THE SIGN OF GALOIS REPRESENTATIONS ATTACHED TO AUTOMORPHIC FORMS FOR UNITARY GROUPS

JOËL BELLAÏCHE AND GAËTAN CHENEVIER

Abstract. Let K be a CM number field and G_K its absolute Galois group. A representation of G_K is said polarized if it is isomorphic to the contragredient of its outer complex conjugate, up to a twist by a power of the cyclotomic character. Absolutely irreducible polarized representations of G_K have a sign ± 1, generalizing the fact that a self-dual absolutely irreducible representation is either symplectic or orthogonal. If Π is a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$, and if ρ is a p-adic Galois representation attached to Π, then ρ is polarized and we show that all of its polarized irreducible constituents have sign $+1$. In particular, we determine the orthogonal/symplectic alternative for the Galois representations associated to the regular algebraic, essentially self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ when F is a totally real number field.

1. Introduction

1.1. **The sign of a representation.** Let L be a field of characteristic 0 or greater than 2. Let G be a group, and $g \mapsto g^c$ an involution of G. For ρ a representation $G \to \mathrm{GL}_n(L)$, we define $\rho^\perp : G \to \mathrm{GL}_n(L)$, $g \mapsto {}^t\rho(g^c)^{-1}$. The equivalence class of the representation $\rho^\perp$ only depends on the equivalence class of ρ.

We fix $\chi : G \to L^*$ a character such that $\chi(g) = \chi(g^c)$ for all g. This property ensures that $\rho \mapsto \rho^\perp\chi^{-1}$ is an involution. In the applications, G will be the absolute Galois group of a CM number field K, c the outer automorphism defined by the non trivial element in $\mathrm{Gal}(K/F)$ where F is the maximal totally real subfield of K, and χ will be a power of the cyclotomic character.

Let ρ be a semi-simple representation $G \to \mathrm{GL}_n(L)$ such that

$$\rho^\perp \simeq \rho\chi. \quad (1)$$

This property is obviously stable by extension of the field of coefficients L.

We shall now attach to any *absolutely irreducible* ρ satisfying (1) an invariant, that we call its *sign*. The invariant can take the value $+1$ or -1. By Schur's lemma there exists a unique (up to a scalar) matrix $A \in \mathrm{GL}_n(L)$ such that

$$\rho^\perp = A\rho A^{-1}\chi. \quad (2)$$

Applying this relation twice, we see that A^tA^{-1} commutes with $\rho^\perp$, hence by Schur's lemma again is a scalar matrix λ. So ${}^tA = \lambda A$ and $\lambda = \pm 1$. This sign is called the

We thank Laurent Clozel, Michael Harris and Jean-Pierre Labesse, for many useful conversations and for their constant support. This paper relies on the book project [GRFAbook], and we thank all its authors for having made it possible. During the elaboration and writing of this paper, Joël Bellaïche was supported by the NSF grant DMS 05-01023, and Gaëtan Chenevier was supported by the C.N.R.S.

sign of ρ (with respect to χ). Note that it is necessarily 1 if n is odd, since there is no invertible antisymmetric matrix in odd dimension.

If $\rho' := Q^{-1}\rho Q$ for some $Q \in \mathrm{GL}_n(L)$, then ρ' satisfies (2) with $A' = {}^tQAQ$, so the sign of ρ only depends on the isomorphism class of ρ. Moreover, it is obvious that it remains unchanged under arbitrary extensions of the coefficient field L. However, it depends on χ in general: if $\rho \simeq \rho \otimes \varepsilon$ for some non-trivial character ε, then the sign of ρ with respect to χ and $\chi\varepsilon$ may differ.[1]

1.2. **Galois representations attached to unitary groups.** Let F be a totally real field and K a totally imaginary quadratic extension, $c \in \mathrm{Gal}(K/F)$ the non-trivial automorphism. Let Π be a cuspidal automorphic representation for GL_n over K, and assume that Π is polarized, i.e. the contragredient $\Pi^\vee$ of Π is isomorphic to $\Pi \circ c$, and that Π_∞ is algebraic regular (see [ChH, General Hypotheses 2.1]).

Under those hypotheses, Shin [Sh] and the many coauthors of this two-volumes book [GRFAbook] have shown the existence of a compatible system of Galois representations attached to Π (see [ChH, Theorem 3.2.5]):

Theorem 1.1. *There is a number field $E(\Pi)$ and a compatible system $\rho_{\Pi,\lambda} : G_K \to GL(n, E(\Pi)_\lambda)$ of semisimple λ-adic representations, where λ runs through finite places of $E(\Pi)$, such that for all finite primes v of K of residue characteristic prime to $N_{E(\Pi)/\mathbb{Q}}(\lambda)$, and such that Π_v is unramified,*

$$(\rho_{\Pi,\lambda}|_{G_v})^{F-ss} \simeq \mathrm{L}(\Pi_v \otimes |\bullet|_v^{\frac{1-n}{2}}),$$

where G_v is a decomposition group of K at v and $L(\bullet)$ is the local Langlands correspondence.

The given property suffices to characterize uniquely $\rho_{\Pi,\lambda}$ up to isomorphism and implies that $\rho_{\pi,\lambda}$ satisfies (1); more precisely, let c be a complex conjugation in K, that is an element of $G_F - G_K$ of order 2. We set $g^c = cgc^{-1} = cgc$ for $g \in G_K$: this is an automorphism of order 2. For that automorphism, we have

$$\rho^\perp(g) = {}^t\rho(g^c)^{-1} \simeq \rho(g)\omega(g)^{n-1}$$

where ω is the cyclotomic character.

The theorem also includes other specifications on $\rho_{\Pi,\lambda}$, including the determination of the Hodge-Tate weights of $\rho_{\Pi,\lambda}$ at places of same residual characteristic as λ (see also §1.6 below). This description implies, since Π_∞ is cohomological, that these weights are distinct integers, hence that $\rho_{\Pi,\lambda}$ is a direct sum of non-isomorphic absolutely irreducible representations of G_K.

1.3. **The result.** The object of this article is to prove

Theorem 1.2. *For every finite prime λ of $E(\Pi)$, every irreducible factor r of $\rho_{\Pi,\lambda}$ that satisfies $r^\perp \simeq r \otimes \omega^{n-1}$ has sign $+1$.*

[1] For example, let $G \subset \mathrm{GL}_2(L)$ be the normalizer of the diagonal matrices, ρ the natural inclusion and $g^c := g\det(g)^{-1}$. Then ρ has sign -1 for the trivial character, as ${}^t(g^c)^{-1} = wgw^{-1}$ for any $g \in \mathrm{GL}_2(L)$ and for $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But ρ has sign $+1$ for the order 2 character ε which is trivial on the diagonal matrices, as $g = dgd^{-1}\varepsilon(g)$ for $g \in G$ and d the diagonal matrix $(1, -1)$.

In this statement, it is understood that this sign is computed with respect to the character $\chi := \omega^{n-1}$. It is expected that $\rho_{\Pi,\lambda}$ is absolutely irreducible (this is known if $n \leq 3$ by [BlRo1] and in many cases if $n = 4$ by an unpublished work of Ramakrishnan). If it is so, $\rho_{\Pi,\lambda}$ has only one factor and satisfies (1), and our theorem simply asserts that its sign is $+1$: this is obvious when n is odd, but new when n is even.

The theorem above has an important corollary concerning essentially self-dual Galois representations of a totally real field F. Precisely, let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that:

(a) $\Pi^\vee \simeq \Pi \otimes \eta$, where η is a Hecke character of F such that $\eta_v(-1)$ does not depend on the real place v of F,

(b) Π_v is cohomological for each real place v of F.

In this case as well the aforementionned works show that for some coefficient number field $E(\Pi)$ there is a compatible system of λ-adic semisimple representations $\rho_{\Pi,\lambda} : G_F \rightarrow \mathrm{GL}_n(E(\Pi)_\lambda)$ which are compatible with the Frobenius-semisimplified local Langlands correspondence twisted by $|.|^{\frac{1-n}{2}}$ at each prime not dividing the residue characteristic of λ and unramified for Π ([ChH, Thm. 4.2]). In particular, we have

$$\rho_{\Pi,\lambda}^\vee \simeq \rho_{\Pi,\lambda} \otimes \omega^{n-1}\eta_\lambda,$$

where η_λ is the λ-adic realization of η (note that η is necessarily algebraic by (a) and (b)). As Π is cuspidal, $\rho_{\Pi,\lambda}$ is conjecturally irreducible, but as before this is not known in general (however, each irreducible constituent of $\rho_{\Pi,\lambda}$ has multiplicity one and is absolutely irreducible). The counterpart of the sign in this situation is the standard alternative orthogonal/symplectic : if $r : G_F \rightarrow \mathrm{GL}_d(L)$ is absolutely irreducible and satisfies $r^\vee \simeq r \otimes \omega^{n-1}\eta_\lambda$, then the unique G_F-equivariant pairing $r \otimes r \rightarrow E(\Pi)_\lambda \omega^{1-n}\eta_\lambda^{-1}$ is either symplectic or orthogonal.

The sign $\eta_v(-1)$ in (a) will be denoted by $\eta_\infty(-1)$. The signs $\eta_\infty(-1)$ and $\eta_\lambda(c)$ are related as follows: there is a unique $q \in \mathbb{Z}$ such that $\eta|.|^{-q}$ is an Artin character, thus

$$\eta_\lambda(c) = (-1)^q \eta_\infty(-1).$$

If $\mathfrak{z}$ denotes the central character of Π, then $\mathfrak{z}$ is an algebraic Hecke character of F and $\mathfrak{z}^{-2} = \eta^n$. In particular, η is the square of an algebraic Hecke character when n is odd, thus $\eta_\infty(-1) = (-1)^q = \eta_\lambda(c) = 1$ in this case.

Corollary 1.3. (Totally real field case) *If n is even and $\eta_\lambda(c) = 1$, then any irreducible constituent r of $\rho_{\Pi,\lambda}$ such that $r^\vee \simeq r \otimes \omega^{n-1}\eta_\lambda$ is symplectic. Otherwise, any such constituent is orthogonal.*

Proof — Let K be a totally imaginary quadratic extension of F which is ramified above some finite place v of F whose residue characteristic is prime to the one of λ, and such that Π_v is unramified. Let $\epsilon_{K/F}$ be the non trivial character of $\mathrm{Gal}(K/F)$ or its associated Hecke character of F. As $\Pi \not\simeq \Pi \otimes \epsilon_{K/F}$ (at v), Arthur-Clozel's base change Π_K of Π to $\mathrm{GL}_n(\mathbb{A}_K)$ is cuspidal. Moreover, for each irreducible constituent r of $\rho_{\Pi,\lambda}$, $r_{|G_K}$ remains absolutely irreducible, still as $r \not\simeq r \otimes \epsilon_{K/F}$ (at v).

By [ClHT, Lemma 4.1.4], we may find some algebraic Hecke character ψ of K such that $\psi \circ N_{K/F} = \eta \circ N_{K/F}$. In particular, $\eta_{\lambda|G_K} = \psi_\lambda(\psi_\lambda{}^\perp)^{-1}$, $\Pi' := \Pi_K \otimes \psi$ is polarized (and algebraic regular) and $\rho_{\Pi',\lambda} = \rho_{\Pi,\lambda|G_K} \otimes \psi_\lambda$. Theorem 1.2 ensures that for each r as in the statement, $r_{|G_K} \otimes \psi_\lambda$ has sign $+1$ with respect to ω^{n-1}. By Lemma 2.1 below, $r_{|G_K}$ has sign $+1$ with respect to $\psi_\lambda(\psi_\lambda^\perp)^{-1}\omega^{n-1} = \eta_{\lambda|G_K}\omega^{n-1}$.

Fix a complex conjugation $c \in G_F$ and choose a matrix realization $r : G_F \to \mathrm{GL}_d(F)$ such that $r(c)$ is diagonal, so $r(c) = {}^t r(c) = r(c)^{-1}$. For some $P \in \mathrm{GL}_d(L)$,

$$ {}^t r(g)^{-1} = P r(g) P^{-1} \eta_\lambda(g) \omega(g)^{n-1}, \quad \forall\, g \in G_F. $$

Applying this to c gives

$$ r(c)P = Pr(c)\eta_\lambda(c)(-1)^{n-1}. \tag{3} $$

On the other hand, an immediate computation shows that for all $g \in G_K$,

$$ {}^t r(cgc)^{-1} = A r(g) A^{-1} \eta_\lambda(g) \omega(g)^{n-1} $$

with $A = r(c)P$. By the preceding paragraph we have ${}^tA = A$, so

$$ {}^t P = (-1)^{n-1} \eta_\lambda(c)\ P $$

by (3) and the corollary follows as $\eta_\lambda(c) = 1$ for odd n. □

1.4. **Historical remarks.** The question of the sign of Galois representations attached to polarized automorphic representations of GL_n on a totally real or CM field is out at least since Clozel, building on the work of Kottwitz, proved their existence in many cases in the mid nineties. More recently, this question has been extensively discussed in [ClHT] where some cases of the above theorem, concerning Galois representations with some constraining properties ensuring they have a nice and workable deformation theory, are proved by a very indirect method – indeed the whole long and hard paper is written with an unknown sign ϵ and only near the end, after the Taylor-Wiles method has been adapted to unitary groups, is it shown that $\epsilon = -1$ leads to a contradiction!

Theorem 1.2 appears, without its proof, in the concluding remarks of our book [BCh] (see [BCh, theorem 9.5.1]) that was made public late 2006. We knew the proof that follows then[2], and told it to a few colleagues, but decided to wait for a more advanced version of the present book project, on which it depends, before writing it.

Meanwhile, one of us, Gaëtan Chenevier, together with Laurent Clozel, have found a completely different proof of a special case of Corollary 1.3, namely when $\eta = 1$ and Π is square integrable at some finite place. In this case, $\rho_{\Pi,\lambda}$ is known to be irreducible by works of Harris-Taylor and Taylor-Yoshida, and they show that it is symplectic for n is even. Their proof was actually conditional to the computation of some archimedean orbital integrals, which has since been done by Chenevier and Renard in [ChR]. The method of the proof in [ChCl] is less expensive in difficult tools than ours, using "simply" the new insight in the trace formula they discovered.

[2]At least for places λ of residual characteristic p split in K.

However, it does not seem that it can be extended to the case of a CM field, or even to the case of an automorphic representation that does not satisfy any local square-integrability hypothesis.

Let us mention also that in a recent preprint [Gr], B. Gross introduces a general notion of *odd* Galois representations and conjectures that the expected Galois representations attached to definite reductive groups G are odd in his sense. Our theorem proves his conjecture when G is the a unitary group attached to a CM extension K/F, in which case it has the following meaning.

Let $\tilde{G}$ be the semi-direct product of $\mathrm{Gal}(K/F) = \langle c \rangle = \mathbb{Z}/2\mathbb{Z}$ by $\mathrm{GL}_n(L) \times L^*$ with respect to the order two automorphism $(x, y) \mapsto (y^t x^{-1}, y)$ (see [ClHT, Ch.I] for similar considerations). Assume that $\rho : G_K \to \mathrm{GL}_n(L)$ satisfies (1), is absolutely irreducible, and fix A a matrix as in (2), and $\epsilon = \pm 1$ the sign of ρ. Consider the morphism $G_K \to \mathrm{GL}_n(L) \times L^*$ defined by $g \mapsto (\rho(g), \chi(g)^{-1})$. A simple computation shows that this map extends to a morphism $\tilde{\rho} : G_F \to \tilde{G}$ if we set $\tilde{\rho}(c) = ({}^tA^{-1}, \epsilon)c$. Assume now that $\rho = \rho_{\Pi,\lambda}$. The map $\tilde{\rho}$ is the analog in our situation of the map denoted ρ whose existence is conjectured in [Gr, page 8] and Gross predicts that the conjugation by $\tilde{\rho}(c)$ on $\mathrm{Lie}(\mathrm{GL}_n)$ is a Cartan involution, that is, has the form $X \mapsto -P^tXP^{-1}$ with P a *symmetric* invertible matrix. In our situation, the conjugation by $\tilde{\rho}(c)$ on the Lie algebra is the map $X \mapsto -{}^tA^{-1\,t}X^tA$. So we see that Gross' prediction amounts to "A is symmetric", which is exactly our theorem. [3]

1.5. **Idea of the proof.** The idea of the proof is very simple. Assume that we know that the representation $\rho_{\Pi,\lambda}$ is irreducible. Then there is nothing to prove if n is odd. When n is even, we can reduce to the odd case, as follows: descend Π to a unitary group in n variables, transfer the result to an automorphic representation π of a unitary groups in $n+1$ variables which is compact at infinity, using a special case of endoscopic transfer proved by Clozel, Harris and Labesse. Use eigenvarieties to deform π into a family of automorphic forms whose Galois representations are generically irreducible. For those Galois representations, the sign is $+1$ since their dimension is odd. Specialize this result to deduce that the components of the representation attached to π, including $\rho_{\Pi,\lambda}$, have sign $+1$.

There are several technical difficulties that make the proof a little bit more indirect: in the current state of science, we do not know that $\rho_{\Pi,\lambda}$ is (absolutely) irreducible, and we cannot descend Π to $\mathrm{U}(n)$ or transfer it to $\mathrm{U}(n+1)$ without supplementary assumptions on K/F and Π. Moreover, we cannot always deform a representation π in a family whose Galois representation is generically irreducible. But this is not a big issue, since, as was already observed in [BCh, §7.7], we can actually do so in two steps, deforming π in a family whose generic members can themselves be deformed irreducibly. Similarly the obstacle posed by the conditions on descent and endoscopic transfer can be solved by base change techniques inspired by the ones used in [ChH].

[3]When ρ is not assumed to be irreducible anymore, note that theorem 1.2 still implies that we may find some symmetric A such that (2) holds, hence a $\tilde{\rho}$ as above satisfying Gross' conjecture.

1.6. **Notations and conventions.** Our general convention will be that the local Langlands correspondence is normalized so that geometric Frobeniuses correspond to uniformizers (and as in [HT]). If π is an unramified complex representation of $\mathrm{GL}_n(E)$ with E a p-adic local field, or more generally an irreducible smooth representation with a nontrivial vector fixed by a Iwahori subgroup, we shall often denote by $L(\pi)$ the semisimple conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ of the geometric Frobenius in the L-parameter of π.

If K is a field, we shall denote by G_K its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$; when K is a number field and v a place of K we also write G_v for G_{K_v}.

We shall use the following notions of p-adic Hodge theory. Let us fix E a finite extension of $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$, and let V be a p-adic representation of G_E of dimension n over $\overline{\mathbb{Q}}_p$. To such a representation Sen attaches a monic polynomial $P_{\mathrm{sen}}(T) \in (\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E)[T]$ of degree n, whose roots will be called the Hodge-Tate weights of V (even when they are not natural integers). Our normalization of the Sen polynomial is the one such that the Hodge-Tate weight of the cyclotomic character $\overline{\mathbb{Q}}_p(1)$ is $-1 \in \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E$. Under the natural identification $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E = \overline{\mathbb{Q}}_p^{\mathrm{Hom}(E,\overline{\mathbb{Q}}_p)}$, we shall often write them as a collection $\{k_{i,\sigma}\}$ for all $i \in \{1, ..., n\}$ and all $\sigma \in \mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$, ordered so that for each embedding σ we have

$$k_{1,\sigma} \leq k_{2,\sigma} \leq \cdots \leq k_{n,\sigma}.$$

We shall need to consider various partial sums of those weights, for which the following definitions will be useful. For I a subset of $\{1, \ldots, n\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, we denote by k_I the sum $\sum_{(i,\sigma)\in I} k_{i,\sigma}$. When $I = \{i\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, we write k_i instead of k_I. Thus $k_i = \sum_\sigma k_{i,\sigma}$.

Assume now that V is crystalline in the sense of Fontaine. Let $E_0 \subset E$ be the maximal unramified extension of $\mathbb{Q}_p$ inside E, and let $\mathbf{v} : \overline{\mathbb{Q}}_p \to \mathbb{Q}$ be the valuation *normalized so that* $\mathbf{v}(p) = e$, where e is the absolute ramification index of E. Fontaine attaches to V an E_0-vector space $D_{\mathrm{crys}}(V)$ with a semilinear action of the crystalline Frobenius φ (commuting with $\overline{\mathbb{Q}}_p$), and which is free of rank n over $E_0 \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. If $f = [E_0 : \mathbb{Q}_p]$, then φ^f is $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E_0$-linear and commutes with φ, so its characteristic polynomial $P_\varphi(T)$ actually belongs to $\overline{\mathbb{Q}}_p[T]$. This polynomial will be referred as the *characteristic polynomial* of φ, its roots are the *eigenvalues* of the crystalline Frobenius, and their valuations (with respect to $\mathbf{v}$) its *slopes*.[4] With these notations, if the $k_{i,\sigma}$ are the Hodge-Tate weights of V, then the weak admissibility property of $D_{\mathrm{crys}}(V)$ implies in particular that

$$\mathbf{v}(P_\varphi(0)) = \sum_{i,\sigma} k_{i,\sigma}.$$

We can now explain a bit more precisely the p-adic part of theorem 1.1. Assume that w is a finite place of K with the same residual characteristic as λ, and assume than Π_w is unramified. Let $P_w(T) \in E(\Pi)[T]$ be the characteristic polynomial of $L(\Pi_w|.|^{(1-n)/2})$. Then a refinement of theorem 1.1 asserts that $\rho_{\Pi,\lambda}|G_w$ *is a*

[4]This definition is slightly different from the usual definition of the slopes of an isocrystal (which are ours divided by $[E : \mathbb{Q}_p]$), but it will be convenient to us.

crystalline representation and that the characteristic polynomial $P_\varphi \in E(\Pi)_\lambda[T]$ of its crystalline Frobenius coincides with the image of $P_w(T)$ in $E(\Pi)_\lambda[T]$, see [ChH, Thm. 3.2.5 (c)].[5]

2. Sorites on the sign

2.1. **The notion of a good representation.** For a representation $\rho : G_K \to \mathrm{GL}_n(L)$ that is a direct sum of absolutely irreducible and pairwise non isomorphic representations, and that satisfies (1) for some fixed character χ, say that ρ is *good* (with respect to χ) if for every irreducible factor of ρ that appears with multiplicity one and satisfies (1), the sign of this factor is $+1$.

In this language, the theorem amounts to prove that $\rho_{\Pi,\lambda}$ is good, which is good.

2.2. **Some trivial lemmas.** In this paragraph, $\rho : G_K \to \mathrm{GL}_n(L)$ is a direct sum of absolutely irreducible representations, and satisfies (1).

Lemma 2.1. *If $\rho : G_K \to GL_n(L)$ is good with respect to χ, and if $\psi : G_K \to L^*$ is a character, then $\rho\psi$ is good with respect to $\chi\psi^{-1}\psi^\perp$.*

In particular, if m is an integer and if $\psi^\perp = \psi\omega^{m-n}$, then ρ is good with respect to ω^{n-1} if and only if $\rho\psi$ is good with respect to ω^{m-1}.

Proof — Note first that since ρ satisfies (1) for χ, then $\rho\psi$ still satisfies (1) for the character $\chi' = \chi\psi^{-1}\psi^\perp$ (which still satisfies $\chi'(g^c) = \chi'(g)$) and is a sum of absolutely irreducible pairwise non isomorphic factors, namely the $\rho_i\psi$ where the ρ_i are the factors of ρ. Now if ρ_i is an irreducible factor that satisfies (1) for χ', a matrix A that satisfies (2) for ρ_i and χ satisfies also (2) for $\rho_i\psi$ and χ', hence the sign of ρ_i and $\rho_i\psi$ are the same, which proves the first assertion. The last part of the lemma follows at once. □

Lemma 2.2. *If $\rho : G_K \to GL_n(L)$ is good, and ρ' is a sub-representation of ρ that satisfies (1), then ρ' is good too (with respect to the same character).*

Proof — That one is *really* trivial. □

Lemma 2.3. *Let F' be a totally real extension of F, and $K' = KF$. If $\rho|G_{K'}$ has the same number of irreducible components as ρ, and if those components are absolutely irreducible, then $\rho|G_{K'}$ is good with respect to $\chi_{|G_{K'}}$ if and only if ρ is good with respect to χ.*

Proof — If ρ_i is an (absolutely) irreducible factor of ρ that satisfies (1), then $\rho_i|G_{K'}$ is still absolutely irreducible by hypothesis, still satisfies (1), and has obviously the same sign as ρ_i. The lemma follows. □

[5] We will actually use this identity $P_\varphi = P_w$ only under the following extra assumptions for which it holds by construction: assumptions (H1) and (H2) stated in §3.1 below on K/F and Π are satisfied.

2.3. **A specialization result.** In this paragraph, $\mathcal{O}$ is a henselian discrete valuation domain with fraction field L and residue field k, such that $2 \in \mathcal{O}^*$. We set also $G = G_K$ and assume that the character $\chi : G_K \to L^*$ actually falls into $\mathcal{O}^*$, thus it makes sense to talk about condition (1) for k or L-valued representations of G (by a slight abuse of language, we shall also denote by χ the residual character $G_K \to k^*$). A simple but crucial observation for our proof is the following:

Proposition 2.4. *Assume that $\rho : G \to GL_n(\mathcal{O})$ is such that $\rho \otimes L$ and $\bar{\rho}^{ss}$ are a sum of absolutely irreducible pairwise non isomorphic representations and satisfy (1). If $\rho \otimes L$ is good with respect to χ, then so is $\bar{\rho}^{ss}$.*

Moreover, the converse holds if $\bar{\rho}^{ss}$ has the same number of irreducible factors as $\rho \otimes L$.

Of course, in this statement $\bar{\rho}^{\rm ss}$ denotes the semisimplification of the reduction $\bar{\rho} := \rho \otimes_{\mathcal{O}} k$ of ρ.

Proof — Let $\bar{\rho}_1$ be a factor of $\bar{\rho}^{\rm ss}$ that satisfies (1). Let $\tau_1, \dots, \tau_k$ be the irreducible factors of $\rho \otimes L$. For each of them we can choose a stable $\mathcal{O}$-lattice, and see them as representations of G over $\mathcal{O}$. We have $\bar{\rho}^{\rm ss} = \oplus_{i=1}^k \overline{\tau_i}^{\rm ss}$ so $\bar{\rho}_1$ appears in exactly one of the $\overline{\tau_i}^{\rm ss}$, say $\overline{\tau_1}^{\rm ss}$. Moreover, $\overline{\tau_1}^\perp$ is isomorphic to $\overline{\tau_i}\chi$ for some $i \in \{1, \dots, k\}$. But it follows that $\overline{\tau_i}^{\rm ss}$ contains $\bar{\rho}_1$ (since $\bar{\rho}_1$ satisfies (1)), so the only possibility is that $i = 1$. In other words, τ_1 satisfies (1), and replacing ρ by τ_1, we are reduced to prove the lemma with the supplementary assumption that $\rho \otimes L$ is absolutely irreducible. In that case, the proposition is [BCh, Lemma 1.8.8].

Since this proposition is really one of the main tool used in our proof, and since the proof of [BCh, Lemma 1.8.8] is a little bit difficult to separate from the other concerns of [BCh, §1.8], let us sketch it here for the convenience of the reader, trying to be as pedagogical as possible.

Note first that the basic point that makes the result not obvious is that there is no reason so that we can find a matrix A for $\rho \otimes L$ as in (2) with $A \in \mathrm{GL}_n(\mathcal{O})$. A priori we just have $A \in \mathrm{GL}_n(L)$, and it is therefore not possible to reduce (2) mod m.

There is one case, however, where a simple proof is possible: assume that $\bar{\rho}^{\rm ss}$ is absolutely irreducible. In this case, the representations $\rho^\perp$ and $\rho\chi$ over $\mathcal{O}$, being isomorphic over L and residually absolutely irreducible, are isomorphic over $\mathcal{O}$ by a theorem of Serre and Carayol. In other words, we can find a matrix $A \in \mathrm{GL}_n(\mathcal{O})$ such that (2) holds, and reducing this modulo m, we get that $\rho \otimes L$ and $\bar{\rho}$ have the same sign in this case. Note that this proves also the last assertion of the theorem (in all cases!).

The proof of the general case consists in reducing to the residually irreducible case. This is not possible, however, if we keep working with representations of groups only. We have to work in the larger world of representations of algebras instead. As we saw, we may assume that $\rho \otimes L$ is absolutely irreducible, and we set $\bar{\rho}^{\rm ss} = \oplus_i \bar{\rho}_i$.

Let R be the algebra $\mathcal{O}[G]$, and $S = \rho(R) \subset M_n(\mathcal{O})$. We have $S \otimes_{\mathcal{O}} L = M_n(L)$. The algebra S is provided with a natural $\mathcal{O}$-algebra anti-automorphism τ, induced by the one on R defined on $g \in G$ by $g \mapsto \chi(g)^{-1}(g^c)^{-1}$. Explicitly, by (2), we have for $M \in S$,

$$\tau(M) = {}^tA^{-1\,t}M^tA, \tag{4}$$

and by our sign assumption ${}^tA = A$: the involution τ is a symmetric involution of the matrix algebra $S \otimes_{\mathcal{O}} L = M_n(L)$.

On the other hand, let $\overline{S}$ denote the image of $k[G]$ in the k-endomorphisms of the representation $\bar{\rho}^{\rm ss} = \oplus_i \bar{\rho}_i$. Then $\overline{S} \simeq \prod_i M_{n_i}(k)$ ($n_i = \dim \bar{\rho}_i$, $\sum_i n_i = n$) and $\overline{S}$ is also provided with a natural k-algebra anti-automorphism τ as above. Moreover, there is a natural surjective $\mathcal{O}$-algebra map $S \to \overline{S}$ which is τ-equivariant.

Let us denote by $\epsilon_i \in \overline{S}$ be the central idempotent corresponding to $\bar{\rho}_i$. It is well known that ϵ_i can be lifted as an idempotent e_i of S as $\mathcal{O}$ is henselian and S finite over $\mathcal{O}$. However, we need a more precise lifting result. Let us fix an i such that $\bar{\rho}_i$ satisfies (1), then we have $\tau(\epsilon_i) = \epsilon_i$. What we need is an idempotent e_i in S lifting ϵ_i, such that $\tau(e_i) = e_i$. The existence of such an idempotent is easy to prove: first choose any lift $x \in S$ of ϵ_i and let S_0 be the sub-$\mathcal{O}$-algebra generated by $\frac{1}{2}(x+\tau(x))$. Obviously, τ fixes any element of S_0. The restriction of the natural surjection $S \to \overline{S}$ to S_0 is onto a k-subalgebra of $\overline{S}$ that contains the image of $\frac{1}{2}(x+\tau(x))$, that is ϵ_i. Thus, defining e_i as a lift of ϵ_i in S_0 does the job. (This result is the trivial case of [BCh, Lemma 1.8.2].) As $\bar{\rho}_i$ is absolutely irreducible and has multiplicity one in $\bar{\rho}^{\rm ss}$ it actually turns out that the rank of e_i is $n_i = \dim \bar{\rho}_i$, and that $e_iSe_i \simeq M_{n_i}(\mathcal{O})$. Replacing ρ by a conjugate if necessary, we may then assume that e_i is a diagonal idempotent of rank n_i in $M_n(L)$.

Applying (4) to $M = e_i$ we get $Ae_i = {}^te_i{}^tA$, that is Ae_i is symmetric. In other words, τ induces a symmetric involution on $e_iSe_i \simeq M_{n_i}(\mathcal{O})$. As a consequence, τ also induces a symmetric involution on $\epsilon_i\overline{S}\varepsilon_i = \mathrm{End}_k(\bar{\rho}_i)$, which exactly means that the sign of $\bar{\rho}_i$ is $+1$, QED. □

Remark 2.5. The above proposition, or rather its crucial case [BCh, Lemma 1.8.8] is a theorem of Thompson in the case where G is a finite group, $\chi = 1$ and the involution $g \mapsto g^c$ is the identity : see [T, last theorem].

3. Proof of the main theorem

3.1. **Proof of theorem 1.2 under special hypotheses.** We shall first prove the theorem under a set of additional hypotheses on the CM extension K/F, the automorphic representation Π and the place λ.

Let us call p the residual characteristic of λ. Recall that the automorphic representation Π defines an embedding $\iota : E(\Pi) \to \mathbb{C}$. We fix once and for all algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, as well as some embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}}_p \to \mathbb{C}$ such that the induced map $\iota_p\iota_\infty^{-1}\iota : E(\Pi) \to \overline{\mathbb{Q}}_p$ factors through $E(\Pi)_\lambda$.

3.1.1. *Some special hypotheses.*

- (H1) Special Hypotheses 2.2 of [ChH], that is
  - (H1a) K/F is unramified at all finite places
  - (H1b) Π_v is spherical at all non-split non-archimedean places v of K
  - (H1c) The degree $[F:\mathbb{Q}]$ is even.
- (H2) Hypothesis 1.3 of [ClHL], that is for all real places σ of K, the infinitesimal character of Π_σ is sufficiently far from the walls.[6]
- (H3) There[7] is a place v above p in F that splits in K, and for w a place of K above v, Π_w is unramified. Denote by $\{\varphi_1,\dots,\varphi_n\}$ the eigenvalues of $L(\Pi_w|.|^{(1-n)/2})$. Then the Hodge-Tate weights $\{k_{i,\sigma}\}$ of $\rho_{\Pi,\lambda}|G_w$ and the slopes $\mathbf{v}(\varphi_j)$ are in sufficiently general position in the following sense: if
$$c=\max_{i\in\{1,\dots,n\}}\min_{j\in\{1,\dots,n\}}|\mathbf{v}(\varphi_i)-k_j|$$
then for all subsets I and J of $\{1,...,n\}\times\mathrm{Hom}(K_w,\overline{\mathbb{Q}}_p)$ with $|I|=|J|<nd$, we have
$$|k_I-k_J|>(n+1)\cdot c.$$

In (H3) above, $\mathbf{v}:\overline{\mathbb{Q}}_p\to\overline{\mathbb{Q}}$ is the valuation such that $\mathbf{v}(p)$ is the ramification index of p in K_w.

3.1.2. *The theorem.* We want to prove :

Theorem 3.1. *With the supplementary hypotheses (H1), (H2) and (H3), Theorem 1.2 holds.*

The rest of this subsection is entirely devoted to the proof of this theorem.

3.1.3. *Descent and transfer.* Let $m=n$ if n is odd, and $m=n+1$ if n is even, so that m is always odd. Let us call $\mathrm{U}(m)$ be the unitary group over F attached to K in m variables that is quasi-split at every finite places of F and compact at every infinite place. Since m is odd, such a group always exists (uniquely up to isomorphism). Actually, $\mathrm{U}(m)$ is simply the standard unitary group attached to the hermitian form $\sum_{i=1}^m N_{K/F}(z_i)$ on K^m ([BCh, §6.2.2]).

If n is odd, that is if $n=m$, by hypothesis (H1) and Labesse's base change theorem [L, Thm. 5.4], we can descend Π to a representation π of $\mathrm{U}(m)$ with $\pi_v\simeq\Pi_w$ for every place w of K split over v in F (with the natural identification $\mathrm{U}(n)(F_v)\simeq\mathrm{GL}_n(K_w)$), and such that for each complex place w of K above a real place v of F, π_v has the same infinitesimal character as Π_w (under the natural identification $\mathrm{U}(n)(K_w)\simeq\mathrm{GL}_n(\mathbb{C})$).

If n is even, we use a result of endoscopic transfer due to Clozel, Harris, and Labesse [ClHL]. Note first that using $\iota_\infty\iota_p^{-1}$, if $v=ww^c$ is as (H3) we may identify $\mathrm{Hom}(K_w,\overline{\mathbb{Q}}_p)=\mathrm{Hom}(F_v,\overline{\mathbb{Q}}_p)$ with subsets Σ_v and Σ_w of $\mathrm{Hom}(F,\mathbb{R})$ and $\mathrm{Hom}(K,\mathbb{C})$. Let us first fix
$$\mu:K^*\backslash\mathbb{A}_K^*\to\mathbb{C}^*$$

[6]Precisely, this means that the extremal weight of the associated algebraic representation of $\mathrm{GL}_n(K\otimes\mathbb{R})$ does not belong to a wall.

[7]See §1.6 for the notations used in this assumption.

a Hecke character such that $\mu(c(x))^{-1} = \mu(x)$, and such that for each $s \in \Sigma_v$, $\mu_s(z) = (\sigma_s(z)/\overline{\sigma_s(z)})^{\frac{1}{2}}$ where $\sigma_s \in \Sigma_w$ is associated to s as above. This last assumption implies that $\mu|\mathbb{A}_F^*$ coincides with the sign of K/F, and that μ does not come by base change from a Hecke-character of U(1) (see e.g. [BCh, §6.9.2]). Such a Hecke character always exists, and as K/F is unramified at all finite places, we can even assume (and we will) that it is unramified at the finite places of K which are either above p or not split above F. Let us choose another Hecke character

$$\chi : K^*\backslash\mathbb{A}_K^* \to \mathbb{C}^*$$

such that $\chi(c(z))^{-1} = \chi(z)$ but assume now that χ descends to $U(1)$, i.e. that for each real place $s \in \Sigma_v$, $\chi_s(z) = \sigma_s(z/c(z))^{-a_\sigma}$ for some $a_s \in \mathbb{Z}$. Assume also that χ is unramified at the finite places of K which do not split over F. Under hypotheses (H1) and (H2), and if all the a_s are big enough, by [ClHL, Thm 4.7] we can transfer Π to an automorphic representation π of U(m) in such a way that at every place w of K splits over a place v in F, we have

$$L(\pi_v) = L(\Pi_w\mu_w) \oplus L(\chi_v). \tag{5}$$

Moreover, for each real place v of F and each complex place w of K above v, the infinitesimal character of π_v is obtained from the one of $\Pi_w\mu_w$ in the obvious way : in terms of the associated Harish-Chandra's cocharacter, it is the direct sum of the one of $\Pi_w\mu_w$ and the one of χ_w.

In both cases (n even or odd), the aforementioned authors actually construct a π which is moreover unramified at all the finite places of K which are not split over F (we don't really need this, but this fixes ideas).

3.1.4. *Consequences of (H3).* When $n = m$ is odd, we set $\rho_\pi := \rho_{\Pi,\lambda}$. When n is even, the G_K representation of dimension m attached to π is by definition

$$\rho_\pi := \rho_{\Pi,\lambda}(\mu|.|^{-\frac{1}{2}}) \oplus \chi|.|^{(1-m)/2}.$$

Note that $\mu|.|^{-\frac{1}{2}}$ and $\chi|.|^{(1-m)/2}$ are both algebraic Hecke characters of K. We shall identify them here with their p-adic realization given by ι_∞ and ι_p. By assumption, $\mu|.|^{-\frac{1}{2}}$ is actually unramified at the place w, and $\chi|.|^{(1-m)/2}$ is crystalline, and we shall denote by φ_μ and $\varphi_\chi \in \overline{\mathbb{Q}}_p{}^*$ their associated Frobenius eigenvalue.

If n is even, so $m = n+1$, we set for each $\sigma \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$,

$$k_{m,\sigma} := \frac{m-1}{2} + a_\sigma$$

(where σ is viewed as an element of $\mathrm{Hom}(F, \mathbb{R})$ as above). Thus, in any case, the $k_{i,\sigma}$ for $i = 1, \ldots, m$ and $\sigma \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ are the Hodge-Tate weights of $\rho_\pi|G_w$. We shall use for this extended collection $\{k_{i,\sigma}\}$ with all $i \in \{1, \ldots, m\}$, and for a subset I of $\{1, \ldots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, the notation k_I analogous to the one in §1.6.

If n is even, we set $\varphi'_i := \varphi_i\varphi_\mu$ for $i < m$ and $\varphi'_m := \varphi_\chi$. We have $\mathbf{v}(\varphi'_i) = \mathbf{v}(\varphi_i)$ for $i < m$ and $\mathbf{v}(\varphi'_m) = k_m$. When $n = m$ is odd, we shall simply set $\varphi'_i := \varphi_i$. Thus, in both cases, $\varphi'_1, \ldots, \varphi'_m$ are the Frobenius eigenvalues of $L(\pi_w|.|^{(1-m)/2})$.

If n is even, we precise now our choice of χ. We assume that $k_{\sigma,m} = a_\sigma + \frac{d(m-1)}{2}$ are all big with respect to the k_i and $\mathbf{v}(\varphi'_i)$ for $i = 1, \dots, n$, and also that they are set sufficiently far apart so that any non trivial sum of the form $\sum_{\sigma \in S} \pm k_{\sigma,m}$, where $S \subset \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, is big in the same sense as above. This is of course always possible. With those assumptions:

Lemma 3.2. (i) *The representation π_v is a fully induced unramified principal series, and the eigenvalues of $L(\pi_w|.|^{(1-m)/2})$ are $\varphi'_1, \dots, \varphi'_m$.*

(ii) *We have $c = \max_{i \in \{1,\dots,m\}} \min_{j \in \{1,\dots,m\}} |\mathbf{v}(\varphi'_i) - k_j|$, and for all distinct subsets I and J of $\{1, \dots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ and with $|I| = |J| < md$, we have $|k_I - k_J| > m \cdot c$.*

Proof — By (H3) and, if n is even, by (5), π_v is unramified. Moreover, the eigenvalues of $L(\pi_w|.|^{(1-m)/2})$ are $\varphi'_1, \dots, \varphi'_m$, and no quotient of those eigenvalues is equal to q, the cardinal of the residue field of K_w. Indeed, a well-known result of Jacquet-Shalika asserts that for $i = 1 \dots n$, the complex numbers $q^{(1-m)/2}\iota_\infty \iota_p^{-1}(\varphi'_i)$ are $< q^{1/2}$ in absolute value, and $q^{(1-m)/2}\iota_\infty \iota_p^{-1}(\varphi'_m)$ has norm 1 by construction. Hence π_v is a full unramified principal series by Zelevinski's theorem, which is (i).

For (ii), there is nothing to prove if $n = m$ is odd. Assume n even so $m = n+1$. Let us note that for $i = m$, we have $\min_{j \in \{1,\dots,m\}} |\mathbf{v}(\varphi'_i) - k_j| = 0$ since $\mathbf{v}(\varphi'_m) = k_m$. For $i \leq n$, the minimum $\min_{j \in \{1,\dots,m\}} |\mathbf{v}(\varphi'_i) - k_j| = 0$ is not realized for $j = m$ because k_m is much too big.

Hence

$$\begin{aligned} \max_{i \in \{1,\dots,m\}} \min_{j \in \{1,\dots,m\}} |\mathbf{v}(\varphi'_i) - k_j| &= \max_{i \in \{1,\dots,n\}} \min_{j \in \{1,\dots,n\}} |\mathbf{v}(\varphi'_i) - k_j| \\ &= c. \end{aligned}$$

It remains to prove that $|k_I - k_J| > mc = (n+1)c$. Let I_0 (resp. J_0) be the subset of I (resp. of J) of pairs (i, σ) with $i = m$. If $I_0 = J_0$, then $k_I - k_J = k_{I-I_0} - k_{J-J_0}$ and since $I - I_0$, $J - J_0$, are distinct subsets of same cardinality of $\{1, \dots, n\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, the desired inequality comes directly from (H3). If $I_0 \neq J_0$, $k_I - k_J$ contains, in addition of a bounded numbers of terms $\pm k_{i,\sigma}$ for $i \leq n$, a non trivial sum of the form $\sum_S \pm k_{\sigma,m}$, where $S \subset \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, hence $|k_I - k_J|$ is again greater than mc. □

3.1.5. *Eigenvarieties and their families of Galois representations.* We are ready now to start the deformation argument. Let $U = \prod_v U_v \subset \mathrm{U}(n)(\mathbb{A}_{F,f})$ be a compact open subgroup such that $\pi^U \neq 0$, and assume that U_v is a Iwahori subgroup for the place v of (H3), and that U_v is hyperspecial for all places of K that are not split over F.

From now on, we shall reserve the notation v for the place of F of hypothesis (H3), and w for one of the place of K above v. We shall denote by d the degree of the field $F_v = K_w$ over $\mathbb{Q}_p$. To U, the place v and (ι_p, ι_∞), we can attach by [Ch2, Thm. 1.6] (see also [Ch1]) an eigenvariety $X = X_{U,v,(\iota_\infty,\iota_p)}$ for the group $\mathrm{U}(m)/F$ which is a reduced rigid analytic space over $\mathbb{Q}_p$ of equidimension[8] md.

By Labesse's base change theorem [L, Cor. 5.3], if π' is any automorphic representation of $\mathrm{U}(m)$ which is unramified outside the split finite places of K/F, then π' admits a base change to GL_m/K (which is strong at each finite place and cohomological at each archimedean place with the expected infinitesimal character), hence a Galois representation by Theorem 1.1. As explained in [BCh, Chap. 7.5] (or in [Ch1]), this is enough to equip X with a continuous m-dimensional pseudocharacter $T : G_K \to \mathcal{O}(X)$ of dimension m. The eigenvariety X and this T satisfy a number of properties and we will only list below the ones we shall need. If $x \in X(\bar{\mathbb{Q}}_p)$ we note T_x the evaluation of T at x and ρ_x the semi-simple representation $G_K \to \mathrm{GL}_m(\bar{\mathbb{Q}}_p)$ of trace T_x. There is :

(i) Zariski-dense and accumulation subsets $Z^{\mathrm{reg}} \subset Z \subset X(\overline{\mathbb{Q}}_p)$ of *classical points*,
(ii) a set of dm analytic functions[9] $\kappa_{1,\sigma}, \dots, \kappa_{m,\sigma}$ where σ runs over the embeddings $K_w \to \overline{\mathbb{Q}}_p$,
(iii) a set of locally constant functions $s_1, \dots, s_m : X(\bar{\mathbb{Q}}_p) \to \mathbb{Q}$,

satisfying the following conditions :

(a) if $z \in Z$, $\rho_z|G_w$ is crystalline.
(b) if $z \in Z$, the ordered Hodge-Tate weights of $\rho_z|G_w$ are $\{\kappa_{i,\sigma}\}$
(c) let C be any real number and $Z_C := \{z \in Z^{\mathrm{reg}}, |\kappa_I(z) - \kappa_J(z)| > C$ for all distinct subsets I, J of $\{1, \dots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ such that $|I| = |J| < md\}$. Then Z_C is Zariski dense and accumulation in X.

Moreover, the classical points z in Z correspond to pairs $(\pi(z), \mathcal{R}(z))$ where $\pi(z)$ is an automorphic representation of $\mathrm{U}(m)$ such that $\pi(z)^U \neq 0$ and $\mathcal{R}(z) = (\varphi_1, \dots, \varphi_m)$ is an accessible refinement[10] of $\pi(z)_w|.|^{(1-m)/2}$, in the following sense : ρ_z is the Galois representation attached to the base change of $\pi(z)$ to GL_m/K by Theorem 1.1 and for each $i = 1, \dots, m$, $\mathbf{v}(\varphi_i) = s_i(z) + \kappa_i(z)$.

(d) If $z \in Z$ parameterizes $(\pi(z), \mathcal{R}(z) = (\varphi_1, \dots, \varphi_m))$, then for all i we have $\mathbf{v}(\varphi_i) = s_i(z) + \kappa_i(z)$.

The subset $Z^{\mathrm{reg}} \subset Z$ parameterizes refined automorphic representations $(\pi, \mathcal{R})$ satisfying some additional properties, and for our concerns here we shall simply assume that they are those $(\pi, \mathcal{R})$ such that π_v unramified and such that for each real place s inducing v via ι_p and ι_∞, the infinitesimal character of π_s is sufficiently

[8] It is not necessary here to let the weights corresponding to the other possible places of F above p move, but we could have, and the eigenvariety would then have dimension $n[K : \mathbb{Q}]$.

[9] Again, we shall use for this collection $\{\kappa_{i,\sigma}\}$, and for a subset I of $\{1, \dots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ (resp. an $i \in \{1, \dots, m\}$), the notation κ_I (resp. κ_i) analogous to the one in §1.6.

[10] Recall that an *refinement* of an irreducible smooth representation ρ of $\mathrm{GL}_m(K_w)$ such that $\rho^I \neq 0$ for I an Iwahori subgroup is an ordering $(\varphi_1, \dots, \varphi_m)$ of the eigenvalues of $L(\rho)(\mathrm{Frob}_w)$. It is said *accessible* if ρ appears as a *sub-representation* of the induced representation $\mathrm{Ind}_B^{\mathrm{GL}_m(K_w)} \chi \delta_B^{1/2}$ where B is (say) the upper Borel subgroup, δ_B is modulus character, and χ the (unramified) character of the diagonal torus sending $(x_1, \cdots, x_m)$ to $\prod_{i=1}^m \varphi_i^{\mathbf{v}(x_i)}$ (see [BCh, §6.4.4]).

far from the walls. Under this latter condition, the base change of an automorphic representation of $\mathrm{U}(m)$ is not necessarily cuspidal, but always associated to a decomposition $m_1 + \cdots + m_r = m$ and a r-tuple $(\pi_1, \ldots, \pi_r)$ of cuspidal (polarized, cohomological) automorphic representations π_i of $\mathrm{GL}_{m_i}(\mathbb{A}_K)$; moreover each π_i satisfies property (H2) in dimension m_i and is unramified at v. In particular, for a $z \in Z^{\mathrm{reg}}$, the characteristic polynomial of the crystalline Frobenius of $\rho_z|G_w$ coincides with the polynomial $P_w(T)$ associated to $\pi_w|.|^{(1-m)/2}$ by the refinement of Theorem 1.1 recalled in §1.6, and we also have the following :

(d') If $z \in Z^{\mathrm{reg}}$, then the m slopes of the crystalline Frobenius of $\rho_z|G_w$ are the $s_i(z) + \kappa_i(z)$, fo $i = 1, \ldots, m$.

3.1.6. *Choice of a refinement.* Going back to the representation π introduced above, if we choose an accessible refinement $\mathcal{R}$ of π_v, there is a point $z_0 \in Z$ corresponding to $(\pi, \mathcal{R})$.

Lemma 3.3. *There exists a refinement $\mathcal{R}$ of π_v such that the pseudocharacter T is generically irreducible in a neighborhood of the corresponding point z_0.*

Proof — We shall eventually show that the conclusion holds for $T|G_w$. Note that by construction, for all $\sigma \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ and $i \in \{1, \ldots, m\}$, $\kappa_{i,\sigma}(z_0) = k_{i,\sigma}$. Let us first renumber the $\varphi'_i \in \overline{\mathbb{Q}}_p^{\,*}$ so that $|\mathbf{v}(\varphi'_i) - k_i| = \min_j |\mathbf{v}(\varphi'_i) - k_j|$. By Lemma 3.2 (ii) there is one and only one way to do so, and this being done we have $\mathbf{v}(\varphi'_1) < \mathbf{v}(\varphi'_2) < \cdots < \mathbf{v}(\varphi'_m)$ (strict inequalities). Then consider a transitive permutation σ of $\{1, \ldots, m\}$. We choose the refinement

$$\mathcal{R} = (\varphi'_{\sigma(1)}, \ldots, \varphi'_{\sigma(m)}).$$

Since π_v is a full unramified principal series by Lemma 3.2 (i), all the refinements of π_v are accessible, so π together with $\mathcal{R}$ defines a point z_0.

Before proving the irreducibility property of the lemma, let us observe a combinatorial property of this refinement. We have by definition $\kappa_i(z_0) = k_i$ and $s_i(z_0) = \mathbf{v}(\varphi'_{\sigma(i)}) - k_i$. We claim that for any non-empty proper subset $I \subset \{1, \ldots, m\}$,

$$\sum_{i \in I} s_i(z_0) \neq 0. \tag{6}$$

Indeed, we compute

$$\begin{aligned} |\sum_{i\in I} s_i(z_0)| &= |\sum_{j \in J} \mathbf{v}(\varphi'_j) - \sum_{i \in I} k_i| \text{ where } J = \sigma(I) \\ &= |(\sum_{j\in J} k_j - \sum_{i \in I} k_i) + \sum_{j \in J}(\mathbf{v}(\varphi'_j) - k_j)| \\ &\geq |(\sum_{j\in J} k_j - \sum_{i \in I} k_i)| - \sum_{j \in J}|\mathbf{v}(\varphi'_j) - k_j| \\ &> mc - mc \text{ by Lemma 3.2(ii) as } I \neq J \\ &= 0. \end{aligned}$$

Let us choose now some affinoid neighborhood Ω of $z_0 \in X$ on which the s_i are constant and in which Z_C is Zariski-dense for $C = \sum_{i=1}^m |s_i(z_0)|$. We claim that for every point z of $Z_C \cap \Omega$, $\rho_z|G_w$ is irreducible. Indeed if it was not, it would have a sub-representation of dimension $0 < r < m$, and by the weak admissibility of $D_{\mathrm{crys}}(\rho_z|G_w)$ there would exist a subset $I \subset \{1, \dots, m\}$ of cardinal r, and a subset $J \subset \{1, \dots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ with $|J| = rd$, such that

$$\sum_{i \in I} (\kappa_i(z) + s_i(z)) = \kappa_J(z).$$

(here we use that $z \in Z^{\mathrm{reg}}$ and property (d') of eigenvarieties.) Since $z \in Z_C$, we see at once that $I \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p) = J$. But this implies

$$0 = \sum_{i \in I} s_i(z),$$

a contradiction with (6) as $s_i(z) = s_i(z_0)$ for all i. □

3.1.7. *End of the proof.* Let $\Omega \subset X$ be the neighborghood defined above of the point z_0, and let A be a complete discrete valuation ring, with a map of $\mathrm{Spec}\, A$ to the spectrum of the rigid local ring $\mathcal{O}_{z_0}$ of X at z_0 which sends the special point of $\mathrm{Spec}\, A$ to z_0 and the generic point to the generic point of any irreducible component of Ω containing z_0. Let us call L the fraction field of A and m its maximal ideal. By pulling back the pseudocharacter T over A, we get a representation $\rho : G_K \to \mathrm{GL}_m(A)$ such that $\rho \otimes L$ is absolutely irreducible and satisfies (1) (for $\chi = \mathbb{Q}_p(m-1)$) and

$$\bar{\rho}^{\mathrm{ss}} = \begin{cases} \rho_{\Pi,\lambda} & \text{if} \quad m = n, \\ \rho_{\Pi,\lambda}(\mu|.|^{-1/2}) \oplus (\chi|.|^{(1-m)/2}) & \text{if} \quad m = n+1. \end{cases}$$

Since $\rho \otimes L$ is absolutely irreducible and satisfies (1) it has a sign that can only be $+1$. Hence it is good, and so is $\bar{\rho}^{\mathrm{ss}}$ by Prop. 2.4, with respect to ω^{m-1}. Hence $\rho_{\Pi,\lambda}$ is good with respect to ω^{n-1} by Lemmas 2.2 and 2.1, as $\psi := \mu|.|^{-1/2}$ satisfies $\psi^{\perp} = \psi\omega$, QED.

3.2. **Weakening of the hypothesis (H3), removal of (H2).** We consider the following variant of (H3)

(H4) Each place of F above p splits in K, and if w is such a place then Π_w has a non-zero vector invariant by a Iwahori subgroup of $\mathrm{GL}_n(K_w)$.

Theorem 3.4. *With the supplementary hypotheses (H1) and (H4), Theorem 1.2 holds.*

We shall argue by induction on $n \geq 1$, there is nothing to show if $n = 1$.

Let $\mathrm{U}(n)$ be the n-variables unitary group over F attached to K/F that is quasi-split at every finite place and compact at every infinite place. Hasse's principle shows that this group exists, even if n is even, by condition (H1c) (see e.g. [ChH, Lemma 3.1]). Moreover, condition (H1) and Labesse's base change theorem also ensures that Π descends to an automorphic representation π for $\mathrm{U}(n)$. Again, π is unramified at non split finite places of K/F and for each complex place w of K above a real place v of F, π_v has the same infinitesimal character as Π_w (under the natural identification $\mathrm{U}(n)(K_w) \simeq \mathrm{GL}_n(\mathbb{C})$).

Let $U = \prod_v U_v \subset \mathrm{U}(n)(\mathbb{A}_{F,f})$ be a compact open subgroup such that $\pi^U \neq 0$, and assume that $U_v = I_v$ for the each place v above p, and that U_v is hyperspecial for each place v of K that is not split over F. Let X be the eigenvariety attached to U, to *all the finite places of* F *above* p (in the setting of [Ch2, §1.1], S_p is the set of all the places above p), and ι_∞, ι_p. Now X has equidimension $n[F:\mathbb{Q}]$ but all what we said for the eigenvarieties of $\mathrm{U}(m)$ in § 3.1.5 also applies to this X with minor changes, the only difference being that there is no prefered place v above p. Precisely, let us fix once and for all a place v of F above p, as well as a place w of K above v. Then (i), (ii), (iii), (a), (b) and (c) hold (with m replaced by n). To be perfectly correct for (i) and (a), we have to precise that Z (resp. Z^{reg}) parameterizes now the refined automorphic representations $(\pi,\mathcal{R})$ of $\mathrm{U}(n)$ such that $\pi^U \neq 0$ (resp. such that π_v is un ramified and such that the infinitesimal character of π_s is sufficiently regular *for all the real places* s *of* F).

Let $z_0 \in Z$ be the point corresponding to π together with some accessible refinement of π_x for each place x of F above p. As a general fact, there is always such a refinement (for each x) and we choose them anyhow here.

Let c be the maximum of the $|s_i(z_0)|$ and $C > nc$. Let $\Omega \subset X$ be an open affinoid of X containing z_0, in which Z_C is Zariski-dense, and over which the s_i are constant. We claim that for $z \in Z_C$, ρ_z is good. Indeed, let $\Pi(z)$ be Labesse's base change of $\pi(z)$ to $\mathrm{GL}_n(\mathbb{A}_K)$. As $z \in Z^{\mathrm{reg}}$, and as explained in §3.1.5, there exists a decomposition $n = n_1 + \cdots + n_r$ and cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$, satisfying (H1b), (H2) and unramified at v, such that

$$\rho_z = \bigoplus_{i=1}^{r} \rho_{\Pi_i,\lambda} \otimes \chi_i,$$

for some characters $\chi_i : G_K \to \overline{\mathbb{Q}}_p^{\,*}$ such that $\chi_i^\perp = \chi_i \omega^{n-n_i}$. If $r > 1$, then ρ_z is good by induction and Lemma 2.1. If $r = 1$, then $\Pi(z)$ is cuspidal and it satisfies (H2) and (H3) by construction, so ρ_z is good by Theorem 3.1. (To check (H3), remark that for such a z, and for each $i \in \{1,\dots,n\}$, we have $\mathrm{Min}_j(s_i(z) + k_i(z) - k_j(z)) = s_i(z) = s_i(z_0)$.)

Let W be any irreducible component of Ω containing z_0, and $\mathrm{Frac}(W)$ its associated function field. As Z_C is Zariski-dense in Ω we may find a $z \in Z_C \cap W$ such that the pseudocharacters T_z and $T \otimes_{\mathcal{O}(\Omega)} \mathrm{Frac}(W)$ have the same number of irreducible factors. Such factors are necessarily absolutely irreducible here, by [BCh, Thm. 1.4.4 (iii)]. Arguing as in the preceding section, let A be a complete discrete valuation ring with a map of $\mathrm{Spec}\, A$ to $\mathrm{Spec}\, \mathcal{O}_z$ which sends the special point of $\mathrm{Spec}\, A$ to z and its generic point to $\mathrm{Frac}(W)$, and let $\rho : G_K \to \mathrm{GL}_n(A)$ be a representation with trace T such that $\rho \otimes_A L$ is a direct sum of absolutely irreducible, distinct, representations (use e.g. [BCh, Prop. 1.6.1]). As we saw, $\bar{\rho} = \rho_z$ is good, hence so is $\rho \otimes_A L$ by Prop.2.4, as well as $\rho \otimes_A \mathrm{Frac}(W)$ for any irreducible component W containing z_0. But arguing back now at the point z_0 as in the preceding section, we obtain that $\rho_{z_0} = \rho_{\Pi,\lambda}$ itself is good as a specialization of a good representation, and we are done.

3.3. **Removal of Hypotheses (H1) and (H4).** We now prove Theorem 1.2.

Lemma 3.5. *Let L be a finite extension of $\mathbb{Q}_p$ and $\rho : G_K \longrightarrow GL_n(L)$ a continuous representation which is a direct sum of absolutely irreducible representations. There is a finite Galois extension M/K such that for every finite extension K'/K linearly disjoint from M, ρ and $\rho|G_{K'}$ have the same number of irreducible factors, and the irreducible factors of $\rho|G_{K'}$ are absolutely irreducible.*

Proof — We can assume without loss of generality that ρ is absolutely irreducible. In particular, there exists n^2 elements $g_1, \dots, g_{n^2}$ such that the $\rho(g_i)$'s generate $M_n(L)$ as a L-vector space. Since G_K has a basis of neighbourhoods of 1 that are open normal subgroups, and since ρ and the determinant are continuous, there is an open normal subgroup U of G_K such that if for all $i = 1, \dots, n^2$, $g'_i \in g_i U$, then the $\rho(g'_i)$'s still generate $M_n(L)$. Set $M = \bar{K}^U$, so M is a finite Galois extension of K.

If K' is a finite extension of K which is linearly disjoint from M, so is its Galois closure. Hence we may assume that K' is Galois over K. Thus $\mathrm{Gal}(K'M/K')$ is naturally isomorphic to $\mathrm{Gal}(K/M)$. For every i, choose g'_i in $G_{K'}$ whose image in $\mathrm{Gal}(K'M/K')$ is sent to g_i by the above isomorphism. This implies that $g'_i g_i^{-1} \in U$, hence the $\rho|G_{K'}(g'_i)$'s generate $M_n(L)$, and $\rho|G_{K'}$ is absolutely irreducible. □

By [ChH, Prop. 4.1.1 and Thm 4.2.2] for any finite extension M/K there exists a totally real solvable Galois extension F'/F such that $K' = KF'$ is linearly disjoint to M and such that Arthur-Clozel's base change $\Pi_{K'}$ and K'/F' satisfy hypotheses (H1) and (H4). We apply this to an M as in the lemma above. By Theorem 3.4 we know that $(\rho_{\Pi,\lambda})_{|G'_K}$ is good, and by the lemma above, that it has the same number of irreducible components as $\rho_{\Pi,\lambda}$.

Hence by Lemma 2.3 $\rho_{\Pi,\lambda}$ is good, QED.

References

[BCh] J. Bellaïche & G. Chenevier, *Families of Galois representations and Selmer groups*, Soc. Math. France, Astérisque 324 (2009).

[BlRo1] D. Blasius et J.Rogawski *Tate class and arithmetic quotient of two-ball in* The Zeta Function of Picard Modular Surfaces, CRM, pages 421–443.

[Bu] K. Buzzard, *Eigenvarieties*, Proc. of the LMS Durham Conference on L-functions and arithmetic (2007).

[Ch2] G. Chenevier, *Une application des variétés de Hecke des groupes unitaires*, in [GRFAbook, II].

[Ch1] G. Chenevier, *Familles p-adiques de formes automorphes pour GL(n)*, Journal für die reine und angewandte Mathematik 570, 143–217 (2004).

[ChCl] G. Chenevier & L. Clozel, *Corps de nombres peu ramifiés et formes automorphes autoduales*, Journal of the A.M.S. 22 Vol 2, 467–519 (2009).

[ChH] G. Chenevier & M. Harris, *Construction of automorphic Galois representations*, book 2 of [GRFAbook].

[ChR] G. Chenevier & D. Renard, *On the vanishing of some non semisimple orbital integrals*, preprint, available on `http://www.math.polytechnique.fr/~chenevier/pub.html`.

[ClHL] L. Clozel, M. Harris & J.-P. Labesse, *Endoscopic transfer*, in [GRFAbook, Ch. I.5.2].

[ClHT] L. Clozel, M. Harris & R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l representations*, Pub. Math. IHES 108, 1–181 (2008).

[E] M. Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. 164 no.1, 1–84 (2006).

[GRFAbook] The two volumes of the book project of the GRFA seminar of Paris 7 university, http://fa.institut.math.jussieu.fr/node/29.

[Gr] B. Gross, *Odd Galois representations*, preprint, available on www.math.harvard.edu/˜gross

[HT] M. Harris & R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, PUP Annals of Math. Studies. 151 (2001).

[L] J.-P. Labesse, *Changement de base CM et séries discrètes*, in [GRFAbook, Ch. I.5.1].

[Sh] S.-W. Shin, *Galois representations arising from some compact Shimura varieties*, preprint available on http://www.math.ias.edu/~swshin/.

[T] J. G. Thompson, *Some finite groups which appear as* $\mathrm{Gal}(L/K)$ *where* $K \subset \mathbb{Q}(\mu_n)$, in Group Theory, Beijing 1984, LNM 1185, Springer.

E-mail address: jbellaic@brandeis.edu

Joël Bellaïche, Brandeis University, 415 South Street, Waltham, MA 02454-9110, U.S.A

E-mail address: chenevier@math.polytechnique.fr

Gaëtan Chenevier, C.N.R.S, Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex, France