# On restrictions and extensions of cusp forms

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The results of this note[1] are probably well known to specialists, however we have not been able to find their proof in the literature. Some of the arguments below already appear in some form for instance in [LL79, p. 49-52], [HS12, Ch. 4] or [Xu15b, Lemma 5.3]. We are grateful to Olivier Taïbi for pointing out a mistake in a preliminary version of this note.

Let $F$ be a number field. We denote by $\mathcal{O}_F$ its ring of integers, $\mathbb{A}_F$ its adèle ring, and for each place $v$ of $F$, we also denote by $F_v$ the associated completion of $F$ and $\mathcal{O}_v \subset F_v$ its ring of integers. Let $H$ be a reductive linear algebraic group defined over $F$. We set $H_\infty = H(F \otimes_{\mathbb{Q}} \mathbb{R})$, the direct product of the Lie groups $H(F_v)$ for $v$ Archimedean, and denote by $H_f$ the locally profinite group $H(\mathbb{A}_F^f)$ where $\mathbb{A}_F^f$ is the $F$-algebra of finite adèles of $F$: we have $H(\mathbb{A}_F) = H_\infty \times H_f$. Our main reference for automorphic forms will be Borel-Jacquet's paper [BJ79]. If $c$ is a character of the center of $H(\mathbb{A}_F)$, we denote by $\mathcal{A}_{\text{cusp}}(H)_c$ the space of cuspidal automorphic functions $H(\mathbb{A}_F) \to \mathbb{C}$ with central character $c$ and with respect to a choice of maximal compact subgroup $K_\infty$ of $H_\infty$ (in the sense of [BJ79, §4.4]). This is an admissible, semi-simple, $(\operatorname{Lie} H_\infty, K_\infty) \times H_f$-module. For short, a $(\operatorname{Lie} H_\infty, K_\infty) \times H_f$-module will be called an $H(\mathbb{A}_F)$-gk-*module*.

Recall that for any integer $m \geq 1$ such that $(F, m)$ is not *special* in the sense of the statement of the Grunwald-Wang theorem (see [AT, Ch. X]), an element of $F$ is an $m$-th power in $F$ if, and only if, it is an $m$-th power in $F_v$ for each place $v$ of $F$. For instance, if $m \not\equiv 0 \bmod 8$, or if $F = \mathbb{Q}$, then $(F, m)$ is not special.

**Proposition 1.** *Let $\widetilde{G}$ be a reductive linear algebraic group, $T$ a torus, $\nu : \widetilde{G} \to T$ a surjective morphism, $\widetilde{Z}$ the center of $\widetilde{G}$, $G$ the kernel of $\nu$ and $Z$ the center of $G$. We assume that $\widetilde{G}$, $T$ and $\nu$ (hence $\widetilde{Z}$, $G$ and $Z$) are defined over the number field $F$, and that $G$ is connected. Let $\widetilde{c}$ be an automorphic character of $\widetilde{Z}(\mathbb{A}_F)$ and $c$ its restriction to $Z(\mathbb{A}_F)$. Then the following assertions hold:*

(i) *The restriction map $\rho : \varphi \mapsto \varphi_{|G(\mathbb{A}_F)}$ induces an equivariant morphism $\rho_{\text{cusp}} : \mathcal{A}_{\text{cusp}}(\widetilde{G})_{\widetilde{c}} \longrightarrow \mathcal{A}_{\text{cusp}}(G)_c$ of $G(\mathbb{A}_F)$-gk-modules.*

(ii) *If $V$ is a non-zero $\widetilde{G}(\mathbb{A}_F)$-gk-submodule of $\mathcal{A}_{\text{cusp}}(\widetilde{G})_{\widetilde{c}}$ then we have $\rho_{\text{cusp}}(V) \neq 0$.*

*Let $Z^0$ be the neutral component of $Z$ and let $m \geq 1$ denote the exponent of the kernel of the isogeny $\widetilde{Z}/Z^0 \to T$ induced by $\nu$.*

(iii) *Assume $\widetilde{Z}$ is a split $F$-torus and $(F, m)$ is not special. Then the map $\rho_{\text{cusp}}$ defined in* (i) *is surjective.*

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*Proof.* As $\nu$ is trivial on the derived subgroup of $\widetilde{G}$, we have $\widetilde{G} = \widetilde{Z}G$ and $\nu(\widetilde{Z}) = T$. We may thus find a torus $C \subset \widetilde{Z}$ defined over $F$ such that $\nu : C \to T$ is an isogeny. We shall denote by

$$\iota : C \times G \to \widetilde{G}$$

the isogeny $(z, g) \mapsto zg$. For each place $v$ of $F$, note that $G(F_v)$ is a normal subgroup of $\widetilde{G}(F_v)$ with abelian quotient. Moreover, $\nu(C(F_v))$ is an open subgroup of finite index in $T(F_v)$, by the finiteness of the Galois cohomology group $H^1(F_v, C \cap G)$ due to Tate. In particular, the image of the morphism $\iota_v : C(F_v) \times G(F_v) \to \widetilde{G}(F_v)$ induced by $\iota$ is an open subgroup of finite index in $\widetilde{G}(F_v)$. For each Archimedean place $v$, we choose a maximal compact subgroup $K_v$ of $G(F_v)$, as well as a maximal compact subgroup $\widetilde{K}_v$ of $\widetilde{G}(F_v)$ containing $K_v$. We have thus $K_v = G(F_v) \cap \widetilde{K}_v$. We define $K_\infty$ (resp. $\widetilde{K}_\infty$) as the product of the $K_v$ (resp. $\widetilde{K}_v$) for $v$ Archimedean. We may and do choose these maximal compact subgroups in order to define $\mathcal{A}_{\mathrm{cusp}}(\widetilde{G})_{\widetilde{c}}$ and $\mathcal{A}_{\mathrm{cusp}}(G)_c$. If we denote by $\mathfrak{z}(H)$ the $\mathbb{C}$-algebra defined as the center of the complex enveloping algebra $\mathrm{U}(\mathrm{Lie}\, H)$ of the Lie group $H$, then for each Archimedean $v$ the local isomorphism $\iota_v$ defines a $\mathbb{C}$-algebra isomorphism $\mathfrak{z}(C(F_v)) \otimes \mathfrak{z}(G(F_v)) \xrightarrow{\sim} \mathfrak{z}(\widetilde{G}(F_v))$. In particular, the natural inclusion $\mathrm{U}(\mathrm{Lie}\, G(F_v)) \subset \mathrm{U}(\mathrm{Lie}\, \widetilde{G}(F_v))$ induces an injection $\mathfrak{z}(G(F_v)) \subset \mathfrak{z}(\widetilde{G}(F_v))$.

We now prove assertion (i). Let $\varphi \in \mathcal{A}_{\mathrm{cusp}}(\widetilde{G})_{\widetilde{c}}$. For any $g_f \in \widetilde{G}_f$ the function $g_\infty \mapsto \varphi(g_\infty \times g_f)$, $\widetilde{G}_\infty \to \mathbb{C}$, is smooth, $\widetilde{K}_\infty$-finite on the right, $\mathfrak{z}(\widetilde{G}_\infty)$-finite, and slowly increasing (see [BJ79, §1.2]). Moreover, there is a compact open subgroup $\widetilde{K}_f \subset \widetilde{G}_f$ such that $\varphi$ is $\widetilde{K}_f$-invariant on the right. Using a model over an open subset of $\mathrm{Spec}\, \mathcal{O}_F$ of the closed morphism $G \subset \widetilde{G}$, we see that $K_f = \widetilde{K}_f \cap G_f$ is a compact open subgroup of $G_f$. By the inclusions $K_f \subset \widetilde{K}_f$, $K_\infty \subset \widetilde{K}_\infty$, and $\mathfrak{z}(G_\infty) \subset \mathfrak{z}(\widetilde{G}_\infty)$ (justified above), it follows that $\rho(\varphi)$ is $K_f$-invariant and $K_\infty$-finite on the right, as well as $\mathfrak{z}(G_\infty)$-finite. It is also trivially slowly increasing (note *e.g.* that the restriction to $G_\infty$ of a *norm* on $\widetilde{G}_\infty$ is still a norm, in the sense of [BJ79, §1.2]). Of course, $\rho(\varphi)$ is $G(F)$-invariant on the left and has central character $c$. We have proved that $\rho(\varphi)$ is an automorphic form. It only remains to check that it is cuspidal. But for any parabolic subgroup $P$ of $G$ defined over $F$, there is a unique parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ such that $\widetilde{P} \cap G = P$, namely $\widetilde{P} = \widetilde{Z}\, P$. We conclude as the unipotent radical of $\widetilde{P}$ and $P$ coincide (in particular, they are included in $G$), and as $f$ is cuspidal. This shows that $\rho_{\mathrm{cusp}}$ is well defined. By construction, it is $(\mathrm{Lie}\, G_\infty, K_\infty) \times G_f$-equivariant.

We now prove assertion (ii). Let $V$ be as in the statement and $\varphi \in V - \{0\}$. Choose $g = g_\infty g_f \in \widetilde{G}(\mathbb{A}_F)$ such that $\varphi(g) \neq 0$. Up to replacing $\varphi$ by $g_f^{-1} \cdot \varphi \in V$ we may assume $g_f = 1$. As the image $\iota_v$ is an open subgroup of $\widetilde{G}(F_v)$ for $v$ Archimedean, and as $\widetilde{K}_\infty$ meets every connected component of $\widetilde{G}_\infty$ by Cartan's decomposition, we have:

$$G_\infty \cdot \widetilde{Z}_\infty \cdot \widetilde{K}_\infty = \widetilde{G}_\infty.$$

It follows that up to replacing $\varphi$ by $h\cdot\varphi$ with $h\in\widetilde{Z}_\infty\cdot\widetilde{K}_\infty$, which preserves $V$, we may assume that we have $g\in G_\infty\subset G(\mathbb{A}_F)$. In particular, we have $\rho(\varphi)\neq 0$.

Let us now prove (the main) assertion (iii). Note that we have $Z=\widetilde{Z}\cap G$ as $\widetilde{G}=\widetilde{Z}\,G$. Consider the subgroups $G(\mathbb{A}_F)\subset H_1\subset H_2\subset\widetilde{G}(\mathbb{A}_F)$ with

$$H_1=\widetilde{Z}(\mathbb{A}_F)G(\mathbb{A}_F)\ \text{ and }\ H_2=\widetilde{G}(F)H_1.$$

They are normal in $\widetilde{G}(\mathbb{A}_F)$, with abelian quotient, as so is $G(\mathbb{A}_F)$. Fix $\varphi$ in $\mathcal{A}_{\rm cusp}(G)_c$. The assumption $\widetilde{c}_{|Z(\mathbb{A}_F)}=c$ and the obvious inclusion $\widetilde{Z}(\mathbb{A}_F)\cap G(\mathbb{A}_F)\subset Z(\mathbb{A}_F)$ show that there is a unique function $\varphi_1:H_1\to\mathbb{C}$ such that $\varphi_1(zg)=\widetilde{c}(z)\varphi(g)$ for all $(z,g)\in\widetilde{Z}(\mathbb{A}_F)\times G(\mathbb{A}_F)$.

We claim that there is a unique function $\varphi_2:H_2\to\mathbb{C}$ such that $\varphi_2(\gamma h)=\varphi_1(h)$ for all $(\gamma,h)$ in $\widetilde{G}(F)\times H_1$. Indeed, such a function exists if, and only if, $\varphi_1$ is left-invariant under $\widetilde{G}(F)\cap H_1$. It is thus enough to show the inclusion $\widetilde{G}(F)\cap H_1\subset\widetilde{Z}(F)G(F)$. For this it suffices to prove

$$T(F)\cap\nu(\widetilde{Z}(\mathbb{A}_F))\subset\nu(\widetilde{Z}(F)). \tag{1}$$

We have not used that $\widetilde{Z}$ is an $F$-split torus so far, but we shall do so now. It implies that $T$ is split over $F$ (being a quotient of $\widetilde{Z}$), and that we may choose the torus $C$ satisfying furthermore

$$\widetilde{Z}=C\times Z^0. \tag{2}$$

We may assume that we have $T=\mathbb{G}_m^r$ for some integer $r\geq 0$, and by the theory of elementary divisors that there is an isomorphism $\mu:\mathbb{G}_m^r\xrightarrow{\sim}C$ over $F$, and integers $m_1,\dots,m_r$, such that we have

$$\nu\circ\mu(z_1,\dots,z_r)=(z_1^{m_1},\dots,z_r^{m_r}). \tag{3}$$

Note that the integer $m$ of the statement is the lcm of the $m_i$. Each $(F,m_i)$ is not special as $(F,m)$ has this property by assumption (and we have $m_i|m$). The Grunwald-Wang theorem shows thus $T(F)\cap\nu(C(\mathbb{A}_F))=\nu(C(F))$. This proves (1) as we have $\nu(\widetilde{Z}(\mathbb{A}_F))=\nu(C(\mathbb{A}_F))$ and $\nu(\widetilde{Z}(F))=\nu(C(F))$ by (2), hence the existence of $\varphi_2$.

We will now have to extend $\varphi_2$ to some open neighborhood of $H_2$ in $\widetilde{G}(\mathbb{A}_F)$. We need first to fix certain integral models of all the data. Let $S$ be a finite set of places of $F$ containing the Archimedean places. We denote by $\mathcal{O}_{F,S}\subset F$ the subring of $S$-integers. Up to enlarging $S$ if necessary, we may assume that $\widetilde{G}$ is an affine group schemes of finite type defined over $\mathcal{O}_{F,S}$, that $C\subset\widetilde{Z}\subset\widetilde{G}$ are closed subgroup schemes defined over $\mathcal{O}_{F,S}$, that $\nu:G\to\mathbb{G}_m$ is a group scheme homomorphism defined over $\mathcal{O}_{F,S}$, and that $\mu$ is a group scheme isomorphism $\mathbb{G}_m^r\xrightarrow{\sim}C$ over $\mathcal{O}_{F,S}$. We define again $G$ as the kernel of $\mu$; this is an affine group scheme of finite type over $\mathcal{O}_{F,S}$. Set

$$\widetilde{K}^S=1\times\prod_{v\notin S}\widetilde{G}(\mathcal{O}_v).$$

Up to enlarging $S$ if necessary, we claim that we may assume that:

(a) $\varphi$ is $G(\mathcal{O}_v)$-invariant on the right, as well as $\widetilde{c}_v(C(\mathcal{O}_v))=1$, for all $v\notin S$.

(b) For each $i = 1, \ldots, r$, if we have $x \in F^\times$ and $u \in \mathbb{A}_F^\times$ such that $u_v = 1$ for $v \in S$, $u_v \in \mathcal{O}_v^\times$ for $v \notin S$, and such that $xu_v$ is an $m_i$-th power in $F_v^\times$ for each $v$, then $x$ is an $m_i$-th power in $F^\times$.

Indeed, by definition of the adelic topology, if we have a collection of open subsets $U_v \subset \widetilde{G}(F_v)$ for each $v \in S$, then $(\prod_{v \in S} U_v) \times (\prod_{v \notin S} \widetilde{G}(\mathcal{O}_v))$ is an open subset of $\widetilde{G}(\mathbb{A}_F)$, and a similar assertion holds with $\widetilde{G}$ replaced by $G$ and $C$. This proves (a). Part (b) is a consequence of Lemma 1 below.

Next, we claim that we have the following properties:

$$(4) \qquad H_2 \cap \widetilde{K}^S = H_1 \cap \widetilde{K}^S \quad \text{and} \quad H_1 \cap \widetilde{K}^S \subset \prod_{v \notin S} C(\mathcal{O}_v) G(\mathcal{O}_v).$$

We first prove the equality on the left. Any element $h$ of $H_2$ has the form $h = \gamma z g$ for $\gamma \in \widetilde{G}(F)$, $z \in C(\mathbb{A}_F)$ and $g \in G(\mathbb{A}_F)$ by (2). If we have furthermore $h \in \widetilde{K}^S$, then the element $\nu(h) = \nu(\gamma)\nu(z)$ of $T(F)\nu(C(\mathbb{A}_F))$ is in $T(\mathcal{O}_v)$ for all $v \notin S$, and equal to 1 for $v \in S$. By the property (b) above and (3) we have thus $\nu(\gamma) \in \nu(C(F))$, so $\gamma$ is in $C(F)G(F) \subset H_1$, and we have $h \in H_1$.

To prove the inclusion on the right in (4), we fix $v \notin S$. By (2), any element $h$ of $\widetilde{Z}(F_v)G(F_v)$ has the form $h = zg$ with $z \in C(F_v)$ and $g \in G(F_v)$. If $h$ is in $\widetilde{G}(\mathcal{O}_v)$ then $\nu(z) = \nu(h)$ is in $\nu(C(F_v)) \cap T(\mathcal{O}_v)$. But by (3) this latter group is $\nu(C(\mathcal{O}_v))$ as an element of $\mathcal{O}_v^\times$ is an $d$-th power in $F_v^\times$ if, and only if, it is a $d$-th power in $\mathcal{O}_v^\times$. This shows $h \in C(\mathcal{O}_v)G(F)$. As we have $C(\mathcal{O}_v) \subset \widetilde{G}(\mathcal{O}_v)$ and $\widetilde{G}(\mathcal{O}_v) \cap G(F_v) = G(\mathcal{O}_v)$, we have proved $h \in C(\mathcal{O}_v)G(\mathcal{O}_v)$.

Consider now the normal subgroup $H_3 = H_2 \widetilde{K}^S$ in $\widetilde{G}(\mathbb{A}_F)$. By the properties (4) and (a) there is a unique function $\varphi_3 : H_3 \to \mathbb{C}$ such that $\varphi_3(hk) = \varphi_2(h)$ for all $h \in H_2$ and $k \in \widetilde{K}^S$. Note that $H_3$ is open in $\widetilde{G}(\mathbb{A}_F)$. Indeed, it contains the image of $\iota_v$ for each $v \in S$, which is open in $\widetilde{G}(F_v)$ by the first paragraph of the proof, as well as $\widetilde{K}^S$. The last step is to define $\psi : \widetilde{G}(\mathbb{A}_F) \to \mathbb{C}$ by $\psi_{|H_3} = \varphi_3$ and $\psi(h) = 0$ for $h \notin H_3$. This function has central character $c$, is $\widetilde{K}^S$-invariant on the right, $\widetilde{G}(F)$-invariant on the left, and satisfies $\rho(\psi) = f$. For $v \in S$, it is also invariant on the right by some compact open subgroup of $C(F_v) \times G(F_v)$, hence of $\widetilde{G}(F_v)$.

We claim that we have $\psi \in \mathcal{A}_{\text{cusp}}(\widetilde{G})_{\widetilde{c}}$. One first observes that the $K_\infty$-finiteness of $\varphi$ implies that of $\psi_{|H_3}$, hence that of $\psi$ (use that $\psi$ vanishes outside $H_3$ and that $K_\infty$ is a subgroup of $H_3$). As the normal subgroup $K_\infty$ of $\widetilde{K}_\infty$ is easily seen to be of finite index, this implies the $\widetilde{K}_\infty$-finiteness of $\psi$. Let us now fix $g \in \widetilde{G}(\mathbb{A}_F)$. Observe that the map $\psi_g : C_\infty \times G_\infty \to \mathbb{C}$, $(z, h) \mapsto \psi(gzh)$, is either identically 0 or a constant multiple of $\psi_{g'}$ with $g' \in G(\mathbb{A}_F)$. For such a $g'$ we have $\psi_{g'}(z, h) = \widetilde{c}_\infty(z)\varphi(g'h)$. As $\iota_v$ is a smooth finite covering for each Archimedean $v$, and both $\varphi$ and $\widetilde{c}_\infty$ are smooth, then $\psi_g$ is smooth. As we have $\mathfrak{z}(C_\infty) \otimes \mathfrak{z}(G_\infty) = \mathfrak{z}(\widetilde{G}_\infty)$, the map $\psi_g$ is also $\mathfrak{z}(\widetilde{G}_\infty)$-finite. As $\varphi$ and $\widetilde{c}$ are slowly increasing, and as $C_\infty \times G_\infty$ has a finite index in $\widetilde{G}_\infty$, $\psi$ is slowly increasing. We have proved that $\psi$ is an automorphic form.

It only remains to check that $\psi$ is a cuspform. Let $\widetilde{P}$ be a parabolic subgroup of $\widetilde{G}$ defined over $F$, and $N$ its unipotent radical. We have $N \subset G$, hence $N(\mathbb{A}_F) \subset H_3$, so if $g \notin H_3$ we have $\psi(N(\mathbb{A}_F)g) = 0$. If $g = \gamma hzk$ with $\gamma \in \widetilde{G}(F)$, $h \in G(\mathbb{A}_F)$, $z \in \widetilde{Z}(\mathbb{A}_F)$ and $k \in \widetilde{K}^S$, then we have

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \psi(ng)dn = \widetilde{c}(z) \int_{N(F)\backslash N(\mathbb{A}_F)} \varphi(\gamma^{-1}n\gamma h)dn,$$

which vanishes as $\varphi$ is a cuspform and $\gamma^{-1}N\gamma$ is the unipotent radical of the $F$-parabolic subgroup of $\gamma^{-1}\widetilde{P}\gamma \cap G$ of $G$. □

**Lemma 1.** *Let $F$ be a number field and $m \geq 1$ an integer. For any set $S$ of places of $F$, let $G(S)$ be the subgroup of elements of $F^\times$ which are an $m$-th power in $F_v$ for all $v \in S$, and an $m$-th power in $F_v$ times a unit of $\mathcal{O}_v$ for all $v \notin S$. If $(F, m)$ is not special then there is a finite set $S$ of places of $F$ such that $G(S)$ coincides with the subgroup of $m$-th powers in $F^\times$.*

*Proof.* For $\lambda$ in $F^\times$ we denote by $A(\lambda)$ the etale $F$-algebra $F[X]/(X^m - \lambda)$; this algebra has an $F$-algebra morphism to $F$ (resp. $F_v$) if, and only if, $\lambda$ is an $m$-th power in $F$ (resp. $F_v$). In particular, for $\lambda$ in $G(S)$ then $A(\lambda)$ has an $F$-algebra morphism to $F_v$ for all $v \in S$. Let $T$ be the finite set of places of $F$ which are either Archimedean or dividing $m$. For any $\lambda$ in $G(T)$ the $F$-algebra $A(\lambda)$ is a product of number fields unramified outside $m$ and of degree $\leq m$. By the Hermite-Minkowski theorem, there are only finitely many such $F$-algebras up to isomorphism, say $A_1, A_2, \ldots, A_s$ with $A_i = A(\lambda_i)$ and fixed $\lambda_i \in G(S)$. For each $i$ such that $A_i$ has no $F$-algebra morphism to $F$, the Grunwald-Wang theorem applied to $\lambda_i$ asserts that there is a place $v_i$ of $F$ such that $A_i$ has no $F$-algebra morphism to $F_{v_i}$. Choose now $S$ finite containing all of those $v_i$ as well as $T$. We trivially have $G(S) \subset G(T)$ as $S \supset T$, so for any $\lambda$ in $G(S)$ we have $A(\lambda) \simeq A_i$ for some $i$. If $A(\lambda)$ has no $F$-algebra morphism to $F$, it has thus no $F$-algebra morphism to $F_{v_i}$, but this contradict the second sentence of this proof as we have $v_i \in S$. We have proved that any $\lambda \in G(S)$ is an $m$-th power. □

Let $H$ be a linear reductive group defined and quasi-split over $F$. Recall that a *Whittaker datum* for $H$ is a quadruple $D = (B, T, U, \chi)$ where $B$ is a Borel subgroup of $H$ defined over $F$, $T$ is a maximal torus of $B$ defined over $F$, $U$ is the unipotent radical of $B$, and $\chi : U(\mathbb{A}_F) \to \mathbb{C}^\times$ is a continuous character which is trivial on $U(F)$ and *non-degenerate* (see *e.g* [KS99, p. 54]). When $H$ is split, it means *e.g.* that $\chi$ is nontrivial on the root subgroup $U_\alpha \subset U$ for each simple root $\alpha$ relative to $(B, T)$. Recall that an irreducible $H(\mathbb{A}_F)$-gk-submodule $V$ of the space of cuspforms on $H(\mathbb{A}_F)$ is said *generic* if there exists a Whittaker datum $D = (B, T, U, \chi)$, such that for all $\varphi \in V - \{0\}$ the function on $H(\mathbb{A}_F)$ defined by

$$\mathrm{W}^D_\varphi(g) = \int_{N(F)\backslash N(\mathbb{A}_F)} \varphi(ng)\chi(n)dn$$

is not identically zero. As the subspace of $\varphi \in V$ with $\mathrm{W}^D_\varphi = 0$ is an $H(\mathbb{A}_F) -$ gk-submodule of $V$, this latter condition is equivalent to ask that we have $\mathrm{W}^D_\varphi \neq 0$ for *some* $\varphi \in V$, by irreducibility of $V$.

**Proposition 2.** *Keep the assumptions of Proposition 1 and assume furthermore that $G$ is quasi-split over $F$. Let $\widetilde{V} \subset \mathcal{A}_{\text{cusp}}(\widetilde{G})_{\widetilde{c}}$ be a non-zero irreducible $\widetilde{G}(\mathbb{A}_F)$-*gk*-submodule. The following properties are equivalent :*

(i) *$\widetilde{V}$ is generic,*

(ii) *$\rho_{\text{cusp}}(\widetilde{V})$ contains a non-zero generic irreducible $G(\mathbb{A}_F)$-*gk*-submodule.*

*Proof.* Assume first that (i) holds. Let $\widetilde{D} = (\widetilde{B}, \widetilde{T}, U, \chi)$ be a Whittaker datum such that for all $\varphi \in \widetilde{V} - \{0\}$ we have $\mathrm{W}_\varphi^{\widetilde{D}} \neq 0$. Then $D = (\widetilde{B} \cap G, \widetilde{T} \cap G, U, \chi)$ is a Whittaker datum for $G$. Choose a non-zero $\varphi \in \widetilde{V}$. Up to replacing $\varphi$ by some $\widetilde{K}_\infty \times \widetilde{G}(\mathbb{A}_F^f)$-translate as in the proof of Proposition 1 (ii), we may assume that we have $\mathrm{W}_\varphi^{\widetilde{D}} \neq 0$ on $G(\mathbb{A}_F)$. This implies that the cuspform $f := \rho(\varphi) = \varphi_{|G(\mathbb{A}_F)}$ is non-zero and $\mathrm{W}_f^D \neq 0$. The $G(\mathbb{A}_F)$-gk-submodule of $\rho_{\text{cusp}}(\widetilde{V}) \subset \mathcal{A}_{\text{cusp}}(G)_c$ generated by $f$ is semi-simple, hence a finite direct sum of irreducible gk-modules $V_i$. Write $f = \sum_i f_i$ with $f_i \in V_i$. We have $\mathrm{W}_f^D = \sum_i \mathrm{W}_{f_i}^D \neq 0$, so there exists $i$ such that $\mathrm{W}_{f_i}^D \neq 0$, and $V_i$ is a generic irreducible constituent of $\rho_{\text{cusp}}(\widetilde{V})$.

Assume now that $\rho_{\text{cusp}}(\widetilde{V})$ contains some generic irreducible $G(\mathbb{A}_F)$-submodule $V \neq 0$. Let $D = (B, T, U, \chi)$ be a Whittaker datum for $G$ such that $\mathrm{W}_\varphi^D$ is non-zero for all $\varphi \in V - \{0\}$. There exists a unique Whittaker datum $\widetilde{D}$ for $\widetilde{G}$ of the form $(\widetilde{B}, \widetilde{T}, U, \chi)$ with $\widetilde{B} \cap G = B$ and $\widetilde{T} \cap G = T$. As we have $0 \neq V \subset \rho(\widetilde{V})$, we may find some $\varphi \in \widetilde{V}$ such that the cuspform $f = \rho(\varphi) = \varphi_{|\mathrm{G}(\mathbb{A}_F)}$ is non-zero and belongs to $V$. It is enough to show $\mathrm{W}_\varphi^{\widetilde{D}} \neq 0$. But for $g \in G(\mathbb{A}_F)$ we have the identity $\mathrm{W}_\varphi^{\widetilde{D}}(g) = \mathrm{W}_f^D(g)$, and we are done. □

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