The infinite fern of Galois representations of type U(3)

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Let E be a number field, p a prime and let S be a finite set of places of E containing the primes above p and ∞. Consider the set of isomorphism classes of continuous semi-simple representations $\rho : G_{E,S} \rightarrow \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ of some fixed dimension d, where $G_{E,S}$ is the Galois group of a maximal algebraic extension of E unramified outside S. This is the set of $\overline{\mathbb{Q}}_p$-points of a natural rigid analytic space $\mathcal{X}$ over $\mathbb{Q}_p$, an interesting subset of which is the set $\mathcal{X}^g$ of the ρ which are geometric, in the sense that they occur as a subquotient of $H^i_{\mathrm{et}}(X_{\overline{E}}, \overline{\mathbb{Q}}_p)(m)$ for some proper smooth variety X over E, some degree $i \geq 0$ and some Tate twist $m \in \mathbb{Z}$. Here are two basic, but presumably difficult, open questions about $\mathcal{X}^g$:

Does $\mathcal{X}^g$ have some specific structure ? Can we describe its Zariski-closure in $\mathcal{X}$?

A trivial observation is that $\mathcal{X}^g$ is countable, so it contains no subvariety of dimension > 0. When $d = 1$, class-field theory and the theory of complex multiplication describe $\mathcal{X}^g$ and $\mathcal{X}$, in particular $\mathcal{X}^g$ is Zariski-dense in $\mathcal{X}$ if Leopold's conjecture holds at p. When $d > 1$, the situation is actually much more interesting, and has been first studied by Hida, Mazur, Gouvêa and Coleman *when $E = \mathbb{Q}$ and $d = 2$*. A discovery of Gouvêa and Mazur is that in the most "regular" odd connected components of $\mathcal{X}$, which are open unit balls of dimension 3, then $\mathcal{X}^g$ *is still Zariski-dense*. Furthermore, it belongs to an intriguing subset of $\mathcal{X}$ they call the *infinite fern* [4], which is a kind of fractal 2-dimensional object in $\mathcal{X}$ built from Coleman's theory of finite slope families of modular eigenforms.

The aim of this talk is to present an extension of these results to the three-dimensional case $d = 3$, mostly by studying the contribution of $\mathcal{X}^g$ coming from the theory of Picard modular surfaces. From now on E is a quadratic imaginary field, p is an odd prime that splits in E, c is the non trivial element of $\mathrm{Gal}(E/\mathbb{Q})$ and the set S is stable by c. Let q be a power of p and fix a continuous absolutely irreducible Galois representation

$$\overline{\rho} : G_{E,S} \rightarrow \mathrm{GL}_3(\mathbb{F}_q)$$

of type U(3), i.e. such that $\overline{\rho}^\vee \simeq \overline{\rho}^c$ (the latter being the outer conjugate by c). This last condition is equivalent to ask that $\overline{\rho}$ extends to a representation $\tilde{\rho} : G_{\mathbb{Q},S} \rightarrow \mathrm{GL}_3(\mathbb{F}_q) \rtimes \mathrm{Gal}(E/\mathbb{Q})$ inducing the natural map $G_{\mathbb{Q},S} \rightarrow \mathrm{Gal}(E/\mathbb{Q})$ and where c acts on GL_3 via $g \mapsto {}^t g^{-1}$. Let us denote by $R(\overline{\rho})$ the universal $G_{E,S}$-deformation of type U(3) of $\overline{\rho}$ to the category of finite local $\mathbb{Z}_q = W(\mathbb{F}_q)$-algebras with residue field $\mathbb{F}_q$. This ring $R(\overline{\rho})$ might be extremely complicated in general, but we shall not be interested in these complications and rather assume that:

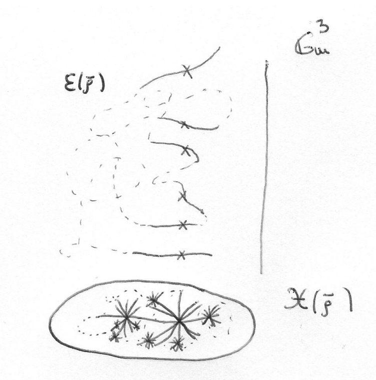
$$(H) \qquad H^2(G_{\mathbb{Q},S}, \mathrm{ad}(\tilde{\rho})) = 0.$$

In this case, one can show that $R(\overline{\rho})$ is formally smooth over $\mathbb{Z}_q$ of relative dimension 6. In particular, its analytic generic fiber $\mathcal{X}(\overline{\rho})$ in the sense of Berthelot is the open unit ball of dimension 6 over $\mathbb{Q}_q$. This space is actually a connected component of the locus of type $U(3)$ of $\mathcal{X}$. By definition its closed points x parameterize the lifts ρ_x of $\overline{\rho}$ such that $\rho_x^\vee \simeq \rho_x^c$. Such an x will be said *modular* if ρ_x is isomorphic to a p-adic Galois representation ρ_Π attached by Rogawski to some cohomological cuspidal automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_E)$ such that $\Pi^\vee \simeq \Pi^c$ and which is unramified outside S and at the two places above p. These Galois representations are cut out from the étale cohomology of (some sheaves over) the Picard modular surfaces of E. We say that $\overline{\rho}$ is modular if there is at least one modular point in $\mathcal{X}(\overline{\rho})$. It might well be the case that each $\overline{\rho}$ is modular (a variant of Serre's conjecture).

Theorem A: *Assume that $\overline{\rho}$ is modular and that (H) holds. Then the modular points are Zariski-dense[1] in $\mathcal{X}(\overline{\rho})$.*

Example: *If A is an elliptic curve over $\mathbb{Q}$, then $\overline{\rho} := (\mathrm{Symm}^2 A[p])(-1)$ is modular of type $\mathrm{U}(3)$. Assume that $E = \mathbb{Q}(i)$, $p = 5$ and let S be the set of primes dividing $10 \cdot \mathrm{condA} \cdot \infty$, then (H) holds whenever A is in the class labeled as $17A$, $21A$, $37B$, $39A$, $51A$, $53A$, $69A$, $73A$, $83A$, or $91B$ in Cremona's tables (this depends on some class number computations by PARI relying on GRH).*

A first important step in the proof of Theorem A is a result from the theory of p-adic families of automorphic forms for the definite unitary group $\mathrm{U}(3)$ ([2],[1]). Fix v a prime of E dividing p, so that $E_v = \mathbb{Q}_p$. Define a *refined modular point* as a pair $(\rho_\Pi, (\varphi_1/p^{k_1}, \varphi_2/p^{k_2}, \varphi_3/p^{k_3}))$ in $\mathcal{X}(\overline{\rho}) \times \mathbb{G}_m^3$ where ρ_Π is a modular Galois representation associated to Π, $k_1 < k_2 < k_3$ are the Hodge-Tate numbers of $\rho_{\Pi,v}$, and where $(\varphi_1, \varphi_2, \varphi_3)$ is an ordering of the eigenvalues of the crystalline Frobenius acting on $D_{\mathrm{cris}}(\rho_{\Pi,v})$ (recall that $\rho_{\Pi,v} := (\rho_\Pi)_{|G_{E_v}}$ is a crystalline representation of $G_{E_v} = G_{\mathbb{Q}_p}$).



Define the *eigenvariety* $\mathcal{E}(\overline{\rho}) \subset \mathcal{X}(\overline{\rho}) \times \mathbb{G}_m^3$ as the Zariski-closure of the refined modular points. The main theorem from the theory of p-adic families of automorphic forms for $U(3)$ asserts that $\mathcal{E}(\overline{\rho})$ *has equi-dimension* 3. By construction the refined modular points are Zariski-dense in $\mathcal{E}(\overline{\rho})$, and even have some accumulation property. The complete *infinite fern of type $U(3)$* is the set theoretic projection of $\mathcal{E}(\overline{\rho})$ in $\mathcal{X}(\overline{\rho})$. At a modular point in $\mathcal{X}(\overline{\rho})$ there are in general 6 branches of the fern passing through it, as there are in general six ways to refine a given modular point, hence 6 points in $\mathcal{E}(\overline{\rho})$ above it, so we get the above picture. (In any dimension d: $\dim \mathcal{X}(\overline{\rho}) = d(d+1)/2$, $\dim \mathcal{E}(\overline{\rho}) = d$ and there are up to $d!$ ways to refine a given modular point).

[1]By Zariski-dense we simply mean here that if $t_1, t_2, \ldots, t_6$ are parameters of the ball $\mathcal{X}(\overline{\rho})$, then there is no nonzero power series in $\mathbb{C}_p[[t_1, \ldots, t_6]]$ converging on the whole of $\mathcal{X}(\overline{\rho})$ and that vanishes at all the modular points.

Theorem B: *There exist modular points $x \in \mathcal{X}(\overline{\rho})$ such that $\rho_{x|G_{E_v}}$ is irreducible and has $\neq$ crystalline Frobenius eigenvalues. If x is such a point, then*

$$\bigoplus_{y \mapsto x, y \in \mathcal{E}(\overline{\rho})} T_y(\mathcal{E}(\overline{\rho})) \longrightarrow T_x(\mathcal{X}(\overline{\rho}))$$

(the map induced on tangent space) is surjective.

Considering the Zariski-closure Z in $\mathcal{X}(\overline{\rho})$ of the modular points satisfying the first part of Theorem B, and applying Theorem B to a smooth such point of Z, we get Theorem A. The first part of Theorem B is a simple application of eigenvarieties, but its second part is rather deep. It relies on a detailled study of the properties at p of the family of Galois representations over $\mathcal{E}(\overline{\rho})$, especially around non-critical refined modular points, as previously studied in [1] (extending some works of Kisin and Colmez in dimension 2). There are several important ingredients in the proof but we end this short note by focusing on a crucial and purely local one.

Let L be a finite extension of $\mathbb{Q}_p$ and let V be a crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of any L-dimension d. Assume V is irreducible, with distinct Hodge-Tate numbers, and that the eigenvalues φ_i of the crystalline Frobenius on $D_{\mathrm{crys}}(V)$ belong to L and satisfy $\varphi_i\varphi_j^{-1} \neq 1, p$ for all $i \neq j$. Let $\mathcal{X}_V$ be the deformation functor of V to the category of local artinian L-algebras with residue field L. It is pro-representable and formally smooth of dimension d^2+1. For each ordering $\mathcal{F}$ of the φ_i (such an ordering is called a *refinement*), we defined in [1] the $\mathcal{F}$-trianguline deformation subfunctor $\mathcal{X}_{V,\mathcal{F}} \subset \mathcal{X}_V$, whose dimension is $d(d+1)/2+1$. Roughly, the choice of $\mathcal{F}$ corresponds to a choice of a triangulation of the (φ, Γ)-module of V over the Robba ring, and $\mathcal{X}_{V,\mathcal{F}}$ parameterizes the deformations such that this triangulation lifts. When the φ-stable complete flag of $D_{\mathrm{cris}}(V)$ defined by $\mathcal{F}$ is in general position compared to the Hodge filtration, we say that $\mathcal{F}$ is *non-critical.*

Theorem C: *Assume that d "well-chosen" refinements of V are non-critical (e.g. all of them), or that $d \leq 3$. Then on tangent spaces we have an equality*

$$\mathcal{X}_V(L[\varepsilon]) = \sum_{\mathcal{F}} \mathcal{X}_{V,\mathcal{F}}(L[\varepsilon]).$$

In other words "any first order deformation of a generic crystalline representation is a linear combination of trianguline deformations". See [3] for proofs of the results of this note.

References

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