{\diamondsuit\clubsuit}(\mathbb{C}) \arrow[r, "\iota"] & \mathrm{GSO}_8(\mathbb{C}) \arrow[r, "\mathrm{std}"] & \mathrm{GL}_8(\mathbb{C}) \\
(\mathrm{GSpin}_4^{\heartsuit\diamondsuit}(\mathbb{C}) \times \mathrm{GSpin}_4^{\spadesuit\clubsuit}(\mathbb{C}))/\mathbb{G}_m^\nabla \arrow[u, "f"] \arrow[r, "\tilde{\iota}"] \arrow[d, "p"'] & \mathrm{GSpin}_8(\mathbb{C}) \arrow[u, "\tilde{\rho}_2^\vee"'] \arrow[d, "p"] & \\
\mathrm{SO}_4(\mathbb{C}) \times \mathrm{SO}_4(\mathbb{C}) \arrow[r, "\iota_\flat"] & \mathrm{SO}_8(\mathbb{C}) \arrow[r, "\mathrm{std}"] & \mathrm{GL}_8(\mathbb{C})
\end{tikzcd} \tag{6.11}$$

where the horizontal arrows are embeddings and the right upward arrow is the same map as in (6.5). While the embeddings ι and $\iota_\flat$ are the natural ones, there are in fact three choices for the embedding $\tilde{\iota}$, determined by the image of $\mu_2^{\heartsuit\spadesuit} = \mathrm{Ker}(f)$ in the center of $\mathrm{Spin}_8(\mathbb{C})$. For the above diagram to be commutative, we have to use the one such that

$$\tilde{\iota}(\mu_2^{\heartsuit\spadesuit}) = \mathrm{Ker}(\tilde{\rho}_2^\vee).$$

This diagram should induce a corresponding diagram of weak Langlands functorial lifting:

$$\begin{tikzcd}
{[\mathcal{A}((\mathrm{GSpin}_4^{\heartsuit\spadesuit} \times \mathrm{GSpin}_4^{\diamondsuit\clubsuit})/\mathbb{G}_m^\nabla)]} \arrow[r, "\iota_*"] & {[\mathcal{A}(\mathrm{GSpin}_8)]} \arrow[r, "\mathrm{std}_*"] & {[\mathcal{A}(\mathrm{GL}_8)]} \\
{[\mathcal{A}(\mathrm{GSO}_4^{\heartsuit\diamondsuit} \times_{\mathrm{sim}} \mathrm{GSO}_4^{\spadesuit\clubsuit})]} \arrow[u, "f_*"] \arrow[r, "\tilde{\iota}_*"] \arrow[d, "p_*"'] & {[\mathcal{A}(\mathrm{GSO}_8)]} \arrow[u, "(\tilde{\rho}_2^\vee)_*"'] \arrow[d, "p_*"] & \\
{[\mathcal{A}(\mathrm{SO}_4 \times \mathrm{SO}_4)]} \arrow[r, "(\iota_\flat)_*"] & {[\mathcal{A}(\mathrm{SO}_8)]} \arrow[r, "\mathrm{std}_*"] & {[\mathcal{A}(\mathrm{GL}_8)]}
\end{tikzcd} \tag{6.12}$$

where $[\mathcal{A}(G)]$ denotes the set of near equivalence classes of irreducible automorphic representations of G. Our goal in this subsection is to understand the composite lifitng $\mathrm{std}_* \circ (\tilde{\rho}_2^\vee)_* \circ \tilde{\iota}_*$. As mentioned above, the endoscopic lifting $\tilde{\iota}_*$ is not known in full generality (see [X2, Thm.

1.2]), but our main concern is with the composite lifting above. For this, the above commutative diagram allows us to construct $\mathrm{std}_* \circ \iota_* \circ f_*$ instead, thus bypassing the lack of $\tilde{\iota}_*$ in full generality.

We now compute f_* and $\mathrm{std}_* \circ \iota_*$:

- Since f is an isogeny, f_* is given by pullback of automorphic forms by Proposition 3.2. More precisely, suppose that
$$\sigma := [(\pi_\heartsuit \otimes \pi_\diamondsuit) \otimes (\pi_\spadesuit \otimes \pi_\clubsuit)] \in [\mathcal{A}(\mathrm{GSO}_4^{\heartsuit\diamondsuit} \times_{\mathrm{sim}} \mathrm{GSO}_4^{\spadesuit\clubsuit})],$$
so that
$$\omega_\heartsuit = \omega_\diamondsuit \quad \text{and} \quad \omega_\spadesuit = \omega_\clubsuit$$
where $\omega_?$ denotes the central character of $\pi_?$. Then Lemma 6.10 shows that
$$f_*(\sigma) = [\pi_\heartsuit \otimes \pi_\spadesuit)] \otimes [\pi_\diamondsuit \otimes \pi_\clubsuit] \in [\mathcal{A}((\mathrm{GSpin}_4^{\heartsuit\spadesuit} \times \mathrm{GSpin}_4^{\diamondsuit\clubsuit})/\mathbb{G}_m^\nabla)].$$
- To determine $\mathrm{std}_* \circ \iota_*$, we note that there is a commutative diagram of morphisms of dual groups:
$$\begin{array}{ccc} \mathrm{GSO}_4^{\heartsuit\spadesuit}(\mathbb{C}) \times_{\mathrm{sim}} \mathrm{GSO}_4^{\diamondsuit\clubsuit}(\mathbb{C}) & \xrightarrow{\iota} & \mathrm{GSO}_8(\mathbb{C}) \\ {\scriptstyle \mathrm{std}\otimes\mathrm{std}}\downarrow & & \downarrow{\scriptstyle \mathrm{std}} \\ \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C}) & \longrightarrow & \mathrm{GL}_8(\mathbb{C}) \end{array}$$
which should give rise to the following diagram of liftings:
$$\begin{array}{ccc} [\mathcal{A}((\mathrm{GSpin}_4^{\heartsuit\spadesuit} \times \mathrm{GSpin}_4^{\diamondsuit\clubsuit})/\mathbb{G}_m^\nabla)] & \xrightarrow{\iota_*} & [\mathcal{A}(\mathrm{GSpin}_8)] \\ {\scriptstyle \mathrm{std}_*\times\mathrm{std}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{std}_*} \\ [\mathcal{A}(\mathrm{GL}_4)] \times [\mathcal{A}(\mathrm{GL}_4)] & \xrightarrow{\boxplus} & [\mathcal{A}(\mathrm{GL}_8)] \end{array}$$
so that
$$\mathrm{std}_* \circ \iota_* = \boxplus \circ (\mathrm{std}_* \times \mathrm{std}_*).$$
On the RHS of this identity, $\boxplus$ is the formation of isobaric sum (i.e. parabolic induction) and
$$\mathrm{std}_* : [\mathcal{A}((\mathrm{GSpin}_4^{\heartsuit\spadesuit})] = [\mathcal{A}(\mathrm{GL}_2^\heartsuit \times_{\det} \mathrm{GL}_2^\spadesuit)] \longrightarrow [\mathcal{A}(\mathrm{GL}_4)]$$
is the Rankin-Selberg lifting $\boxtimes$ from $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 constructed by Ramakrishnan [Ra]. Hence, we have explained the construction of $\mathrm{std}_* \circ \iota_*$.

By the above discussion, we thus have
$$\mathrm{std}_* \circ (\tilde{\rho}_2^\vee)_* \circ \tilde{\iota}_* = \boxplus \circ (\mathrm{std}_* \times \mathrm{std}_*) \circ f_*$$
and all three functorial liftings on the RHS have been constructed. Thus, we have shown:

Proposition 6.13. *Let*
$$\sigma := [(\pi_\heartsuit \otimes \pi_\diamondsuit) \boxtimes (\pi_\spadesuit \otimes \pi_\clubsuit)] \in [\mathcal{A}(\mathrm{GSO}_4^{\heartsuit\diamondsuit} \times \mathrm{GSO}_4^{\spadesuit\clubsuit})].$$
so that
$$(\mathrm{std}_* \circ p_* \circ \tilde{\iota}_*)(\sigma) = [(\pi_\heartsuit \boxtimes \pi_\diamondsuit)) \boxplus (\pi_\spadesuit \boxtimes \pi_\clubsuit)] \in [\mathcal{A}(\mathrm{GL}_8)].$$

Then

$$(\mathrm{std}_* \circ (\tilde{\rho}_2^\vee)_* \circ \tilde{\iota}_*)(\sigma) = [(\pi_\heartsuit \boxtimes \pi_\spadesuit) \boxplus (\pi_\diamondsuit \boxtimes \pi_\clubsuit)] \in [\mathcal{A}(\mathrm{GL}_8)].$$

Hence, the application of triality does not produce a fundamentally new case of functorial lifting here: its effect is to produce a "remixing" as depicted in the following equation:

$$2_\heartsuit \cdot 2_\diamondsuit + 2_\spadesuit \cdot 2_\clubsuit = 2_\heartsuit \cdot 2_\spadesuit + 2_\diamondsuit \cdot 2_\clubsuit.$$

We end this subsection with a remark. In establishing the above proposition, we had only needed to establish the weak functorial lifting $\mathrm{std}_* \circ \iota_*$. In fact, the weak functorial lifting ι_*, which is an endoscopic lifting in the setting of GSpin-groups can also be established using the automorphic descent results of Hundley-Sayag [HS].

More precisely, let

$$\sigma := [\pi_\heartsuit \boxtimes \pi_\spadesuit] \boxtimes [\pi_\diamondsuit \boxtimes \pi_\clubsuit]$$

be an L-packet of cuspidal representations of $(\mathrm{GSpin}_4^{\heartsuit\spadesuit} \times \mathrm{GSpin}_4^{\diamondsuit\clubsuit})/\mathbb{G}_m^\nabla$, so that we have the identity of central characters

$$\omega_\heartsuit \cdot \omega_\spadesuit = \omega_\diamondsuit \cdot \omega_\clubsuit =: \mu.$$

Then as we saw above,

$$(\boxplus \circ (\mathrm{std}_* \times \mathrm{std}_*))(\sigma) = (\pi_\heartsuit \boxtimes \pi_\spadesuit) \boxplus (\pi_\diamondsuit \boxtimes \pi_\clubsuit) \quad \text{on } \mathrm{GL}_8.$$

The two summands in the above equation satisfies

$$(\pi_\heartsuit \boxtimes \pi_\spadesuit)^\vee = (\pi_\heartsuit \boxtimes \pi_\spadesuit) \cdot \mu^{-1} \quad (\pi_\diamondsuit \boxtimes \pi_\clubsuit)^\vee = (\pi_\diamondsuit \boxtimes \pi_\clubsuit) \cdot \mu^{-1},$$

so that the (partial) twisted Rankin-Selberg L-functions

$$L^S(s, (\pi_\heartsuit \boxtimes \pi_\spadesuit) \times (\pi_\heartsuit \boxtimes \pi_\spadesuit) \cdot \mu^{-1}) \quad \text{and} \quad L^S(s, (\pi_\diamondsuit \boxtimes \pi_\clubsuit) \times (\pi_\diamondsuit \boxtimes \pi_\clubsuit) \cdot \mu^{-1})$$

have simple poles at $s = 1$. On the other hand, the twisted exterior square L-function

$$L^S(s, \pi_\heartsuit \boxtimes \pi_\spadesuit, \wedge^2 \times \mu^{-1}) = L^S(s, \pi_\heartsuit, \mathrm{Ad}) \cdot L^S(s, \pi_\spadesuit, \mathrm{Ad})$$

is holomorphic at $s = 1$, and likewise for the twisted exterior square L-function of $\pi_\diamondsuit \boxtimes \pi_\clubsuit$. Hence, the twisted symmetric square L-functions

$$L^S(s, \pi_\heartsuit \boxtimes \pi_\spadesuit, \mathrm{Sym}^2 \times \mu^{-1}) \quad \text{and} \quad L^S(s, \pi_\diamondsuit \boxtimes \pi_\clubsuit, \mathrm{Sym}^2 \times \mu^{-1})$$

have poles at $s = 1$. By [HS], one concludes that $\boxplus \circ (\mathrm{std}_* \times \mathrm{std}_*)(\sigma)$ can be descended back to GSpin_8. In other words, there is a globally generic automorphic representation $\iota_*(\sigma)$ on GSpin_8 such that

$$\mathrm{std}_*(\iota_*(\sigma)) = (\pi_\heartsuit \boxtimes \pi_\spadesuit) \boxplus (\pi_\diamondsuit \boxtimes \pi_\clubsuit) \quad \text{on } \mathrm{GL}_8.$$

6.4. **Where art thou, triality?** Neither GSO_8 nor GSpin_8 has an automorphism inducing triality on PGSO_8 or Spin_8, because of the presence of a unique central $\mathbb{G}_m$ subgroup. As promised in the end of the introduction of Section 6, we now first explain how to restore an order 3 symmetry on a variant of GSpin_8. In this section, our groups are defined and split over an arbitrary field F, say with $\operatorname{char} F \neq 2$.

Let E be the set of order two elements in the center Z of Spin_8. We have $|E| = 3$. For bookkeeping reasons we introduce, for each $e \in E$, a copy of $\mathbb{G}_m$, SO_8 and GSO_8, that we denote respectively by $\mathbb{G}_m^e$, SO_8^e and GSO_8^e; we also use the notation $\mathbb{G}_m^E$ for $\prod_{e\in E} \mathbb{G}_m^e$. We have an embedding

$$\iota : Z \to \mathbb{G}_m^E$$

sending any $e \in E \subset Z$ to the element of $\mathbb{G}_m^E$ with e-component 1, and two other components -1. Using this embedding, we set:

$$ {}^{\circledast}\mathrm{Spin}_8 = (\mathbb{G}_m^E \times \mathrm{Spin}_8)/(\iota \times \mathrm{id})(Z). \tag{6.14}$$

(pronounced *tri-spin*). Let us fix a triality automorphism θ of Spin_8 as in §2.4. It naturally induces a 3-cycle on E. The automorphism θ thus trivially extends to ${}^{\circledast}\mathrm{Spin}_8$ by permuting the factors of $\mathbb{G}_m^E$ according to the same cycle, and we still denote by θ this extension.

Let $e \in E$. It will be convenient to set $e' = \theta(e)$ and $e'' = \theta^2(e) = \theta^{-1}(e)$. We have then $E = \{e, e', e''\}$ and we may write any element $t \in \mathbb{G}_m^E$ as $(t_e, t_{e'}, t_{e''})$. We finally set

$$\mathrm{GSpin}_8^e = (\mathbb{G}_m^e \times \mathrm{Spin}_8)\,/\langle(-1, e)\rangle.$$

There is a unique pair of morphisms j_e and ρ_e as in the sequence below

$$\mathrm{GSpin}_8^e \xrightarrow{\ j_e\ } {}^{\circledast}\mathrm{Spin}_8 \xrightarrow{\ \rho_e\ } \mathrm{GSO}_8^e$$

and satisfying the following properties:

- The morphism j_e induces the identity on the natural Spin_8 subgroups on both sides.
- For $t \in \mathbb{G}_m^e$, the element $j_e(t) \in \mathbb{G}_m^E$ has e-component 1, and other two components t.

Such a morphism j_e exists as we have $(1, -1, -1) \equiv e'e'' = e$ in ${}^{\circledast}\mathrm{Spin}_8$.

- Over the natural subgroup Spin_8 inside ${}^{\circledast}\mathrm{Spin}_8$, the morphism ρ_e coincides with the morphism $\rho_i : \mathrm{Spin}_8 \to \mathrm{SO}_8$ defined in §2.4 for the unique i such that $\rho_i(e) = 1$.
- For $t \in \mathbb{G}_m^e$, the element $\rho_e(t) \in \mathrm{GSO}_8^e$ is the multiplication by the scalar $t_{e'}/t_{e''}$.

Such a morphism ρ_e exists, as for all $f \in E$ and $t = \iota(f)$, the scalar $t_{e'}/t_{e''}\rho_e(f)$ is always 1 (we have $\rho_i(e) = 1$ and $\rho_i(e') = \rho_i(e'') = -1$). Summarizing, we have for any $e \in E$ the following diagram:

$$\begin{tikzcd}
\mathrm{GSpin}_8^e \arrow[dr, "j_e"] & & \mathrm{GSO}_8^e \\
\mathrm{GSpin}_8^{e'} \arrow[r] & {}^{\circledast}\mathrm{Spin}_8 \arrow[ur, "\rho_e"] \arrow[r] \arrow[dr, "\rho_{e''}"'] & \mathrm{GSO}_8^{e'} \\
\mathrm{GSpin}_8^{e''} \arrow[ur, "j_{e''}"'] & & \mathrm{GSO}_8^{e''}
\end{tikzcd}$$

The triality θ of Spin_8 induces for each e an isomorphism $\mathrm{GSpin}_8^e \xrightarrow{\sim} \mathrm{GSpin}_8^{e'}$ that we may also harmlessly denote θ. We then have by construction the identities

$$\rho_e = \rho_{e'} \circ \theta \ \text{ and } \ j_e = j_{e'} \circ \theta. \tag{6.15}$$

Observe moreover that the map

$$\rho_e \circ j_e : \mathrm{GSpin}_8^e \longrightarrow \mathrm{GSO}_8^e$$

is trivial on $\mathbb{G}_m^e$ and factors through the natural inclusion $\mathrm{SO}_8^e \subset \mathrm{GSO}_8^e$ (in particular, it is not an isogeny), whereas the morphism

$$\rho_{e'} \circ j_e : \mathrm{GSpin}_8^e \longrightarrow \mathrm{GSO}_8^{e'}$$

coincides with $\rho_{e'}$ on Spin_8 and maps $t \in \mathbb{G}_m$ to the multiplication by t in $\mathrm{GSO}_8^{e'}$. It follows that $\rho_{e'} \circ j_e$ is an instance of the map $\tilde{\rho}_2$ in (6.1). More precisely, if we label E by $\{1, 2, 3\}$ as in § 2.4, with e labelled by 1 and θ inducing the cycle (1 2 3) on E, then we have

$$\tilde{\rho}_2 = \rho_{e'} \circ j_e.$$

The promised factorisation (6.2) follows then from the identity

$$\rho_{e'} \circ j_e = \rho_e \circ \theta^{-1} \circ j_e,$$

which in turn follows from the first equation in (6.15).

Without going in too much details, we finally discuss how the group ${}^{\circledast}\mathrm{Spin}_8$ and its automorphism θ occur when studying the exceptional group

$$H = \mathrm{GE}_6 = (\mathbb{G}_m \times \mathrm{E}_6^{sc})/\mu_3^{\Delta},$$

whose derived group is simply-connected of type E_6. Looking at the Dynkin diagram of E_6, one sees that there is a (non-maximal) parabolic subgroup $P = MN$ of H whose Levi subgroup M is of semisimple type D_4. Indeed, the derived subgroup of M is isomorphic to Spin_8, and it can be shown that we have

$$M \simeq {}^{\circledast}\mathrm{Spin}_8.$$

The associated Weyl group $N_H(M)/M$ is isomorphic to S_3. Hence, there is an order 3 element h of H normalizing M and we can show that we may choose this element and the isomorphism above so that h induces the automorphism θ of ${}^{\circledast}\mathrm{Spin}_8$. Let us also mention that the adjoint action of M on $\mathrm{Lie}(N)$ decomposes into the direct sum of the three 8-dimensional irreducible representations $\rho_e \oplus \rho_{e'} \oplus \rho_{e''}$ of M.

7. An Arithmetic Application: Siegel modular forms for $\mathrm{Sp}_6(\mathbb{Z})$

In this section, we specialize the results of Section 3.5 to the case of automorphic representations of PGSp_6 over $\mathbb{Q}$ generated by holomorphic Siegel modular forms for the full Siegel modular group $\mathrm{Sp}_6(\mathbb{Z})$. In this setting, we shall show that Theorems 3.8 and 3.10 continue to hold without the genericity (of A-parameters) assumption there, so that the Spin lifting always exists on GL_8. In addition, we shall determine precisely the shape of the A-parameter of the Spin lifting on GL_8; this amounts to showing a *cuspidality criterion* for the Spin lifting. These improvements will be possible by the use of Galois representations arguments, and in particular, of the Minkowski theorem (*a non trivial number field always has a ramified prime*). We conclude with an application to spinor L-functions.

7.1. **Cuspidal representations of Siegel type.** In all of this section, we assume that π is a cuspidal automorphic representation of PGSp_6 over $\mathbb{Q}$ generated by a holomorphic Siegel modular eigenform[2] f for $\mathrm{Sp}_6(\mathbb{Z})$. Such a π will be called *of Siegel type*. In other words, we have π_p unramified for each prime p and π_∞ is a holomorphic discrete series. In the classical language (see e.g. [vdG]), this means that the weights $k_1 \geq k_2 \geq k_3$ of f, which is possibly vector valued, satisfy $k_3 \geq 4$. The relation between those k_i and the infinitesmal character $\mathrm{c}(\pi_\infty) \subset \mathfrak{spin}_7(\mathbb{C})$ of π_∞ is as follows: the 7 eigenvalues of $\mathrm{std}(\mathrm{c}(\pi_\infty))$ are $0, \pm a, \pm b, \pm c$ with

$$(a, b, c) \,=\, (k_1 - 1, k_2 - 2, k_3 - 3),$$

and the 8 eigenvalues of $\mathrm{spin}(\mathrm{c}(\pi_\infty))$ are $\pm w_1, \pm w_2, \pm w_3, \pm w_4$ with

$$(w_1, w_2, w_3, w_4) = (\,(a+b+c)/2,\ (a+b-c)/2,\ (a-b+c)/2,\ |a-b-c|/2). \tag{7.1}$$

As $\sum_i k_i \equiv 0 \bmod 2$, the w_i are in $\mathbb{Z}$. Moreover, one has $w_1 > w_2 > w_3 > w_4 \geq 0$.

Let π be of Siegel type. We denote by $\Psi(\pi, \mathrm{std})$ the standard Arthur parameter of $\pi|_{\mathrm{Sp}_6}$. A detailed examination of all the possibilities for $\Psi(\pi, \mathrm{std})$ has been carried out in [CR, §9.3] (assuming the main result in [AMR]): see also [T1, §4.2.2] and [CL, §8.5.1]. In this situation, in which standard Galois representations are available by the works of many authors (Arthur, Chenevier, Clozel, Labesse, Harris, Shin, Taylor..., see the discussion in [CL, §8.2.16]), it is known that $\Psi(\pi, \mathrm{std})$ is generic if, and only if, the representation π is tempered (Clozel, Shin). Of course, note that π itself is never generic, since π_∞ is not.

7.2. **Generic case.** Let us first assume $\Psi(\pi, \mathrm{std})$ is generic and denote by $\Psi(\pi, \mathrm{spin})$ the spin lift of π to GL_8 furnished by Theorem 3.8.

By [CR, §9.3], there are actually only two possibilities for $\Psi(\pi, \mathrm{std})$: either it is cuspidal (the most important case), or we have (*endoscopic tempered case*)

$$\Psi(\pi, \mathrm{std}) \,=\, (\pi_1 \boxtimes \pi_2) \boxplus \mathrm{Sym}^2 \pi_3, \tag{7.2}$$

where π_1, π_2, π_3 are cuspidal automorphic representations of PGL_2 generated by holomorphic cuspidal eigenforms $\mathrm{SL}_2(\mathbb{Z})$ (see *loc. cit.* for the precise constraints on their weights in terms of the weights of f), with $\pi_1 \not\simeq \pi_2$. Here $\pi_1 \boxtimes \pi_2$ and $\mathrm{Sym}^2\pi_3$ denote of course the automorphic tensor product and symmetric square, constructed respectively by Ramakrishnan [Ra] and Gelbart-Jacquet [GJ], and they are cuspidal.

In both these cases, the automorphic representation $\Theta(\pi')$ produced by global similitude theta lifting in the proof of Theorem 3.8 is a tempered cuspidal representation of PGSO_8 and hence so is its pullback $f_2^*(\Theta(\pi'))$ on SO_8. By Theorem 3.10(iii), we may view both $\Psi(\pi, \mathrm{std})$ and $\Psi(\pi, \mathrm{spin})$ as a formal direct sums of self-dual orthogonal cuspidal automorphic representations π_i of some GL_{n_i} over $\mathbb{Q}$, with $\sum_i n_i = 7$ or 8 accordingly. The π_i have level 1 (*i.e.* are unramified at all finite places), by Thm. 3.10 in the spin case, hence have trivial central characters (they are selfdual). Also, in both cases the π_i's are algebraic, by Thm. 3.10 and (7.1) in the spin case (see [CL, Prop. 8.2.13]).

[2] We mean here that f is an eigenform for the full Hecke algebra of PGSp_6, not only of Sp_6.

7.3. **Shape of Spin lifting: tempered endoscopic case.** We would like to determine precisely what the isobaric sum $\Psi(\pi, \mathrm{spin}) = \boxplus_i \pi_i$ looks like. In this subsection, we handle the case when π satisfies (7.2); the case when $\Psi(\pi, \mathrm{std})$ is cuspidal is handled in §7.4.

Proposition 7.3. *If* $\Psi(\pi, \mathrm{std})$ *satisfies* (7.2)*, then we have*

$$\Psi(\pi, \mathrm{spin}) = (\pi_1 \boxplus \pi_2) \boxtimes \pi_3 = (\pi_1 \boxtimes \pi_3) \boxplus (\pi_2 \boxtimes \pi_3). \tag{7.4}$$

This is a very natural guess, since the spin representation of Spin_7, when naturally restricted to a natural

$$\nu : \mathrm{Spin}_4 \times \mathrm{Spin}_3 \to \mathrm{Spin}_7, \tag{7.5}$$

is isomorphic to the tensor product of the direct sum of the two spin representations of $\mathrm{Spin}_4 \simeq \mathrm{SL}_2 \times \mathrm{SL}_2$, with the spin representation of $\mathrm{Spin}_3 \simeq \mathrm{SL}_2$ (§2.1). But this only implies that, for each prime p, both Satake parameters at p on the left and right sides of (7.4) agree *up to a sign*, and our main claim is that this sign is $+1$.

For the proof of this proposition, and of others below, we will use certain Galois representations that we first briefly review. Fix a prime ℓ and, for convenience, an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. For any π of Siegel type, recall that by the aforementioned collective works, we have a continuous semi-simple Galois representation

$$\mathrm{r}_{\pi,\mathrm{std},\iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{SO}_7(\overline{\mathbb{Q}}_\ell),$$

which is unramified outside ℓ and such that for all $p \neq \ell$ the semi-simplified conjugacy class of $\mathrm{r}_{\pi,\mathrm{std},\iota}(\mathrm{Frob}_p)^{\mathrm{ss}}$ coincides with $\iota(\mathrm{c}(\pi'_p))$, where π' is any level one automorphic constituent of $\pi|_{\mathrm{Sp}_6}$. Recall that we have $\mathrm{c}(\pi'_p) = \rho(\mathrm{c}(\pi_p))$ where $\rho : \mathrm{Spin}_7 \to \mathrm{SO}_7$ is the standard morphism. The representation $\mathrm{r}_{\pi,\mathrm{std},\iota}$ is unique up to conjugacy and known to be crystalline at ℓ in the sense of Fontaine.

On the other hand, by[3] [T2, Thm. 2], there exists also a continuous semisimple morphism[4]

$$\mathrm{r}_{\pi,\mathrm{spin},\iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$$

which is unramified outside ℓ and satisfies

$$\mathrm{r}_{\pi,\mathrm{spin},\iota}(\mathrm{Frob}_p)^{\mathrm{ss}} = \iota(\mathrm{c}(\pi_p)), \ \text{ for all primes } p \neq \ell. \tag{7.6}$$

Again, this morphism $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ is unique up to conjugacy, and we may assume it satisfies $\rho \circ \mathrm{r}_{\pi,\mathrm{spin},\iota} = \mathrm{r}_{\pi,\mathrm{std},\iota}$. Taïbi also shows that $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ is crystalline at ℓ.

Proof. (of Prop. 7.3) Fix ℓ and ι as above. For $i = 1, 2, 3$, let us denote $\mathrm{r}_i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ the semi-simple Galois representation attached by Deligne to π_i, or more precisely, to the algebraic representation $\pi'_i := \pi_i|.|^{1/2}$ of GL_2, and to ι. It is unramified outside ℓ and satisfies $\mathrm{r}_i(\mathrm{Frob}_p)^{\mathrm{ss}} = \iota(\mathrm{c}((\pi'_i)_p))$ for $p \neq \ell$, hence $\mathrm{r}_i^\vee \simeq \mathrm{r}_i \otimes \omega_\ell^{-1}$ where ω_ℓ is the ℓ-adic cyclotomic character. The morphism ν (Formula 7.5) extends to $\nu : \mathrm{GSpin}_4 \times \mathrm{GSpin}_3 \to \mathrm{GSpin}_7$, and we have exceptional isomorphisms

$$\mathrm{GSpin}_4 \simeq (\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det_1 = \det_2} \ \text{ and } \ \mathrm{GSpin}_3 \simeq \mathrm{GL}_2.$$

[3]We stress that the Minkowski theorem is one of the ingredients of Taïbi's construction of $\mathrm{r}_{\pi,\mathrm{spin},\iota}$.

[4]We use here that for $g = 3$, $g(g+1) \equiv 0 \bmod 4$. For general PGSp_{2g}, Taïbi's statement involves GSpin instead of Spin.

Using these isomorphisms, we may view $(\mathrm{r}_1, \mathrm{r}_2)$ and $\mathrm{r}_3^\vee$ as $\mathrm{GSpin}_n(\overline{\mathbb{Q}}_\ell)$-valued with $n = 4, 3$ respectively, hence composing with ν we obtain a semi-simple morphism $\sigma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSpin}_7(\overline{\mathbb{Q}}_\ell)$ satisfying by construction

$$(7.7) \qquad \left\{ \begin{array}{l} \mathrm{std} \circ \sigma \;\simeq\; \mathrm{r}_1 \otimes \mathrm{r}_2^\vee \,\oplus\, \mathrm{Sym}^2 \mathrm{r}_3 \otimes \omega_\ell^{-1} \;\simeq\; \mathrm{std} \circ \mathrm{r}_{\pi,\mathrm{std},\iota}, \\ \\ \mathrm{spin} \circ \sigma \;\simeq\; (\mathrm{r}_1 \oplus \mathrm{r}_2) \otimes \mathrm{r}_3^\vee \;\simeq\; (\mathrm{r}_1 \otimes \mathrm{r}_3^\vee) \,\oplus\, (\mathrm{r}_2 \otimes \mathrm{r}_3^\vee). \end{array} \right.$$

In particular, we have $\mathrm{sim} \circ \sigma = \det \mathrm{r}_1 \det \mathrm{r}_3^\vee = \omega_\ell \cdot \omega_\ell^{-1} = 1$, *i.e.* $\mathrm{Im}\, \sigma \subset \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$. Up to conjugating σ if necessary, we may thus assume $\rho \circ \sigma = \mathrm{r}_{\pi,\mathrm{std},\iota}$. But this implies that there is a continuous character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \{\pm 1\}$ satisfying

$$\sigma \;=\; \chi\, \mathrm{r}_{\pi,\mathrm{spin},\iota}.$$

As σ and $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ are unramified outside ℓ (resp. cristalline at ℓ), so is χ, since χ is a subrepresentation of $(\mathrm{spin} \circ \sigma) \otimes (\mathrm{spin} \circ \mathrm{r})^\vee_{\pi,\mathrm{spin},\iota}$. As χ has finite order, it is thus unramified at p, hence everywhere. But there is no quadratic extension of $\mathbb{Q}$ unramified at all finite primes, so we have $\chi = 1$ and $\sigma = \mathrm{r}_{\pi,\mathrm{spin},\iota}$. Observe that we have

$$(\pi_1 \boxplus \pi_2) \boxtimes \pi_3 = (\pi_1' \boxplus \pi_2') \boxtimes (\pi_3')^\vee.$$

By (7.6) and the definition of σ, this proves $\mathrm{c}(\pi_p) = (\mathrm{c}((\pi_1)p) \boxplus \mathrm{c}((\pi_2)_p) \boxtimes \mathrm{c}((\pi_3)_p)$ for all $p \neq \ell$. We conclude by this same argument applied to any other ℓ. $\square$

7.4. **Shape of Spin lifting: cuspidal $\Psi(\pi, \mathrm{std})$.** We now consider the shape of the Spin lifting $\Psi(\pi, \mathrm{spin})$ when $\Psi(\pi, \mathrm{std})$ is (tempered) cuspidal. Here, the exceptional group G_2 plays a crucial role. Recall that there is an embedding (well-defined up to conjugacy)

$$\eta : \mathrm{G}_2(F) \to \mathrm{Spin}_7(F)$$

over any algebraically closed field F (below, of char. 0). As is well-known, the stabilizers in $\mathrm{Spin}_7(F)$ of the non-isotropic vectors in its Spin representation (which we recall is orthogonal) are exactly the conjugate of $\eta(\mathrm{G}_2(F))$.

Definition 7.8. *Let π be an automorphic representation of GSp_6 over the number field k. We say that π is of* type G_2 *if for almost all finite places v of k, the Satake parameter $\mathrm{c}(\pi_v)$ meets $\eta(\mathrm{G}_2(\mathbb{C}))$.*

If $r : \Gamma \to \mathrm{Spin}_7(F)$ is any group homomorphism, we also say that r is *of type* G_2 if $r(\Gamma)$ is conjugate to a subgroup of $\eta(\mathrm{G}_2(F))$.

Proposition 7.9. *Let π be a cuspidal representation of GSp_6 over $\mathbb{Q}$ of Siegel type. The following are equivalent:*

(i) *π is of type G_2,*

(ii) *$\mathrm{spin}(\mathrm{c}(\pi_p))$ has the eigenvalue 1 for almost all primes p,*

(iii) *$\Psi(\pi, \mathrm{spin}) = \mathbf{1} \boxplus \Psi(\pi, \mathrm{std})$,*

(iv) *for some ℓ and ι, $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ is of type G_2,*

(v) *for all ℓ and ι, $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ is of type G_2.*

Proof. Recall the following two facts :

(a) a semisimple element g in $\mathrm{Spin}_7(F)$ has the eigenvalue 1 in the spin representation if and only if it is conjugate to an element of $\eta(\mathrm{G}_2(F))$;
(b) if std and spin are the standard and spin representations of Spin_7, then the representation $\mathrm{spin} \circ \eta$ of G_2 is isomorphic to $1 \oplus \mathrm{std} \circ \eta$.

Applied to $F = \mathbb{C}$, this shows (i) $\Rightarrow$ (ii), and that they are equivalent to the equality $\mathrm{c}(\pi_v) = 1 \boxplus \mathrm{c}(\Psi(\pi, \mathrm{std})_v$ for all but finitely many v. This, in turn, is equivalent to $\Psi(\pi, \mathrm{spin}) = \mathbf{1} \boxplus \Psi(\pi, \mathrm{std})$ by the Jacquet-Shalika theorem.

Observe that (v) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are trivial, so it only remains to show (i) $\Rightarrow$ (v). Assume (i) and fix ℓ and ι. By the Cebotarev theorem, as well as (a) and (b) above applied to $F = \overline{\mathbb{Q}}_\ell$, we have an isomorphism of semisimple representations $\mathrm{spin} \circ \mathrm{r}_{\pi,\mathrm{spin},\iota} \simeq 1 \oplus \mathrm{std} \circ \mathrm{r}_{\pi,\mathrm{std},\iota}$, showing that $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ is of type G_2, hence (v), and we are done (see [C, Thm. A] for similar ideas). □

Remark 7.10. (i) *For π cuspidal of Siegel type, there is also an equivalence:*

$$\pi \textit{ is of type } \mathrm{G}_2 \iff \pi \textit{ is a functorial lift from (a form of) } \mathrm{G}_2.$$

This follows from [Vo] *for generic* $\Psi(\pi, \mathrm{std})$ *and by* [GS1, Thm 10.1 and Thm 10.2] *in general.*

(ii) *It is not difficult to prove that, for any ℓ and ι, $\mathrm{r}_{\pi,\mathrm{spin},\iota}$ is of type G_2 if, and only if, the image of $\mathrm{r}_{\pi,\mathrm{std},\iota}$ in $\mathrm{SO}_7(\overline{\mathbb{Q}}_\ell)$ is contained in a subgroup isomorphic to $\mathrm{G}_2(\overline{\mathbb{Q}}_\ell)$.*

The following proposition determines the possible shape of $\Psi(\pi, \mathrm{spin})$.

Proposition 7.11. *Assume $\Psi(\pi, \mathrm{std})$ is cuspidal, then $\Psi(\pi, \mathrm{spin})$ is cuspidal if, and only if, π is not a functorial lift from G_2.*

Proof. Assume $\Psi(\pi, \mathrm{std})$ is cuspidal and set $r = \mathrm{r}_{\pi,\mathrm{std},\iota}$. By Theorem D in [PT], we may choose ℓ and ι such that $\mathrm{std} \circ r$ is irreducible. Set $F = \overline{\mathbb{Q}}_\ell$ and let $\Gamma \subset \mathrm{SO}_7(F)$ be the Zariski-closure of the image of r. As r is crystalline at ℓ and unramified outside ℓ, so are all tensor powers $r^{\otimes n}$ with $n \geq 1$, and the Minkowski theorem implies that Γ is connected.

As Γ acts irreducibly in std, it is well-known that we have either $\Gamma = \mathrm{SO}_7(F)$, or Γ is a principal $\mathrm{PGL}_2(F)$ in $\mathrm{SO}_7(F)$, or Γ is a conjugate of $\eta(\mathrm{G}_2(F))$. Set now $r' = \mathrm{r}_{\pi,\mathrm{std},\iota}$ and $\Gamma' = \mathrm{Im}\, r'$. Then Γ' is connected as well, for the same reason as above, so we must have $\Gamma' = \mathrm{Spin}_7(F)$, or Γ' is a principal $\mathrm{SL}_2(F)$ in $\mathrm{Spin}_7(F)$, or Γ is a conjugate of $\eta(\mathrm{G}_2(F))$, respectively. In the first two situations, $\mathrm{spin} \circ r'$ is thus irreducible. But if we write

$$\Psi(\pi, \mathrm{spin}) = \pi_1 \boxplus \cdots \boxplus \pi_k,$$

the representation $\mathrm{spin} \circ r'$ is also the direct sum of that associated to the π_i's (which are algebraic, selfdual and essentially regular by (7.1)). This forces $k = 1$, and we are done. In the remaining case, r is of type G_2 and we conclude by Prop. 7.9. □

7.5. **Non-generic case.** Finally, we consider the case where the A-parameter $\Psi(\pi, \mathrm{std})$ is not generic, which is not a priori covered by Thm. 3.8. By [CR, §9.3], we have[5]

$$\Psi(\pi, \mathrm{std}) \;=\; \pi_1 \boxtimes S_2 \boxplus \mathrm{Sym}^2 \pi_3, \tag{7.12}$$

[5] For our purposes here, we could replace S_2 by $|.|^{1/2} \boxplus |.|^{-1/2}$, but Arthur's notation is more suggestive.

where π_1 and π_3 are cuspidal automorphic representations of PGL_2 generated by holomorphic cuspidal eigenforms for $\mathrm{SL}_2(\mathbb{Z})$ (again, the precise constraints on the weights of π_1 and π_3 are given *loc. cit.*). In this situation we simply *define*

$$\Psi(\pi, \mathrm{spin}) \ := \ (\pi_1 \boxplus S_2) \boxtimes \pi_3 \ = \ (\pi_1 \boxtimes \pi_3) \boxplus \pi_3 \boxtimes \mathrm{S}_2. \tag{7.13}$$

In Theorem 7.14 below, we verify that $\Psi(\pi, \mathrm{spin})$ is indeed the A-parameter of a Spin lifting of π. Together with Theorem 3.8, this establishes the existence of the Spin lifting $\Psi(\pi, \mathrm{spin})$ *for all* π *of Siegel type.*

Theorem 7.14. *For all cuspidal* π *of Siegel type,* $\Psi(\pi, \mathrm{spin})$ *is the A-parameter of a Spin lifting of* π*. Moreover, this lifting is strong at all unramified places, as well as at the Archimedean place in the sense of infinitesimal characters.*

Proof. We may assume $\Psi(\pi, \mathrm{std}) = \pi_1 \boxtimes S_2 \boxplus \mathrm{Sym}^2\pi_3$ as above. We apply exactly the same argument as in the proof of Prop. 7.3, with π_2 there replaced with the trivial representation of PGL_2, and setting $\mathrm{r}_2 := 1 \oplus \omega_\ell$. Defining $\sigma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$ as in that proof, the argument shows verbatim that σ is conjugate to $\mathrm{r}_{\pi,\mathrm{spin},\iota}$, hence that $\Psi(\pi, \mathrm{spin})$ is a spin lifting of π, which is strong at all finite places except maybe ℓ (the assertion about infinitesimal characters is obvious from the shape of $\Psi(\pi, \mathrm{spin})$). Using another ℓ gives the full result. $\square$

7.6. **Application to Spin L-functions.** We end with an interesting corollary about spinor L-functions. Assume π is of Siegel type and generated by a Siegel cuspidal eigenform of weights $k_1 \geq k_2 \geq k_3 \geq 4$. Following Langlands, recall that for any prime p the local spin L-factor is defined as

$$\mathrm{L}(s, \pi_p, \mathrm{spin}) = \frac{1}{\det(1 - \mathrm{spin}(\mathrm{c}(\pi_p))p^{-s})}.$$

Set $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$, with $\Gamma(s)$ the Euler Γ-function, define the w_i as in (7.1), and set

$$\mathrm{L}(s, \pi_\infty, \mathrm{spin}) \ := \ \prod_{i=1}^{4} \Gamma_\mathbb{C}(s + w_i).$$

We then define $\mathrm{L}(s, \pi, \mathrm{spin})$ as the product, over all places v of $\mathbb{Q}$, of the local L-factors $\mathrm{L}(s, \pi_v, \mathrm{spin})$. We know since Langlands that this Euler product is absolutely convergent for $\mathrm{Re}(s)$ big enough. Recall also the standard L-function $\mathrm{L}(s, \pi, \mathrm{std})$ of π, whose analytic properties are now well-known (for any genus).

Theorem 7.15. *Assume* π *is a cuspidal of Siegel type. Then:*

(i) $\mathrm{L}(s, \pi, \mathrm{spin})$ *has a meromorphic continuation to all of* $\mathbb{C}$*, with at most a simple pole at* $s = 0$ *and* 1*, and no other poles. It satisfies* $\mathrm{L}(s, \pi, \mathrm{spin}) = \mathrm{L}(1 - s, \pi, \mathrm{spin})$.

(iii) *Moreover,* $\mathrm{L}(s, \pi, \mathrm{spin})$ *has a pole at* $s = 1$ *if, and only if,* π *is of type* G_2*, in which case* $\mathrm{L}(s, \pi, \mathrm{spin}) = \zeta(s) \cdot \mathrm{L}(s, \pi, \mathrm{std})$.

There has been an number of past works on the spinor L-functions of cuspidal automorphic representations π of PGSp_6 over number fields. For a globally generic π, a partial representation by a Rankin-Selberg type integral was found in [BG] and studied in [Vo]. For our Siegel type π's, a weaker statement had also been proved by Pollack in [Po1, Thm. 1.2], who assumed that the associated Siegel modular form has a nonzero Fourier coefficient at the maximal order of a definite quaternion algebra. Our method here is quite different, and ultimately relies on the properties of the Godement-Jacquet L-functions; it also provides the most complete result for Siegel type π.

Proof. In all cases, there are integers $k \geq 1$ and $n_i, d_i \geq 1$, with $\sum_{i=1}^k n_i d_i = 8$, and

$$\Psi(\pi, \mathrm{spin}) = \boxplus_{i=1}^k \pi_i \boxtimes S_{d_i},$$

for some selfdual level 1 cuspidal automorphic representations π_i of GL_{n_i}. We have either $d_i = 1$ and π_i is orthogonal, or $d_i = 2$ and π_i is symplectic (with $n_i = 2$). Recall that the Godement-Jacquet standard L-function $\mathrm{L}(s, \pi_i)$ is entire if $\pi_i \neq \mathbf{1}$, and is equal to the completed $\zeta(s)$ otherwise. Moreover, it satisfies the functional equation $\mathrm{L}(s, \pi_i) = \epsilon(\pi_i, 1/2)\mathrm{L}(1-s, \pi_i)$ for some sign $\epsilon(\pi_i, 1/2) = \pm 1$, equal to 1 if π_i is orthogonal (Arthur).

When $\Psi(\pi, \mathrm{spin})$ is generic, *i.e.* $d_i = 1$ for all i, Formula (7.1) and *Clozel's purity lemma* (see [CL, Prop. 8.2.13]) show that the Langlands parameter of $\boxplus_{i=1}^k (\pi_i)_\infty$ is $\oplus_{i=1}^4 \mathrm{I}_{w_i}$ with $\mathrm{I}_w = \mathrm{Ind}_{\mathrm{W}_\mathbb{C}}^{\mathrm{W}_\mathbb{R}}(z \mapsto (z/|z|)^{2w})$ for $w \in \frac{1}{2}\mathbb{Z}$. We have thus (equality at all places)

$$\mathrm{L}(s, \pi, \mathrm{spin}) = \prod_{i=1}^k \mathrm{L}(s, \pi_i). \tag{7.16}$$

As $\mathbf{1}$ cannot appears twice as a π_i (*e.g.* at the Archimedean place), this proves (i). Part (ii) follows from $\mathrm{L}(1, \pi_i) \neq 0$ (Jacquet-Shalika) and Prop. 7.9.

Assume now $\Psi(\pi, \mathrm{spin})$ is nongeneric, *i.e.* satisfies (7.13), or in Langlands form

$$\Psi(\pi, \mathrm{spin}) \;=\; \pi_1 \boxtimes \pi_3 \;\boxplus\; \pi_3|.|^{1/2} \;\boxplus\; \pi_3|.|^{-1/2}. \tag{7.17}$$

Denote by $\pm u$, with $u \in \frac{1}{2}\mathbb{Z}_{>0} \smallsetminus \mathbb{Z}$, the 2 eigenvalues of $\mathrm{c}((\pi_3)_\infty)$. An inspection of the infinitesimal character shows that we may write $\{w_1, w_2, w_3, w_4\} = \{u_1, u_2, u + \frac{1}{2}, u - \frac{1}{2}\}$. The Langlands parameter of $\Psi(\pi, \mathrm{spin})_\infty$ is thus $\mathrm{I}_{u_1} \oplus \mathrm{I}_{u_2} \oplus \mathrm{I}_u|.|^{1/2} \oplus \mathrm{I}_u|.|^{-1/2}$, whose standard L-function coincides with our definition of $\mathrm{L}(s, \pi_\infty, \mathrm{spin})$, showing again

$$\mathrm{L}(s, \pi, \mathrm{spin}) = \mathrm{L}(s, \pi_1 \boxtimes \pi_3)\mathrm{L}(s - 1/2, \pi_3)\mathrm{L}(s + 1/2, \pi_3).$$

Again, an unramified character $|.|^s$ appears at most once in the L-parameter of $\Psi(\pi, \mathrm{spin})_\infty$, and only for $s = 0$. The ϵ-factor of $\mathrm{L}(s, \pi, \mathrm{spin})$ is $\epsilon(\pi_1 \otimes \pi_3, 1/2) \cdot \epsilon(\pi_3, 1/2)^2 = 1 \cdot (\pm 1)^2 = 1$. We conclude as above. $\square$

8. **Appendix A: Similitude Theta Correspondence**

In this appendix, we collect and establish some results from the theory of local theta correspondence for similitude classical groups that we need in the main body of the article, especially concerning the theta lifts of unramified representations. Throughout this appendix, F will denote a non-Archimedean local field with ring of integers $\mathcal{O}_F$ and residue field of cardinality q.

8.1. **Setup.** We briefly recall the setup of isometry and similitude theta correspondences over the local field F. Fix a nontrivial additive charatcer $\psi : F \to \mathbb{C}^\times$. Suppose that $(W, \langle -, -\rangle_W)$ is a symplectic vector space of dimension $2n$ and (V, q) a quadratic space of dimension $2m$ with associated symmetric bilinear form $\langle -, -\rangle_V$. For simplicity, we shall assume the following running hypotheses throughout this appendix:

- (V, q) is split, so that $\mathrm{disc}(V, q) = 1 \in F^\times/F^{\times 2}$ and the orthogonal group $\mathrm{O}(V)$ is split;
- $m = \dim V/2 > n = \dim W/2$.

Then we have the isometry dual pair

$$i : \mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Sp}(V \otimes W).$$

Depending on ψ, the map ι can be lifted to the metaplectic cover $\mathrm{Mp}(V \otimes W)$:

$$i_\psi : \mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Mp}(V \otimes W).$$

The Weil representation ω_ψ of $\mathrm{Mp}(V \otimes W)$ can then be pulled back via i_ψ to yield the Weil representation $\omega_{V,W,\psi}$ of $\mathrm{Sp}(W) \times \mathrm{O}(V)$.

For a representation $\pi \in \mathrm{Irr}(\mathrm{Sp}(W))$, we consider its big theta lift to $\mathrm{O}(V)$, defined by:

$$\Theta_\psi(\pi) = (\omega_{V,W,\psi} \otimes \pi^\vee)_{\mathrm{Sp}(W)},$$

which is a smooth representation of $\mathrm{O}(V)$ (for a representation U of a group G we denote by U_G the largest quotient of U over which G acts trivially). The Howe duality theorem says that $\Theta_\psi(\pi)$ has finite length (possibly zero) and a unique irreducible quotient $\theta_\psi(\pi)$ (possibly zero). Hence, one has a map

$$\theta_\psi : \mathrm{Irr}(\mathrm{Sp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{O}(V)) \cup \{0\}.$$

Moreover, the Howe duality theorem further asserts that this map is injective on the domain where it does not vanish. The map θ_ψ is the local theta lifting for isometry groups.

Under our hypothesis that $m > n$, it turns out that if $\theta_\psi(\pi) \in \mathrm{Irr}(\mathrm{O}(V))$ is nonzero, then it remains irreducible when restricted to the special orthogonal group $\mathrm{SO}(V)$. Hence, composing with restriction to $\mathrm{SO}(V)$, one has in fact a map

$$\theta_\psi : \mathrm{Irr}(\mathrm{Sp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{SO}(V)) \cup \{0\}.$$

We would like to extend the above theory and results to the setting of similitude groups. These similitude groups are defined by:

$$\mathrm{GSp}(W) = \{(g, \lambda) \in \mathrm{GL}(W) \times \mathbb{G}_m : g^*(\langle -, -\rangle_W) = \lambda \cdot \langle -, -\rangle_W\}$$

and

$$\mathrm{GO}(V) = \{(h, \lambda) \in \mathrm{GL}(V) \times \mathbb{G}_m : h^*(\langle -, -\rangle_V) = \lambda \cdot \langle -, -\rangle_V\}.$$

The similitude characters

$$\mathrm{sim} : \mathrm{GSp}(W) \longrightarrow \mathbb{G}_m \quad \text{and} \quad \mathrm{sim} : \mathrm{GO}(V) \longrightarrow \mathbb{G}_m$$

are given by the second projection. By our hypotheses, these similitude characters are surjective onto $F^\times$ on taking F-valued points.

Consider now the group

$$R = (\mathrm{GSp}(W) \times \mathrm{GO}(V))^{\mathrm{sim}} = \{((g,h) \in \mathrm{GSp}(W) \times \mathrm{GO}(V) : \mathrm{sim}(g) \cdot \mathrm{sim}(h) = 1\}.$$

One has the short exact sequences:

$$1 \longrightarrow \mathrm{Sp}(W) \longrightarrow (\mathrm{GSp}(W) \times \mathrm{GO}(V))^{\mathrm{sim}} \xrightarrow{p} \mathrm{GO}(V) \longrightarrow 1.$$

$$1 \longrightarrow \mathrm{O}(V) \longrightarrow (\mathrm{GSp}(W) \times \mathrm{GO}(V))^{\mathrm{sim}} \xrightarrow{q} \mathrm{GSp}(W) \longrightarrow 1$$

where p and q are the natural projections onto the relevant factors. Moreover, observe that one has an exact sequence

$$1 \longrightarrow \mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow R \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

It turns out that the Weil representation $\omega_{V,W,\psi}$ can be extended to the slightly larger group $R = R(F)$. The extension is not unique but we fix one as in [GT1] (which differs from the normalization in [Ro]). Then we define the similitude Weil representation as:

$$\Omega_{V,W} := \mathrm{ind}_R^{\mathrm{GSp}(W) \times \mathrm{GO}(V)} \omega_{V,W,\psi}.$$

It turns out that this representation is independent of the choice of ψ (because V has trivial discriminant). For a representation $\pi \in \mathrm{Irr}(\mathrm{GSp}(W))$, we can then define its big theta lift (which is a smooth representation of $\mathrm{GO}(V)$) by

$$\Theta(\pi) := (\Omega_{V,W} \otimes \pi^\vee)_{\mathrm{GSp}(W)}.$$

One may also define $\Theta(\pi)$ as

$$\Theta(\pi) = (\omega_{V,W,\psi} \otimes \pi^\vee)_{\mathrm{Sp}(W)}.$$

It follows from [Ro, GT1] that the analog of the Howe duality theorem holds for similitude groups. In other words, for each $\pi \in \mathrm{Irr}(\mathrm{GSp}(W))$, $\Theta(\pi)$ has finite length and unique irreducible quotient $\theta(\pi)$ (if nonzero). Hence, one obtains a map

$$\theta : \mathrm{Irr}(\mathrm{GSp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{GO}(V)) \cup \{0\},$$

which is injective on the domain of nonvanishing. This map θ is the local theta lifting for similitude groups. Moreover, under our definition of the extension of ω_ψ to R, it turns out that $\Theta(\pi)$ has central character equal to that of π. As in the isometry case, for $\pi \in \mathrm{Irr}(\mathrm{GSp}(W))$, $\theta(\pi)$ is irreducible (or zero) when restricted to $\mathrm{GSO}(V)$ under our hypothesis that $m > n$. Hence, we have a map

$$\theta : \mathrm{Irr}(\mathrm{GSp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{GSO}(V)) \cup \{0\}. \tag{8.1}$$

8.2. **Unramified representations.** We will need to give an explicit description of the maps θ_ψ and θ on the subset of unramified representations. Let us recall this notion more precisely.

Suppose that G is a split connected reductive group over F. Then G has a reductive model (the Chevalley model) over the ring of integers $\mathcal{O}_F$, so that $K := G(\mathcal{O}_F)$ is a hyperspecial maximal compact subgroup. Fix the tuple

$$T \subset B \subset G$$

consisting of a maximal split torus T contained in a Borel subgroup B defined over $\mathcal{O}_F$.

The subset $\mathrm{Irr}_K(G(F)) \subset \mathrm{Irr}(G(F))$ of K-unramified representations consists of those irreducible representations π such that $\pi^K \neq 0$, in which case $\dim \pi^K = 1$. The following are the main facts about unramified representations we need; they are largely consequences of the so-called Satake isomorphism.

- If $\pi \in \mathrm{Irr}_K(G(F))$, then π is a constituent of a principal series representation $\mathrm{Ind}_B^G(\chi)$, where
$$\chi : T(\mathcal{O}_F)\backslash T(F) = X_*(T) \longrightarrow \mathbb{C}^\times$$
is an unramified character of $T(F)$, well defined up to the action of the Weyl group. Moreover, $\dim \mathrm{Ind}_B^G(\chi)^K = 1$, so that π is the unique K-unramified subquotient of $\mathrm{Ind}_B^G(\chi)$.
- With $T^\vee$ as the dual torus of T, note that
$$\mathrm{Hom}(X_*(T), \mathbb{C}^\times) = X^*(T) \otimes \mathbb{C}^\times = X_*(T^\vee) \otimes \mathbb{C}^\times = T^\vee(\mathbb{C}).$$
Hence the Weyl-orbit of the unramified character χ of $T(F)$ corresponds to the Weyl-orbit of an element $c(\pi) \in T^\vee(\mathbb{C})$, or equivalently a semisimple conjugacy class in the Langlands dual group $G^\vee(\mathbb{C})$ of G. This semisimple class $c(\pi)$ is the Satake parameter of π.

As an example, the trivial representation of $G(F)$ is certainly unramified and its Satake parameter is described as follows. Let
$$\iota_{\mathrm{reg}} : \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee(\mathbb{C})$$
be a principal or regular SL_2. Then the Satake parameter $c(\mathrm{triv})$ of the trivial representation is the conjugacy class of
$$c(\mathrm{triv}) = \iota_{\mathrm{reg}} \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix},$$
where q is the cardinality of the residue field of F.

8.3. **Unramified isometry theta correspondence.** Now let's return to our setting where W and V are symplectic and quadratic spaces, with $\dim W = 2n$ and $\dim V = 2m$. Since V and W are both split with trivial discriminants, we may fix self-dual lattices
$$\Lambda \subset W \quad \text{and} \quad \Lambda' \subset V.$$
These lattices endow the groups $\mathrm{GSp}(W)$, $\mathrm{Sp}(W)$, $\mathrm{GSO}(V)$ and $\mathrm{SO}(V)$ with its Chevalley structure over $\mathcal{O}_F$. The hyperspecial maximal compact subgroups
$$K = \mathrm{GSp}(W)(\mathcal{O}_F) \quad \text{and} \quad K' = \mathrm{GSO}(V)(\mathcal{O}_F)$$
are the stabilizers of these lattices in the respective similitude groups. Likewise, we have the hyperspecial maximal compact subgroups
$$K_\flat = K \cap \mathrm{Sp}(W) \quad \text{and} \quad K'_\flat = K' \cap \mathrm{SO}(V)$$
in the respective isometry groups.

Let us fix a Witt basis
$$\{e_1, \ldots, e_n, f_n, \ldots, f_1\} \quad \text{of } \Lambda$$

so that $\langle e_i, f_j\rangle_W = \delta_{ij}$. Likewise, we may fix such a Witt basis for the lattice Λ'. These Witt bases define maximal tori over $\mathcal{O}_F$:

$$T \subset \mathrm{GSp}(W) \quad \text{and} \quad T_\flat = T \cap \mathrm{Sp}(W) \subset \mathrm{Sp}(W)$$

and

$$T' \subset \mathrm{GSO}(V) \quad \text{and} \quad T'_\flat = T' \cap \mathrm{SO}(V) \subset \mathrm{SO}(V),$$

with the property that the relevant Witt basis is a basis of eigenvectors for the action of the relevant maximal torus. Thus, the Witt bases provide isomorphisms

$$T_\flat \simeq \prod_{i=1}^n \mathbb{G}_m \quad \text{and} \quad T \simeq \left(\prod_{i=1}^n \mathbb{G}_m\right) \times \mathbb{G}_m \tag{8.2}$$

where the first isomorphism is given by the eigencharacters of T on the ordered basis $\{e_1, \ldots, e_n\}$ and likewise for the projection of the second isomorphism to the first factor; the second projection $T \to \mathbb{G}_m$ is given by the similitude character. In particular, this provides a complementary $\mathbb{G}_m$ to $T_\flat$ in T:

$$T = T_\flat \times \mathbb{G}_m. \tag{8.3}$$

In this identification, the center Z of $\mathrm{GSp}(W)$ is given by

$$\big((z, \ldots, z), z^2\big) \in T_\flat \times \mathbb{G}_m, \quad z \in \mathbb{G}_m.$$

In view of (8.2) and (8.3), we may write a character $\chi_\flat : T_\flat(F) \to \mathbb{C}^\times$ as $\chi_\flat = \chi_1 \times \cdots \times \chi_n$ and a character χ of $T(F)$ as $\chi = \chi_\flat \times \mu$. Restricted to the center Z of $\mathrm{GSp}(W)$,

$$\chi|_Z = \mu^2 \cdot \prod_{i=1}^n \chi_i.$$

Of course, we have the analogous discussion for the tori $T'_\flat$ and T' in $\mathrm{SO}(V)$ and $\mathrm{GSO}(V)$.

Containing each maximal torus above, we also have a Borel subgroup over $\mathcal{O}_F$, which is upper triangular with respect to the relevant Witt basis, so that we have:

$$T \subset B \subset \mathrm{GSp}(W) \quad \text{and} \quad T_\flat = T \cap \mathrm{Sp}(W) \subset B_\flat \subset \mathrm{Sp}(W),$$

and likewise

$$T' \subset B' \subset \mathrm{GSO}(V) \quad \text{and} \quad T'_\flat \subset B'_\flat \subset \mathrm{SO}(V).$$

Given a character $\chi_\flat = \chi_1 \times \cdots \times \chi_n$ of $T_\flat(F)$, we will denote the associated (normalized) principal series representation as:

$$i_{B_\flat}(\chi_1 \times \cdots \times \chi_n) := \mathrm{Ind}_{B_\flat}^{\mathrm{Sp}(W)} \chi_\flat. \tag{8.4}$$

Likewise, we will write $i_B(\chi_1 \times \cdots \times \chi_n \times \mu)$ for a principal series representation of $\mathrm{GSp}(W)$.

After the preparation above, we can now recall the following result for isometry theta correspondence.

Proposition 8.5. *Assume that $m \geq n+1$.*

(i) Let $\pi_\flat \in \mathrm{Irr}(\mathrm{Sp}(W))$ be a constituent of a principal series representation $i_{B_\flat}(\chi_1 \times \chi_2 \times \cdots \times \chi_n)$. If $\theta_\psi(\pi_\flat) \neq 0$, then $\theta_\psi(\pi_\flat)$ is a constituent of the induced representation

$$i_{B'_\flat}(\chi_1 \times \chi_2 \times \cdots \times \chi_n \times |-|^{m-n-1} \times |-|^{m-n-2} \times \cdots \times 1)$$

of $\mathrm{SO}(V)$.

(ii) Suppose that ψ has conductor $\mathcal{O}_F$. For $\pi_\flat \in \mathrm{Irr}_{K_\flat}(\mathrm{Sp}(W))$, we have $\theta_\psi(\pi_\flat) \in \mathrm{Irr}_{K'_\flat}(\mathrm{SO}(V))$ (in particular it is nonzero).

(iii) In the context of (ii), the Satake parameters of $\pi_\flat$ and $\theta_\psi(\pi_\flat)$ are related as follows. Consider the natural embedding

$$\iota_\flat : \mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathrm{SO}_{2m-2n-1}(\mathbb{C}) \longrightarrow \mathrm{SO}_{2m}(\mathbb{C}) = \mathrm{SO}(V)^\vee.$$

Then

$$c(\theta_\psi(\pi_\flat)) = \iota(c(\pi_\flat), c(\mathrm{triv}))$$

where $c(\mathrm{triv})$ is the Satake parameter of the trivial representation of the split group $\mathrm{Sp}(W')$, with $\dim W' = 2m-2n-2$, whose dual group is $\mathrm{SO}_{2m-2n-1}(\mathbb{C})$.

Proof. The statement (i) is a special case of a result of Kudla [K, Thm. 2.5, Cor. 2.6 and Cor. 2.7], obtained as a consequence of his computation of the Jacquet modules of the Weil representation [K, theorem 2.8]. Statements (ii) and (iii) are results of Howe and Rallis [R1, §6]. □

For the sake of concreteness, let us describe the map $\iota_\flat$ in Proposition 8.5(iii) on the level of maximal tori, as this is all that is needed for the unramified correspondence. Recall that we have fixed identifications:

$$T_\flat = \prod_{i=1}^{n} \mathbb{G}_m \quad \text{and} \quad T'_\flat = \prod_{i=1}^{m} \mathbb{G}_m$$

so that one has

$$T_\flat^\vee = (\mathbb{C}^\times)^n \quad \text{and} \quad T_\flat'^\vee = (\mathbb{C}^\times)^m.$$

We likewise have a maximal torus $S_\flat \subset \mathrm{Sp}(W')$ where W' is as in Proposition 8.5(iii), with fixed identification

$$S_\flat = \prod_{i=1}^{m-n-1} \mathbb{G}_m \quad \text{and hence} \quad S_\flat^\vee = (\mathbb{C}^\times)^{m-n-1}.$$

Restricted to these maximal tori, the map

$$\iota_\flat : T_\flat^\vee \times S_\flat^\vee \longrightarrow T_\flat'^\vee \tag{8.6}$$

is given explicitly by:

$$\iota\left((t_1, \ldots, t_n), (s_1, \ldots, s_{m-n-1})\right) = (t_1, \ldots, t_n, s_1, \ldots, s_{m-n-1}, 1).$$

Now, if $\pi_\flat$ is an unramified representation of $\mathrm{Sp}(W)$ contained in a principal series representation $i_{B_\flat}(\chi_1, \dots, \chi_n)$, then its Satake parameter is the conjugacy class of

$$c(\pi_\flat) = (\chi_1(\varpi), \dots, \chi_n(\varpi)) \in (\mathbb{C}^\times)^n.$$

On the other hand, the Satake parameter of the trivial representation is

$$c(\mathrm{triv}) = (q^{m-n-1}, \dots, q) \in (\mathbb{C}^\times)^{m-n-1}.$$

Hence,

$$\iota_\flat(c(\pi_\flat), c(\mathrm{triv})) = (\chi_1(\varpi), \dots, \chi_n(\varpi), q^{m-n-1}, \dots, q, 1) \in (\mathbb{C}^\times)^m.$$

In view of Proposition 3.5(i), this is precisely the Satake parameter of $\theta_\psi(\pi_\flat)$.

8.4. **Unramified similitude theta correspondence.** We would now like to establish the analog of the above proposition for the similitude theta correspondence. Before that, let us make an observation about the interaction between the principal series representations for the similitude and isometry groups.

Since $B_\flat \backslash \mathrm{Sp}(W) = B \backslash \mathrm{GSp}(W)$, the natural restriction map of functions define a $\mathrm{Sp}(W)$-equivariant isomorphism

$$i_B(\chi_1 \times \cdots \times \chi_n \times \mu) \simeq i_{B_\flat}(\chi_1 \times \cdots \times \chi_n).$$

Suppose now that the characters χ_i and μ are unramified, then one has

$$1 = \dim i_B(\chi_1 \times \cdots \times \chi_n \times \mu)^K \le \dim i_B(\chi_1 \times \cdots \times \chi_n \times \mu)^{K_\flat} = \dim i_{B_\flat}(\chi_1 \times \cdots \times \chi_n)^{K_\flat} = 1,$$

so that equality holds. We record the consequence of this observation as a lemma.

Lemma 8.7. *Suppose that each χ_i and μ are unramified characters. Then*

$$\dim i_B(\chi_1 \times \cdots \times \chi_n \times \mu)^K = \dim i_B(\chi_1 \times \cdots \times \chi_n \times \mu)^{K_\flat} = 1$$

In particular, if π is a constituent of $i_B(\chi_1 \times \cdots \times \chi_n \times \mu)$, then

$$\dim \pi^K = \dim \pi^{K_\flat} = 0 \quad \textit{or} \quad 1.$$

The analogous lemma holds in the context of $\mathrm{SO}(V)$ and $\mathrm{GSO}(V)$.

Here is the main result of this appendix.

Proposition 8.8. *Assume that $m \ge n+1$.*

(i) Let $\pi \in \mathrm{Irr}(\mathrm{GSp}(W))$ be a constituent of a principal series representation

$$i_B(\chi_1 \times \chi_2 \times \cdots \times \chi_n \times \mu).$$

If $\Theta(\pi) \neq 0$, then every irreducible subquotient of $\Theta(\pi)$ is a subquotient of the principal series representation

$$i_B(\chi_1 \times \chi_2 \times \cdots \times \chi_n \times |-|^{m-n-1} \times |-|^{m-n-2} \times \cdots \times 1 \times \mu |-|^{-\frac{(m-n)(m-n-1)}{4}})$$

of $\mathrm{GSO}(V)$.

(ii) For $\pi \in \mathrm{Irr}_K(\mathrm{GSp}(W))$, we have $\theta(\pi) \in \mathrm{Irr}_{K'}(\mathrm{GSO}(V))$ (in particular it is nonzero).

(iii) In the context of (ii), the Satake parameters $c(\pi)$ and $c(\theta(\pi))$ of π and $\theta(\pi)$ are related as follows. One has a natural commutative diagram

$$\begin{array}{ccc} \mathrm{GSpin}_{2n+1}(\mathbb{C}) \times \mathrm{GSpin}_{2m-2n-1}(\mathbb{C}) & \xrightarrow{\iota} & \mathrm{GSpin}_{2m}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathrm{SO}_{2m-2n-1}(\mathbb{C}) & \xrightarrow{\iota_\flat} & \mathrm{SO}_{2m}(\mathbb{C}). \end{array} \tag{8.9}$$

Then

$$c(\theta(\pi)) = \iota(c(\pi), c(\mathrm{triv})),$$

where $c(\mathrm{triv})$ *is the Satake parameter of the trivial representation of the split group* $\mathrm{GSp}(W')$*, with* $\dim W' = 2m - 2n - 2$*, whose dual group is* $\mathrm{GSpin}_{2m-2n-1}(\mathbb{C})$*.*

Proof. (i) This follows from the results of [GT2]. More precisely, [GT2, Thms A.1 and A.2] determine the Jacquet modules of the induced Weil representation with respect to the maximal parabolic subgroups of $\mathrm{GSp}(W)$ and $\mathrm{GO}(V)$, analogous to what Kudla did in [K, Thm. 2.8] in the setting of isometry theta correspondence. With these Jacquet modules at hand, the same argument as in the proof of [K, Thm. 2.5 and Cor. 2.6] allows one to determine the behaviour of cuspidal support under the similitude theta correspondence. The statement (i) is a special case of this result.

We shall give the proof of (i) for the sake of completeness, proceeding by induction on $n = 1/2 \cdot \dim W$. The base case when $n = 0$ holds trivially. Assume now that $n \geq 1$ and (after applying a Weyl group element if necessary), we may assume that

$$\pi \hookrightarrow i_B(\chi_1 \times \cdots \times \chi_n \times \mu).$$

By induction in stages, we have

$$\pi \hookrightarrow \mathrm{Ind}_{P_1}^G \chi_1 \boxtimes \pi'$$

where

- P_1 is the maximal parabolic subgroup of $\mathrm{GSp}(W)$ stabilizing the isotropic line $X_1 = Fe_1$ in W, which has Levi subgroup $\mathrm{GL}(X_1) \times \mathrm{GSp}(W')$, with $W' = \langle e_2, \ldots, e_n, f_n, \ldots, f_2 \rangle$;
- $\pi' \in \mathrm{Irr}(\mathrm{GSp}(W'))$ is a subquotient of $i_{B'}(\chi_2 \times \cdots \times \chi_n \times \mu)$.

Now it follows that

$$0 \neq \Theta(\pi)^\vee \subset \Theta(\pi)^* = \mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, \pi) \subset \mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, \mathrm{Ind}_{P_1}^G \chi_1 \boxtimes \pi'),$$

where the superscript $^\vee$ indicates contragredient of a $\mathrm{GSO}(V)$-module whereas the superscript * indicates the full linear dual of a vector space. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, \mathrm{Ind}_{P_1}^G \chi_1 \boxtimes \pi') = \mathrm{Hom}_{\mathrm{GL}(X_1) \times \mathrm{GSp}(W')}(R_{P_1}(\Omega_{V,W}), \chi_1 \boxtimes \pi'),$$

where $R_{P_1}(\Omega_{V,W})$ denotes the normalized Jacquet module of $\Omega_{V,W}$ with respect to P_1. This normalized Jacquet module is what [GT2, Thm A.2] computes.

More precisely, [GT2, Thm A.2] shows that there is a short exact sequence

$$0 \longrightarrow J_0 \longrightarrow R_{P_1}(\Omega_{V,W}) \longrightarrow J_1 \longrightarrow 0 \tag{8.10}$$

of representations of $\mathrm{GSO}(V) \times \mathrm{GL}(X_1) \times \mathrm{GSp}(W')$. The submodule J_0 and quotient J_1 are described as follows:

- Let $Q_1 \subset \mathrm{GSO}(V)$ be the maximal parabolic subgroup stabilizing an isotropic line Y_1, so that its Levi subgroup is of the form $\mathrm{GL}(Y_1)\times\mathrm{GSO}(V')$, with $\dim V' = 2m-2$. Then
$$J_0 = \mathrm{Ind}_{Q_1\times\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}^{\mathrm{GSO}(V)\times\mathrm{GL}(X_1)\times\mathrm{GSp}(W')} C_c^\infty(F^\times)\otimes\Omega_{V',W'}$$
where the action on the inducing data is as follows:
  - $\Omega_{V',W'}$ is the induced Weil representation of $\mathrm{GSO}(V')\times\mathrm{GSp}(W')$;
  - $(a,h,b)\in\mathrm{GL}(Y_1)\times\mathrm{GSO}(V')\times\mathrm{GL}(X_1)$ acts on $S(F^\times)$ via:
$$(a,h,b)\cdot f(x) = f(a^{-1}\mathrm{sim}_{V'}(h)\cdot x\cdot b).$$
- the quotient J_1 is given by:
$$J_1 = |-|^{m-n}\cdot|\mathrm{sim}_{W'}|^{\frac{n-m}{2}}\cdot\Omega_{V,W'}$$
where $\Omega_{V,W'}$ is the induced Weil representation of $\mathrm{GSO}(V)\times\mathrm{GSp}(W')$ and $|-|^{m-n}$ is a character of $\mathrm{GL}(X_1)$.

Applying $\mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(-,\chi_1\boxtimes\pi')$ to the short exact sequence (8.10), one obtains the exact sequence of (not-necessarily smooth) $\mathrm{GSO}(V)$-modules:

(8.11)
$$\begin{array}{c} 0 \longrightarrow \mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(J_1,\chi_1\boxtimes\pi') \longrightarrow \mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(R_{P_1}(\Omega_{V,W}),\chi_1\boxtimes\pi') \\ \big\downarrow \\ \mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(J_0,\chi_1\boxtimes\pi'). \end{array}$$

On considering the subspace of $\mathrm{GSO}(V)$-smooth vectors in the above exact sequence, one deduces that $\Theta(\pi)^\vee$ lies in a short exact sequence
$$0\longrightarrow A_1\longrightarrow\Theta(\pi)^\vee\longrightarrow A_0\longrightarrow 0$$
where A_1 (resp. A_0) is a smooth submodule of the first (resp. last) Hom-space in (8.11). Hence, if σ is any irreducible subquotient of $\Theta(\pi)$, then σ is a subquotient of $A_0^\vee$ or $A_1^\vee$. To establish the inductive step of the argument, it remains to verify that any irreducible subquotient of $A_0^\vee$ or $A_1^\vee$ is an irreducible subquotient of
$$i_B(\chi_1\times\chi_2\times\cdots\times\chi_n\times|-|^{m-n-1}\times|-|^{m-n-2}\times\cdots\times1\times\mu|-|^{-\frac{(m-n)(m-n-1)}{4}}).$$
To this end, we now explicitly determine the $\mathrm{GSO}(V)$-modules
$$\mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(J_1,\chi_1\boxtimes\pi')\quad\text{and}\quad\mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(J_0,\chi_1\boxtimes\pi')$$
in (8.11) in turn.

For the first Hom space in (8.11), we see that for
$$\mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(J_1,\chi_1\boxtimes\pi')\neq0,$$
we need
$$\chi_1 = |-|^{m-n}.$$
When this holds,
$$\mathrm{Hom}_{\mathrm{GL}(X_1)\times\mathrm{GSp}(W')}(J_1,\chi_1\boxtimes\pi') = \mathrm{Hom}_{\mathrm{GSp}(W')}(\Omega_{V,W'},\pi'\cdot|\mathrm{sim}_{W'}|^{\frac{m-n}{2}}) = \Theta(\pi'\cdot|\mathrm{sim}_{W'}|^{\frac{m-n}{2}})^*,$$

so that

$$\Theta(\pi' \cdot |\mathrm{sim}_{W'}|^{\frac{m-n}{2}}) \twoheadrightarrow A_1^\vee.$$

By induction hypothesis (applied to $\mathrm{GSO}(V) \times \mathrm{GSp}(W')$), every irreducible subquotient of $\Theta(\pi' \cdot |\mathrm{sim}_{W'}|^{\frac{m-n}{2}})$ is a subquotient of the principal series

$$\begin{aligned} & i_B(\chi_2 \times \cdots \times \chi_n \times |-|^{m-n} \times |-|^{m-n-1} \times \cdots \times 1 \times \mu|-|^{-\frac{(m-n+1)(m-n)}{4}} \cdot |-|^{\frac{m-n}{2}}) \\ = \;& i_B(\chi_2 \times \cdots \times \chi_n \times \chi_1 \times |-|^{m-n-1} \times \cdots \times 1 \times \mu|-|^{-\frac{(m-n-1)(m-n)}{4}}). \end{aligned}$$

Hence, we see that (after permuting the χ_i's), any irreducible subquotient of $A_1^\vee$ is an irreducible subquotient of

$$i_B(\chi_1 \times \cdots \times \chi_n \times |-|^{m-n-1} \times \cdots \times 1 \times \mu|-|^{-\frac{(m-n-1)(m-n)}{4}})$$

as desired.

For the last Hom-space in (8.11), we have

$$\mathrm{Hom}_{\mathrm{GL}(X_1)\times \mathrm{GSp}(W')}(J_0, \chi_1 \boxtimes \pi') = \left(\mathrm{Ind}_{Q_1}^{\mathrm{GSO}(V)} \chi_1^{-1} \otimes \Theta_{V',W'}(\pi') \cdot (\chi_1 \circ \mathrm{sim}_{V'})\right)^*,$$

so that

$$A_0 \subset \left(\mathrm{Ind}_{Q_1}^{\mathrm{GSO}(V)} \chi_1^{-1} \otimes \Theta_{V',W'}(\pi') \cdot (\chi_1 \circ \mathrm{sim}_{V'})\right)^\vee,$$

or equivalently

$$\mathrm{Ind}_{Q_1}^{\mathrm{GSO}(V)} \chi_1^{-1} \otimes \Theta_{V',W'}(\pi') \cdot (\chi_1 \circ \mathrm{sim}_{V'}) \twoheadrightarrow A_0^\vee.$$

By induction hypothesis (applied to the similitude theta correspondence for $\mathrm{GSO}(V') \times \mathrm{GSp}(W')$), one sees that any irreducible subquotient of $\Theta_{V',W'}(\pi')$ is a subquotient of the principal series representation

$$i_{B'}(\chi_2 \times \cdots \times \chi_n \times |-|^{m-n-1} \times \cdots \times 1 \times \mu|-|^{-\frac{(m-n)(m-n-1)}{4}}).$$

Hence, any irreducible subquotient of $\Theta_{V',W'}(\pi') \cdot (\chi_1 \circ \mathrm{sim}_{V'})$ is an irreducible subquotient of

$$i_{B'}(\chi_2 \times \cdots \times \chi_n \times |-|^{m-n-1} \times \cdots \times 1 \times \chi_1\mu|-|^{-\frac{(m-n)(m-n-1)}{4}}).$$

It follows that any irreducible subquotient of $A_0^\vee$ is an irreducible subquotient of

$$i_B(\chi_1^{-1} \times \chi_2 \times \cdots \times \chi_n, |-|^{m-n-1} \times \cdots \times 1 \times \chi_1\mu|-|^{-\frac{(m-n)(m-n-1)}{4}}).$$

Applying a Weyl element (exchanging e_1 and f_1), we see that the above principal series representation has the same irreducible subquotients as

$$i_B(\chi_1 \times \chi_2 \times \cdots \times \chi_n, |-|^{m-n-1} \times \cdots \times 1 \times \mu|-|^{-\frac{(m-n)(m-n-1)}{4}}),$$

as desired.

We have thus established the inductive step of the argument and completed the proof of (i).

(ii) Suppose that $\pi \in \mathrm{Irr}_K(\mathrm{GSp}(W))$ is a submodule of an unramified principal series representation

$$i_B(\chi_1 \times \chi_2 \times \cdots \times \chi_n \times \mu).$$

By Lemma 8.7, as an $\mathrm{Sp}(W)$-module, π contains a unique irreducible summand which is $K_\flat$-unramified. By Proposition 3.5(iii), $\pi_\flat$ has nonzero theta lift to an irreducible $K'_\flat$-unramified representation $\theta_\psi(\pi_\flat)$ of $\mathrm{GSO}(V)$. It follows by [GT1, Lemma 2.2] that $\theta(\pi)$ is nonzero irreducible, has the same central character as π and contains $\theta_\psi(\pi_\flat)$ as a $\mathrm{SO}(V)$-summand.

It remains to see that $\theta(\pi)$ is K'-unramified. By (i), $\theta(\pi)$ is a constituent of the K'-unramified principal series representation

$$i_{B'}(\chi_1 \times \chi_2 \times \cdots \times \chi_n \times |-|^{m-n-1} \times |-|^{m-n-2} \times \cdots \times 1 \times \mu|-|^{-(m-n)(m-n-1)/4}).$$

Since $\theta(\pi)$ contains $\theta_\psi(\pi_\flat)$ as a $\mathrm{SO}(V)$-summand, one has

$$\dim \theta(\pi)^{K'_\flat} \neq 0.$$

It follows by Lemma 8.7 (or rather its analog for $\mathrm{GSO}(V)$) that $\theta(\pi)$ is K'-unramifiied.

(iii) To understand the theta lift of unramified representations in terms of Satake parameters, let us describe the map ι on the level of maximal tori. Recall from (8.3) that we have an identification $T = T_\flat \times \mathbb{G}_m$. This induces an identification

$$T^\vee(\mathbb{C}) = T_\flat^\vee(\mathbb{C}) \times \mathbb{C}^\times$$

so that the natural map $T^\vee \to T_\flat^\vee$ induced by the inclusion $T_\flat \hookrightarrow T$ is given by the first projection. If W' is a symplectic space of dimension $2m-2n-2$, with associated maximal torus $S \subset \mathrm{GSp}(W')$, then we likewise have

$$S^\vee(\mathbb{C}) = S_\flat(\mathbb{C}) \times \mathbb{C}^\times.$$

Similarly, for the group $\mathrm{GSO}(V)$, one has

$$T'^\vee(\mathbb{C}) = T_\flat'^\vee(\mathbb{C}) \times \mathbb{C}^\times.$$

On the level of isometry groups, one has the inclusion

$$\iota_\flat : T_\flat^\vee \times S_\flat^\vee \longrightarrow T_\flat'^\vee(\mathbb{C})$$

described by (8.6). Using the above description of the dual tori, the map ι is given by

$$\begin{array}{ccc} T^\vee \times S^\vee(\mathbb{C}) & \xrightarrow{\iota} & T'^\vee(\mathbb{C}) \\ \| & & \| \\ (T_\flat^\vee \times \mathbb{C}^\times) \times (S_\flat^\vee \times \mathbb{C}^\times) & \xrightarrow{\iota_\flat \times \mathrm{mult}} & T_\flat'^\vee \times \mathbb{C}^\times \end{array}$$

where $\iota_\flat$ is as given in (8.6) and $\mathrm{mult} : \mathbb{C}^\times \times \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ is the multiplication map.

Now one has:

$$c(\pi) = ((\chi_1(\varpi), \ldots, \chi_n(\varpi)), \mu(\varpi)) \in T_\flat^\vee \times \mathbb{C}^\times = T^\vee$$

and

$$c(\mathrm{triv}) = \left((q^{m-n-1}, \ldots, q), q^{-\frac{(m-n)(m-n-1)}{4}}\right) \in S_\flat^\vee \times \mathbb{C}^\times,$$

so that

$$\iota(c(\pi), c(\mathrm{triv})) = \left((\chi_1(\varpi), \ldots, \chi_n(\varpi), q^{m-n-1}, \ldots, q, 1), \mu(\varpi) \cdot q^{-\frac{(m-n)(m-n-1)}{4}}\right) \in T_\flat'^\vee \times \mathbb{C}^\times.$$

By (i), this is the Satake parameter of $\theta(\pi)$. $\square$

8.5. **Proof of Proposition 3.5.** When we specialize Proposition 8.8 to the case $m = n+1$ and take into account of (8.1), we obtain Proposition 3.5.

References

[AMR] N. Arancibia, C. Moeglin and D. Renard, *Paquets d'Arthur des groupes classiques et unitaires*, Ann. Fac. Sci. Toulouse Math. (6) 27 no. 5 (2018), 1023–110.

[A] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups.* American Mathematical Society Colloquium Publications **61**, A. M. S., Providence, RI (2013).

[AS] M. Asgari and F. Shahidi, *Generic transfer for general spin groups*, Duke Math. J. 132 (2006), no. 1, 137-190.

[AGIKMS] H. Atobe, W.T. Gan, A. Ichino, T. Kaletha, A. Minguez and S.W. Shin, *Local intertwining relations and cotempered A-packets of classical groups*, arxiv preprint, arXiv:2410.13504.

[BG] D. Bump & D. Ginzburg, *Spin L-functions on symplectic groups*, IMRN 8 (1992), 153–160.

[CFK] Y.Q. Cai, S. Friedberg and E. Kaplan, *Doubling constructions: global functoriality for non-generic cuspidal representations*, Ann. of Math. (2) 200 (2024), no. 3, 893-966.

[CZ] R. Chen and J.L. Zou, *Arthur multiplicity formula for even orthogonal and unitary groups*, to appear in Journal EMS, arXiv:2103.07956.

[C] G. Chenevier, *Subgroups of* $\mathrm{Spin}(7)$ *or* $\mathrm{SO}(7)$ *with each element conjugate to some element of* G_2*, and applications to automorphic forms*, Documenta Math. 24 (2019), 95–161.

[CL] G. Chenevier and J. Lannes, *Automorphic forms and even unimodular lattices*, Ergebnisse der Mathematik und ihrer Grenzgebiete 69, Springer Verlag, xxi + 417 pp. (2019).

[CR] G. Chenevier and D. Renard, *Level one algebraic cusp forms of classical groups of small rank*, Mem. Am. Math. Soc., 237 (2015), no. 1121, v+122.

[CKPSS] J.W. Cogdell, H. Kim, I. Piatetski-Shapiro, F. Shahidi, *Functoriality for the classical groups.* Publ. Math. IHES. No. 99 (2004), 163–233.

[D] M. Demazure, *Groupes réductifs : déploiements, sous-groupes, groupes quotients*, Exposé XIII in M. Demazure, A. Grothendieck & al., SGA3 , 1st ed. Spinger (1970), new ed. SMF Doc. Math. 8 (2011).

[G] W.T. Gan, *Automorphic forms and the theta correspondence*, in *Automorphic forms beyond* GL_2, 59-112, Math. Surveys Monogr., 279, Amer. Math. Soc., Providence, RI (2024).

[GS1] W.T. Gan and G. Savin, *An exceptional Siegel-Weil formula and poles of the Spin L-function of* PGSp_6. Compos. Math. 156 (2020), no. 6, 1231-1261.

[GS2] W.T. Gan and G. Savin, *The local Langlands conjecture for* G_2, Forum Math. Pi 11 (2023), Paper No. e28, 42 pp.

[GT1] W.T. Gan and S. Takeda, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2) 173 (2011), no. 3, 1841-1882.

[GT2] W.T. Gan and S. Takeda, *Theta correspondences for* $\mathrm{GSp}(4)$. Represent. Theory 15 (2011), 670-718.

[GT3] W.T. Gan and S. Takeda, *On the regularized Siegel-Weil formula (the second term identity) and non-vanishing of theta lifts from orthogonal groups*, J. Reine Angew. Math. 659 (2011), 175-244.

[GJ] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(3)$, Ann. Sci. É. N. S., Ser. 4 vol 11 (1978), 471–542.

[GRS1] D. Ginzburg, S. Rallis and D. Soudry, *Periods, poles of L-functions and symplectic-orthogonal theta lifts*, J. Reine Angew. Math. 487 (1997), 85-114.

[GRS2] D. Ginzburg, S. Rallis and D. Soudry, *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2011).

[HS] J. Hundley and E. Sayag, *Descent construction for* GSpin *groups*, Mem. Amer. Math. Soc. 243 (2016), no. 114.

[I] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) 154 (2001), no. 3, 641-681.

[IY] T. Ikeda and S. Yamana, *On the lifting of Hilbert cusp forms to Hilbert-Siegel cusp forms.*, Ann. Sci. ENS (4) 53 (2020), no. 5, 1121-1181.

[KMRT] M. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The Book of Involutions.* AMS Colloquium Publications, Vol. 44 (1998).

[K] S. Kudla, *On the local theta-correspondence*, Invent. Math. 83 (1986), no. 2, 229-255.

[Li] J.S. Li, *The correspondences of infinitesimal characters for reductive dual pairs in simple Lie groups*, Duke Math. J. 97 (1999), no. 2, 347-377.

[MW1] C. Moeglin and J.L. Waldspurger, *Stabilisation de la formule des traces tordue. Vol. 1*, Progress in Mathematics, 316. Birkhäuser/Springer, Cham (2016), xviii+587 pp.

[MW2] C. Moeglin and J.L. Waldspurger, *Stabilisation de la formule des traces tordue. Vol. 2*, Progress in Mathematics, 317. Birkhäuser/Springer, Cham (2016).

[PT] S. Patrikis & R. Taylor, *Automorphy and irreducibility of some ℓ-adic representations*, Compositio Math. 151 (2013).

[Po1] A. Pollack, *The Spin L-function on* GSp_6 *for Siegel modular forms*, Compositio Math, 153 (2017), no. 7, 1391–1432.

[Po2] A. Pollack & S. Shah, *The Spin L-function of* GSp_6 *via non unique model*, American Journal of Mathematics 140 (2018), no. 3, 753–788.

[Pr] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. 70 (1996), no. 1, 93-102.

[R1] S. Rallis, *Langlands' functoriality and the Weil representartion*, American Journal of Math., Vol. 104 (1982), No. 3, 469-515.

[R2] S. Rallis, *L-functions and the oscillator representation*, Lecture Notes in Math. 1245, Springer-Verlag (1987).

[Ra] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $\mathrm{SL}(2)$, Ann. of Math. (2) 152 (2000), no. 1, 45-111.

[Ro] B. Roberts, *The theta correspondence for similitudes*, Israel J. Math. 94 (1996), 285-317.

[T1] O. Taïbi, *Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula*, Ann. Sci. ENS 50 (2017), no. 2, 269-344.

[T2] O. Taïbi, *The Euler characteristic of ℓ-adic local systems on $\mathcal{A}_n$*, arXiv:2510.00656 (2025).

[Ti] J. Tits, *Reductive groups over local fields*, in Proc. Symp. in Pure Math. 33, *Automorphic forms, Representations and L-functions, Part I*, A. Borel & W. Casselman ed., A.M.S. (1977).

[vdG] G. van der Geer, *Siegel modular forms and their applications*, in *The 1–2–3 of modular forms*, Universitext, Springer Verlag, (2008), 181–245.

[Vo] S. C. Vo, *The spin L-function on the symplectic group* $\mathrm{GSp}(6)$, Israel J. Math. 101, 1–71 (1997).

[X1] Bin Xu, *L–packets of quasisplit* $\mathrm{GSp}(2n)$ *and* $\mathrm{GO}(2n)$. Math. Ann. 370 (2018), no. 1-2, 71-189.

[X2] Bin Xu, *Global L-packets of quasi-split* $\mathrm{GSp}(2n)$ *and* $\mathrm{GSO}(2n)$, Amer. J. Math. 147 (2025), no 2, 401-464.

Département de mathématiques et applications, École Normale Supérieure PSL, 45 rue d'Ulm, 75230 Paris Cedex, France

E-mail address: gaetan.chenevier@math.cnrs.fr

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road Singapore 119076

E-mail address: matgwt@nus.edu.sg