The infinite fern of Galois representations of type $U(3)$

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Let $E$ be a number field, $p$ a prime and let $S$ be a finite set of places of $E$ containing the primes above $p$ and $\infty$. Consider the set of isomorphism classes of continuous semi-simple representations $\rho : G_{E,S} \to \text{GL}_d(\mathbb{Q}_p)$ of some fixed dimension $d$, where $G_{E,S}$ is the Galois group of a maximal algebraic extension of $E$ unramified outside $S$. This is the set of $\mathbb{Q}_p$-points of a natural rigid analytic space $X$ over $\mathbb{Q}_p$, an interesting subset of which is the set $X^g$ of the $\rho$ which are geometric, in the sense that they occur as a subquotient of $H^i_\text{et}(X_{\overline{E}}, \mathbb{Q}_p)(m)$ for some proper smooth variety $X$ over $E$, some degree $i \geq 0$ and some Tate twist $m \in \mathbb{Z}$. Here are two basic, but presumably difficult, open questions about $X^g$:

Does $X^g$ have some specific structure? Can we describe its Zariski-closure in $X$?

A trivial observation is that $X^g$ is countable, so it contains no subvariety of dimension $> 0$. When $d = 1$, class-field theory and the theory of complex multiplication describe $X^g$ and $X$, in particular $X^g$ is Zariski-dense in $X$ if Leopold’s conjecture holds at $p$. When $d > 1$, the situation is actually much more interesting, and has been first studied by Hida, Mazur, Gouvêa and Coleman when $E = \mathbb{Q}$ and $d = 2$. A discovery of Gouvêa and Mazur is that in the most "regular" odd connected components of $X$, which are open unit balls of dimension 3, then $X^g$ is still Zariski-dense. Furthermore, it belongs to an intriguing subset of $X$ they call the infinite fern [4], which is a kind of fractal 2-dimensional object in $X$ built from Coleman’s theory of finite slope families of modular eigenforms.

The aim of this talk is to present an extension of these results to the three-dimensional case $d = 3$, mostly by studying the contribution of $X^g$ coming from the theory of Picard modular surfaces. From now on $E$ is a quadratic imaginary field, $p$ is an odd prime that splits in $E$, $c$ is the non trivial element of $\text{Gal}(E/\mathbb{Q})$ and the set $S$ is stable by $c$. Let $q$ be a power of $p$ and fix a continuous absolutely irreducible Galois representation $\overline{\rho} : G_{E,S} \to \text{GL}_3(\mathbb{F}_q)$ of type $U(3)$, i.e. such that $\overline{\rho}^\vee \simeq \overline{\rho}$ (the latter being the outer conjugate by $c$). This last condition is equivalent to ask that $\overline{\rho}$ extends to a representation $\overline{\rho} : G_{\mathbb{Q},S} \to \text{GL}_3(\mathbb{F}_q) \rtimes \text{Gal}(E/\mathbb{Q})$ inducing the natural map $G_{\mathbb{Q},S} \to \text{Gal}(E/\mathbb{Q})$ and where $c$ acts on $\text{GL}_3$ via $g \mapsto cyg^{-1}$. Let us denote by $R(\overline{\rho})$ the universal $G_{\mathbb{Q},S}$-deformation of type $U(3)$ of $\overline{\rho}$ to the category of finite local $\mathbb{Z}_q = W(\mathbb{F}_q)$-algebras with residue field $\mathbb{F}_q$. This ring $R(\overline{\rho})$ might be extremely complicated in general, but we shall not be interested in these complications and rather assume that:

$$(H) \quad H^2(G_{\mathbb{Q},S}, \text{ad}(\overline{\rho})) = 0.$$ 

In this case, one can show that $R(\overline{\rho})$ is formally smooth over $\mathbb{Z}_q$ of relative dimension 6. In particular, its analytic generic fiber $X(\overline{\rho})$ in the sense of Berthelot is the open unit ball of dimension 6 over $\mathbb{Q}_p$. This space is actually a connected component of the locus of type $U(3)$ of $X$. By definition its closed points $x$ parameterize the lifts $\rho_x$ of $\overline{\rho}$ such that $\rho_x^\vee \simeq \rho_x^c$. Such an $x$ will be said modular.
if \( \rho \) is isomorphic to a \( p \)-adic Galois representation \( \rho_\Pi \) attached by Rogawski to some cohomological cuspidal automorphic representation \( \Pi \) of \( GL_3(A_E) \) such that \( \Pi^\vee \cong \Pi \) and which is unramified outside \( S \) and at the two places above \( p \). These Galois representations are cut out from the étale cohomology of (some sheaves over) the Picard modular surfaces of \( E \). We say that \( \bar{\rho} \) is modular if there is at least one modular point in \( X(\bar{\rho}) \). It might well be the case that each \( \bar{\rho} \) is modular (a variant of Serre’s conjecture).

**Theorem A:** Assume that \( \bar{\rho} \) is modular and that \((H)\) holds. Then the modular points are Zariski-dense in \( X(\bar{\rho}) \).

**Example:** If \( A \) is an elliptic curve over \( \mathbb{Q} \), then \( \bar{\rho} := (\text{Symm}^2 A([-])((-1)) \) is modular of type \( U(3) \). Assume that \( E = \mathbb{Q}(i) \), \( p = 5 \) and let \( S \) be the set of primes dividing \( 10 \cdot \text{cond}_A \cdot \infty \), then \((H)\) holds whenever \( A \) is in the class labeled as \( 17A, 21A, 37B, 39A, 51A, 53A, 69A, 73A, 83A, \) or \( 91B \) in Cremona’s tables (this depends on some class number computations by PARI relying on GRH).

A first important step in the proof of Theorem A is a result from the theory of \( p \)-adic families of automorphic forms for the definite unitary group \( U(3) \) ([2],[1]).

Fix \( v \) a prime of \( E \) dividing \( p \), so that \( E_v = \mathbb{Q}_p \). Define a refined modular point as a pair \((\rho_\Pi, (\varphi_1/p^{k_1}, \varphi_2/p^{k_2}, \varphi_3/p^{k_3}))\) in \( X(\bar{\rho}) \times \mathbb{G}_m^3 \) where \( \rho_\Pi \) is a modular Galois representation associated to \( \Pi \), \( k_1 < k_2 < k_3 \) are the Hodge-Tate numbers of \( \rho_{\Pi,v} \), and where \((\varphi_1, \varphi_2, \varphi_3)\) is an ordering of the eigenvalues of the crystalline Frobenius acting on \( D_{\text{cris}}(\rho_{\Pi,v}) \) (recall that \( \rho_{\Pi,v} := (\rho_\Pi)|_{G_{E_v}} \) is a crystalline representation of \( G_{E_v} = G_{\mathbb{Q}_p} \)).

Define the eigenvariety \( E(\bar{\rho}) \subset X(\bar{\rho}) \times \mathbb{G}_m^3 \) as the Zariski-closure of the refined modular points. The main theorem from the theory of \( p \)-adic families of automorphic forms for \( U(3) \) asserts that \( E(\bar{\rho}) \) has equi-dimension 3. By construction the refined modular points are Zariski-dense in \( E(\bar{\rho}) \), and even have some accumulation property. The complete infinite fern of type \( U(3) \) is the set theoretic projection of \( E(\bar{\rho}) \) in \( X(\bar{\rho}) \). At a modular point in \( X(\bar{\rho}) \) there are in general 6 branches of the fern passing through it, as there are in general six ways to refine a given modular point, hence 6 points in \( E(\bar{\rho}) \) above it, so we get the above picture. (In any dimension \( d \): \( \text{dim} X(\bar{\rho}) = d(d+1)/2 \), \( \text{dim} E(\bar{\rho}) = d \) and there are \( d! \) ways to refine a given modular point).

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1By Zariski-dense we simply mean here that if \( t_1, t_2, \ldots, t_6 \) are parameters of the ball \( X(\bar{\rho}) \), then there is no nonzero power series in \( \mathbb{C}_p[[t_1, \ldots, t_6]] \) converging on the whole of \( X(\bar{\rho}) \) and that vanishes at all the modular points.
Theorem B: There exist modular points $x \in \mathcal{X}(\overline{\mathbb{F}})$ such that $\rho_{x|G_{E}}$ is irreducible and has $\neq$ crystalline Frobenius eigenvalues. If $x$ is such a point, then

$$\bigoplus_{y \rightarrow x, y \in \mathcal{E}(\overline{\mathbb{F}})} T_y(\mathcal{E}(\overline{\mathbb{F}})) \rightarrow T_x(\mathcal{X}(\overline{\mathbb{F}}))$$

(the map induced on tangent space) is surjective.

Considering the Zariski-closure $Z$ in $\mathcal{X}(\overline{\mathbb{F}})$ of the modular points satisfying the first part of Theorem B, and applying Theorem B to a smooth such point of $Z$, we get Theorem A. The first part of Theorem B is a simple application of eigenvarieties, but its second part is rather deep. It relies on a detailed study of the properties at $p$ of the family of Galois representations over $\mathcal{E}(\overline{\mathbb{F}})$, especially around non-critical refined modular points, as previously studied in [1] (extending some works of Kisin and Colmez in dimension 2). There are several important ingredients in the proof but we end this short note by focusing on a crucial and purely local one.

Let $L$ be a finite extension of $\mathbb{Q}_p$ and let $V$ be a crystalline representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of any $L$-dimension $d$. Assume $V$ is irreducible, with distinct Hodge-Tate numbers, and that the eigenvalues $\varphi_i$ of the crystalline Frobenius on $D_{\text{crys}}(V)$ belong to $L$ and satisfy $\varphi_i \varphi_j^{-1} \neq 1, p$ for all $i \neq j$. Let $\mathcal{X}_V$ be the deformation functor of $V$ to the category of local artinian $L$-algebras with residue field $L$. It is pro-representable and formally smooth of dimension $d^2 + 1$. For each ordering $\mathcal{F}$ of the $\varphi_i$ (such an ordering is called a refinement), we defined in [1] the $\mathcal{F}$-trianguline deformation subfunctor $\mathcal{X}_{V, \mathcal{F}} \subset \mathcal{X}_V$, whose dimension is $d(d + 1)/2 + 1$. Roughly, the choice of $\mathcal{F}$ corresponds to a choice of a triangulation of the $(\varphi, \Gamma)$-module of $V$ over the Robba ring, and $\mathcal{X}_{V, \mathcal{F}}$ parameterizes the deformations such that this triangulation lifts. When the $\varphi$-stable complete flag of $D_{\text{crys}}(V)$ defined by $\mathcal{F}$ is in general position compared to the Hodge filtration, we say that $\mathcal{F}$ is non-critical.

Theorem C: Assume that $d$ "well-chosen" refinements of $V$ are non-critical (e.g. all of them), or that $d \leq 3$. Then on tangent spaces we have an equality

$$\mathcal{X}_V(L[z]) = \sum_{\mathcal{F}} \mathcal{X}_{V, \mathcal{F}}(L[z]).$$

In other words "any first order deformation of a generic crystalline representation is a linear combination of trianguline deformations". See [3] for proofs of the results of this note.

References