Unimodular lattices of rank 29 and related even genera of small determinant

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January 27, 2026

Abstract

We classify the unimodular Euclidean integral lattices of rank 29 by developing an elementary, yet very efficient, inductive method. As an application, we determine the isometry classes of even lattices of rank ≤ 28 and prime (half-)determinant ≤ 7. We also provide new isometry invariants allowing for independent verification of the completeness of our lists, and we give conceptual explanations of some *unique orbit phenomena* discovered during our computations. Some of the genera classified here are orders of magnitude larger than any genus previously classified. In a forthcoming companion paper, we use these computations to study the cohomology of $\mathrm{GL}_n(\mathbb{Z})$.

1. Introduction

1.1. The classification of unimodular lattices

Let us denote by X_n the set of isometry classes of *unimodular integral Euclidean* lattices of rank $n \geq 1$ (see Sect. 2.1). The simplest example of an element of X_n is the (class of the) *standard*, or *cubic*, lattice $\mathrm{I}_n := \mathbb{Z}^n$. We know from reduction theory or the geometry of numbers (Hermite, Minkowski) that X_n is a finite set; we even know its *mass* in the sense of Smith-Minkowski-Siegel.

Determining the exact cardinality and finding representatives of X_n is, however, a difficult problem and a classical topic in number theory. Its origin goes back at least to the works of Lagrange and Gauss on counting the number of representations of an integer as a sum of squares, a question concerning the single lattice I_n a priori but intimately linked to the whole of X_n, e.g. by the *Siegel-Weil* formula. The known values of $|\mathrm{X}_n|$ are gathered in Table 1.1 below.

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n	≤ 7	8	9	10	11	12	13	14	15	16	17
$\|X_n\|$	1	2	2	2	2	3	3	4	5	8	9
n	18	19	20	21	22	23	24	25	26	27	28
$\|X_n\|$	13	16	28	40	68	117	297	665	2 566	17 059	374 062

Table 1.1: Size of X_n for $n \leq 28$.

These classification results for X_n with $n \leq 25$ were obtained through a long series of works in the last century [Mo38, Ko38, W41, Kn57, Nie73, CoS82, Bor84]: see [CoS99] for some historical perspectives. Notable tools include Kneser's neighboring method introduced in [Kn57] (to settle the cases $10 \leq n \leq 16$), root lattices and the gluing method [W41, Nie73, CoS82], and the use of Lorentzian lattices in [Bor84, Bor00] in the case $24 \leq n \leq 25$. We also mention that if $X_n^R \subset X_n$ denotes the subset of lattices with root system isomorphic to R (and say, with no norm 1 vectors), King computed the mass of X_n^R for all R and $n \leq 30$ in [Ki03], substantially improving the mass formula lower bound on $|X_n|$ for $26 \leq n \leq 30$.

The remaining cases in Table 1.1 were obtained in the recent series of works [Ch22, Ch25] ($26 \leq n \leq 27$) and [ACh25] ($n = 28$), using a biased enumeration of the Kneser neighbors of I_n (developing ideas in [BV01]), King's aforementioned work, refinements of the Plesken-Souvignier algorithm [PlS97], and substantial computer calculations. A remarkable, though not yet understood, byproduct of [ACh25] is the fact that a certain invariant of vectors of norm ≤ 3, inspired by [BV01] and which we denote by BV, is both fast to compute and distinguishes all unimodular lattices of rank ≤ 28. Our first main result is:

Theorem A. *There are exactly* 38 966 352 *classes in* X_{29}*, all distinguished by their* BV *invariants, and with Gram matrices given in* [ChTa].

King's lower bound $|X_{29}| \geq 37\,938\,009$ was thus remarkably close. Our method to prove Theorem A is different from those above. Indeed, the neighbor enumeration and analysis in [ACh25] for the case $n = 28$ already required more than 70 years of CPU time on a single core, and would not be reasonable in the much larger case $n = 29$. The methods in [Ch25, ACh25] allow in principle to determine X_{29}^R for each R (assuming the choice of invariant is fine enough, a serious assumption), but each choice of R requires some specific handlings, and there are 11085 contributing R by [Ki03]. The set $X_{29}^{\emptyset}$ was actually already determined in [ACh25], and we use here the same strategy only in the cases $R = \mathbf{A}_1$ and $R = \mathbf{A}_2$ (see Proposition 4.3).

All the remaining lattices contain a pair of orthogonal roots, and for those we use an entirely different strategy, that we develop in Sect. 3. It is based on the elementary fact that for any $n \geq 1$, there is a natural groupoid equivalence between: (i) pairs (L, e) with L a rank n unimodular lattice and $e \in L/2L$ satisfying $e \cdot e \equiv 2 \bmod 4$, and (ii) pairs $(U, \{\alpha, \beta\})$ with U a rank $n+2$ unimodular lattice and $\{\alpha, \beta\}$ an unordered pair of orthogonal roots of U with $\frac{\alpha+\beta}{2} \notin U$. This paves the way for a recursive exhaustion of X_{n+2}, by listing first the orbits of mod 2 vectors of each element in X_n.

The main drawback of this method is that it produces each rank $n+2$ unimodular lattice U as many times as the number of $O(U)$-orbits of pairs of orthogonal roots in U: see Sect. 3 for a few simple techniques to reduce these redundancies. The worst case occurs when the root system of a generic U is of the form $a\mathbf{A}_1\, b\mathbf{A}_2$ with large $a+b$, and unfortunately these lattices constitute a significant proportion of the cases in practice when n grows (presumably, a manifestation of the "law of conservation of complexity").

Nevertheless, despite this issue, we realized that this method, combined with *unimodular hunting* for empty, $\mathbf{A}_1$ or $\mathbf{A}_2$ root systems, is by far the most efficient strategy to recursively reconstruct all unimodular lattices from scratch. For instance, using our current algorithms, it allowed us to recover all unimodular lattices of rank ≤ 28 in only about a week of computations. Of course, such computations are always much easier *a posteriori*, but the superiority of this method over the neighbor method (which requires significantly more redundancies) is clear.

In Sect. 4 we put the theory into practice and use this method to prove Theorem A. As in rank ≤ 28, the BV invariant turned out to miraculously distinguish all elements of X_{29}, and this is a fundamental ingredient in the proof. We also refer to this section for detailed information about the final list of rank 29 unimodular lattices and for further details about the computation process.

Our computations were performed using the open-source computer algebra system Pari/GP [PARI]. For efficiency, we were led to improve or reimplement several key algorithms. These important algorithmic contributions, due to the second author, are briefly described in Section 2.17 and will be the subject of a separate paper. They include: an exact implementation of the Fincke-Pohst algorithm [FP85], an improvement of the Plesken-Souvignier algorithm for lattices with roots [T24], a faster implementation of the BV invariant, a probabilistic algorithm for finding "good" Gram matrices (improving the one in [ACh25]), and a specific algorithm computing orbits of mod 2 vectors. The source codes are, however, already available in [ChTa].

The entire computation took about 20 months of CPU time (single core equivalent). It was run on a system with 2×64-core AMD EPYC 7763 CPU (zen3) running at 1.5 GHz, with 1024 GB of RAM.[1] The given Gram matrices in [ChTa], together with the invariant BV and the mass formula, allow for an independent check that our list is complete, and which only requires about 80 days of CPU time: see Sect. 4.

We conclude this section with a discussion of the case $n = 30$. King's lower bound indicates that X_{30} has more than 20 billion of classes! Using the unimodular hunting techniques [Ch25, ACh25], and the improvements above of our algorithms, we were able to show the following (see Sect. 8).

Theorem B. *The size of* X_{30}^R *for* $R = \emptyset$, $\mathbf{A}_1$ *or* $\mathbf{A}_2$ *is given by the table below. Neighbor forms for representatives for all of those lattices are given in* [ChTa]. *Moreover, all those lattices are distinguished by their* BV *invariant.*

R	$\emptyset$	$\mathbf{A}_1$	$\mathbf{A}_2$
$\lvert\mathrm{X}_{30}^R\rvert$	82 323 107	357 495 297	12 708 298

[1] For reference, all CPU times reported in this paper are relative to this system. While this system allowed efficient parallel computations, the algorithms described are equally performant on standard personal computers.

Although we have no doubt that we could go further and determine X_{30}^R for many other root systems R, we do not pursue this direction here. Indeed, it seems more promising to directly study the more fundamental (and famous) genus X_{32}^{even} of rank 32 even unimodular lattices, which should contain "only" about 1.2 billion lattices according to King, and from which X_{30} can be theoretically deduced.

The results established in this section have several interesting consequences for the study of X_{32}^{even}, such as a classification of all lattices containing $\mathbf{A}_3$. To the best of our knowledge, they provide the first indication in the literature that a classification of X_{32}^{even} may now be within reach, the sizes of X_{29} and of the X_{30}^R above being of the same order of magnitude as the expected size of X_{32}^{even}. A key remaining difficulty in studying X_{32}^{even} is that the natural analogue of the BV invariant in this case is computationally much more expensive due to the large number of norm 4 vectors.

We will study X_{32}^{even} more thoroughly in a forthcoming work. We stress that the study of such large genera is not as futile as it may seem, and is not solely motivated by the computational challenge. Indeed, as shown in a series of recent works by the authors (such as [ChL19, ChR15, T17, ChT20, ChTb]), it would also have significant consequences for the theory of automorphic forms for $\mathrm{GL}_n(\mathbb{Z})$: see Sect. 1.4 for more information about these aspects.

1.2. Even lattices of small and prime (half-)determinant p

For an integer $n \geq 1$ and p an odd prime, we consider the genus $\mathcal{G}_{n,p}$ of even Euclidean lattices of rank n and determinant d, with $d = p$ for n even and $d = 2p$ for n odd. See Section 7 for examples and basic properties of these genera. The genus $\mathcal{G}_{n,p}$ is nonempty if and only if either n is odd or $n + p \equiv 1 \bmod 4$. Our main result is the following.

Theorem C. *Each (p, n)-entry given in the Tables 1.2 and 1.3 provides the number of isometry classes in the genus $\mathcal{G}_{n,p}$. Moreover, Gram matrices for representatives of these isometry classes are given*[2] *in* [ChTa].

$p \setminus n$	2	4	6	8	10	12	14	16	18	20	22	24	26	28
3	1		1		1		2		6		31		678	
5		1		1		2		5		27		352		2 738 211
7	1		1		2		4		20		153		44 955	

Table 1.2: Number of isometry classes of even lattices of even rank $2 \leq n \leq 28$ and odd prime determinant $p \leq 7$ (zero if blank).

$p \setminus n$	1	3	5	7	9	11	13	15	17	19	21	23	25	27
3	1	1	1	1	2	2	3	5	10	19	64	290	2 827	285 825
5	1	1	1	1	3	3	5	10	21	55	210	1 396	38 749	24 545 511
7	1	1	1	2	3	5	8	14	37	119	513	5 535	341 798	659 641 434

Table 1.3: Number of isometry classes of even lattices of odd rank $1 \leq n \leq 27$ and determinant $2p$ with p a prime ≤ 7.

[2] We did not store the (huge) list in the case $(p, n) = (7, 27)$.

The results for $p = 3$ and even $n \leq 14$ in these tables go back to Kneser [Kn57]. Kneser deduced them from his aforementioned classification of unimodular lattices of rank ≤ 16 by studying embeddings into unimodular lattices of slightly larger rank.[3] Applying similar ideas, along with a more systematic use of root lattices and of the gluing method, Conway and Sloane determined in [CoS88a] all the values in the tables above which are ≤ 2, or with $n \leq 10$. We also mention the almost complete[4] study in [Bor00, §4.7] of the cases $(n, p) = (25, 3)$ and $(26, 3)$ by a quite different approach using Lorentzian lattices.

We will prove Theorem C in Sect. 7.2. Our method, in the spirit of Kneser's, consists in deducing all the genera $\mathcal{G}_{n,p}$ above from the classification of unimodular lattices of rank ≤ 29 and successive applications of *the orbit method*, that we now discuss.

We say that a genus $\mathcal{G}'$ may be deduced from another genus $\mathcal{G}$ by *the orbit method of type t* if the map $(L, v) \mapsto L' := v^\perp \cap L$ induces a bijection between the isometry classes of pairs (L, v), with L in $\mathcal{G}$ and $v \in L$ of type t, and the isometry classes of L' in $\mathcal{G}'$. *Type t vectors* will typically be the set of all primitive vectors of a certain norm satisfying possibly some extra conditions. Several natural instances of such triples $(\mathcal{G}, \mathcal{G}', t)$ are recalled in Sect. 2.6 (see also Proposition 5.10 for a possibly new example). If $\mathcal{L}$ is a set of representatives for the isometry classes in $\mathcal{G}$, finding representatives of the isometry classes in $\mathcal{G}'$ then amounts to determining, for each $L \in \mathcal{L}$, the $\mathrm{O}(L)$-orbits of vectors of type t in L, hence the terminology. If the number of type t vectors in these lattices L is not too large,[5] and if the Plesken-Souvignier algorithm succeeds in finding generators for $\mathrm{O}(L)$, this orbit computation is a simple task for a computer.

The proof of Theorem C is fairly long and intricate. An overview of the techniques we used is shown in Figure 1 below. In this figure, even genera are pictured in blue, and odd ones are in yellow.[6] The presence of an arrow $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ means that $\mathcal{G}'$ can be deduced from $\mathcal{G}$ by the orbit method of type t. An important advantage of the orbit method is that it is unnecessary to provide any sharp isometry invariant for the lattices in the target genus to produce each isometry class once and only once. Nevertheless, we found it highly desirable to have such an invariant (that is fast to compute), at least for our larger genera. For instance, having such invariants allows one to directly check that the lattices given in our lists are pairwise non-isomorphic, hence form a complete list of representatives, by applying the known mass formula for $\mathcal{G}_{n,p}$ [CoS88b], and the Plesken-Souvignier algorithm. It is also useful for potential Hecke operator computations. This is achieved by the following result.

Theorem D. *For each (p, n)-entry in black in the Tables 1.2 and 1.3, the isometry classes in $\mathcal{G}_{n,p}$ are distinguished by their root systems. For the entries in red,[7] they are distinguished by their $\mathrm{BV}_{n,p}$-invariant.*

[3] Actually, this idea had already been used by Gauss in the last section of his *Disquisitiones*, in which he relates the representations of an integer n as a sum of three squares to the isometry classes in a certain genus of integral binary forms of determinant n.

[4] The few indeterminacies for $(26, 3)$ in Borcherds's list were settled in [Me18] (see [Ch25, §9] for a different approach).

[5] This holds for all cases but two cases: see the end of Sect. 7.2.

[6] We focus in this figure on the most important cases $n \geq 23$. The cases $n \leq 22$ are only easier: see the information in [ChTa].

[7] The case $(p, n) = (7, 27)$ could presumably be treated as well, but we did not try.

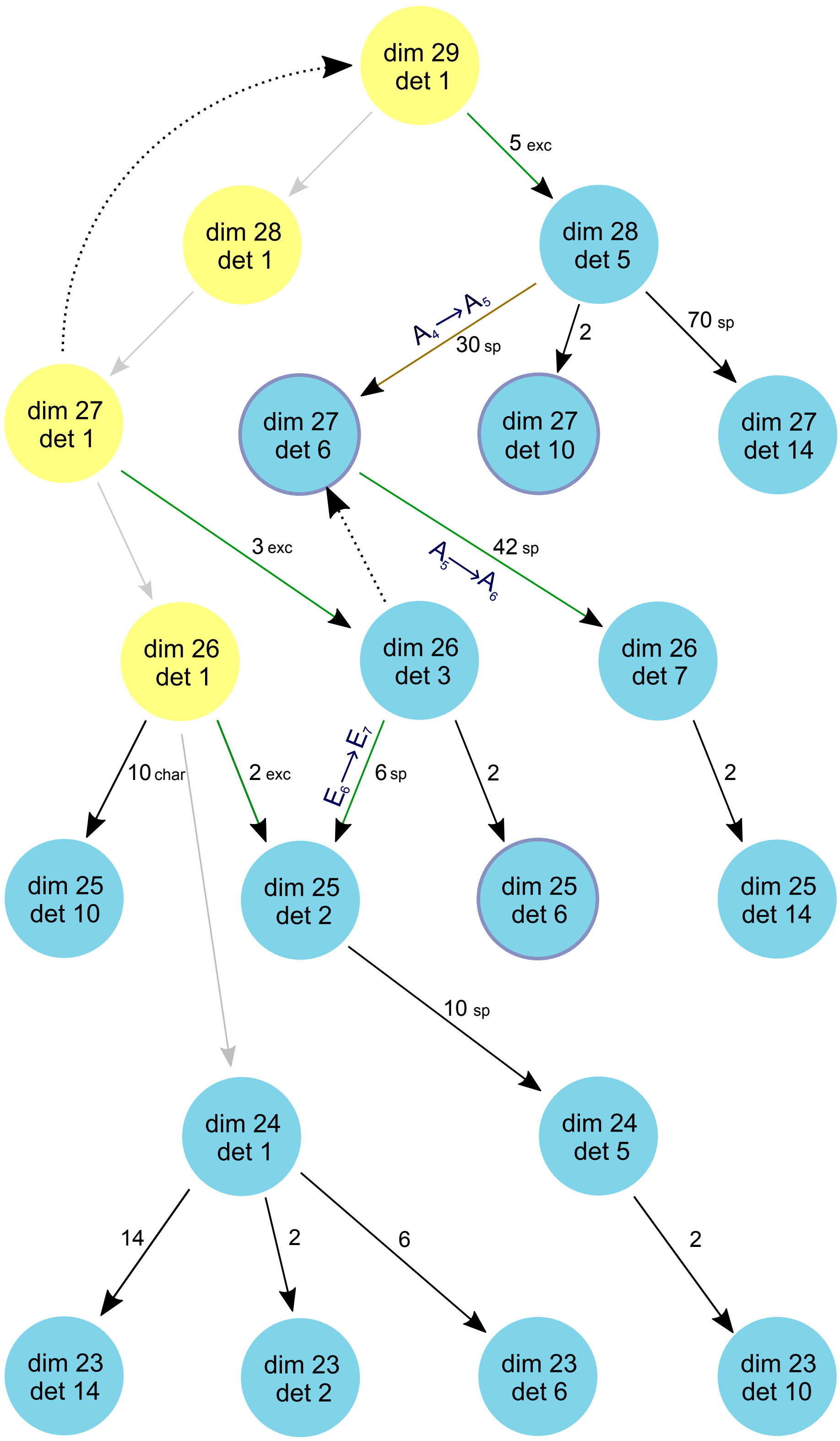


Figure 1: Unimodular lattices rule!
(*Leitfaden* for the proof of Theorem C)

We refer to Sect. 7.14 for the proof, and a (rather case-by-case) definition of $\mathrm{BV}_{n,p}$. These invariants are special cases of new invariants that we introduce in Sect. 2.14 and that we call the *marked* BV *invariants.* The marked BV invariants are defined more generally for arbitrary isometry classes of pairs (L, ι), where L is a lattice and ι is an embedding of a fixed lattice A into L; they refine the BV invariant of L by taking into account the embedding ι. Our strategy is then to identify the groupoid $\mathcal{G}_{n,p}$ with a suitable groupoid of such pairs (there are many different ways to do so using gluing constructions), and define the invariant of a lattice in $\mathcal{G}_{n,p}$ as the marked BV invariant of the corresponding pair. The amazing efficiency of these invariants, demonstrated by the tens of millions of lattices they distinguish, is an unexplained miracle to us.

Going back to the statement of Theorem C, we mention that for each lattice in our lists in [ChTa], we provide not only a (good) Gram matrix, but also its root system, its (reduced) mass, generators of its reduced isometry group, and for red (p, n)-entries, its (hashed) $\mathrm{BV}_{n,p}$ invariant. It is then easy to check that our lists are complete, thanks to the mass formula.

1.3. Unique orbit property for "exceptional" vectors

During the numerical applications of the orbit method described in § 1.2, and strikingly often, we observed that the set of vectors of a certain type in certain lattices L (or their dual $L^\sharp$) forms a single $\mathrm{O}(L)$-orbit. *This applies to the set of type t vectors of each $L \in \mathcal{G}$ for each green arrow $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ in Figure 1*, making the orbit method particularly straightforward to implement in these cases.

We naturally tried to find conceptual reasons for this: this is the topic of Sect. 5 and Sect. 6. Here is a striking example. Let L be a unimodular lattice. Following [BV01], we denote by $\mathrm{Exc}\, L$ the set of characteristic vectors v of L with norm $v \cdot v < 8$ (the *exceptional* vectors), and call L *exceptional* if $\mathrm{Exc}\, L$ is non-empty. The following result is proved in Sect. 6.

Theorem E. *Assume L is an exceptional unimodular lattice of rank n with $n \not\equiv 6, 7 \bmod 8$. Then $\mathrm{O}(L)$ acts transitively on $\mathrm{Exc}\, L$.*

This explains the green arrows of type `exc` in Figure 1. Known simple descriptions of $\mathrm{Exc}\, L$ for $n \equiv 0, 1, 2, 3, 4 \bmod 8$ actually make the theorem straightforward to prove in these cases. However, the case $n \equiv 5 \bmod 8$, of importance here since $29 \equiv 5 \bmod 8$, is much harder. In this case we prove a stronger result (Theorem 6.5): *if furthermore L has no norm 1 vector, then we have $|\mathrm{Exc}\, L| \leq 2n$ and the subgroup of $\mathrm{O}(L)$ generated by -1 and the Weyl group of L acts transitively on $\mathrm{Exc}\, L$.* For instance, we have $|\mathrm{Exc}\, L| = 2$ if L has no root, a previously unexplained observation in [ACh25]. Note that the statement of Theorem E does not hold for $n \equiv 6, 7 \bmod 8$ (but see Remark 6.11).

A general framework in which the number of orbits can be controlled is introduced in Sect. 5. Fix a root system R, with associated root lattice $\mathrm{Q}(R)$. We are interested in an even lattice M in a genus that is *opposite* to that of $\mathrm{Q}(R)$, in the sense that there is an isometry of finite quadratic spaces

$$\eta : M^\sharp/M \xrightarrow{\sim} -\mathrm{Q}(R)^\sharp/\mathrm{Q}(R).$$

We say that a class c in $\mathrm{Q}(R)^\sharp/\mathrm{Q}(R)$ is *fertile* if the minimum $\nu(c)$ of $\xi \cdot \xi$, over all $\xi \in \mathrm{Q}(R)^\sharp$ in the class c, is < 2. To each fertile c is attached a canonical root system $R_c \supset R$, whose Dynkin diagram is obtained from that of R by adding a single node and some edges; each possible such diagram arises for one and only one fertile c. Fix a fertile class c, as well as (M, η) as above. We are interested in the following set of (short) vectors in $M^\sharp$

$$\mathrm{Exc}_{c,\eta} M \;=\; \{v \in M^\sharp \;\mid\; \eta(v) = -c \text{ and } v \cdot v = 2 - \nu(c)\},$$

that we call the (c, η)-*exceptional vectors* of M.[8] A key fact is that these vectors are in natural bijection with the extensions to $\mathrm{Q}(R_c)$ of the natural embedding $\mathrm{Q}(R) \to U$, where $U \supset M \perp \mathrm{Q}(R)$ denotes the even unimodular lattice associated to the pair (M, η) by the gluing construction (Proposition 5.6). Studying the $\mathrm{W}(M)$-orbits of (c, η)-exceptional vectors of M then becomes equivalent to studying Weyl group orbits of embeddings of root systems into each other, a combinatorial problem which in favorable cases leads to unique orbit properties.

The proof of Theorem E in the case $n \equiv 5 \bmod 8$, consists in applying these ideas to the even part M of the unimodular lattice L with $R \simeq \mathbf{A}_3$ and to the fertile classes c leading to $\mathrm{R}_c \simeq \mathbf{A}_4$ or $\mathbf{D}_4$. Note that the classification of exceptional unimodular lattices of rank 29 easily follows from Theorem A: see Theorem 6.10 for a detailed study. By Theorem E the number of such lattices is exactly the same as the number of isometry classes in the genus $\mathcal{G}_{28,5}$, explaining the important $(5, 28)$-entry in Table 1.2: see Remark 7.6.

A second consequence of the general theory above is a relationship between the genera opposite to the genus of $\mathrm{Q}(R)$ and those opposite to the genus of $\mathrm{Q}(R_c)$, for each fertile class c of R (Corollary 5.8). When an arrow $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ in Figure 1 is a special case of this relationship, we add the label $R \to R_c$ to it. Here is an example occurring twice in the figure. For all $n \geq 1$ the pair $(\mathbf{A}_n, \mathbf{A}_{n+1})$ occurs as some (R, R_c), so we deduce a groupoid equivalence between:

(i) pairs (M, e), with M an even lattice of determinant $n + 1$ and $e \in M^\sharp$ a primitive vector satisfying $e \cdot e = \frac{n+2}{n+1}$,

(ii) pairs (N, w), with N an even lattice of determinant $n+2$ and w a generator of the group $N^\sharp/N$ satisfying $w \cdot w \equiv -\frac{n+1}{n+2} \bmod 2\mathbb{Z}$.

A generalization of this statement is given in Proposition 5.10. We have not been able to locate such statements elsewhere in the literature.

The unique orbit property for the pair $(R, R_c) = (\mathbf{A}_n, \mathbf{A}_{n+1})$ is discussed in Proposition 7.7. It explains the green arrow $\mathbf{A}_5 \to \mathbf{A}_6$ in Figure 1, as well as the brown arrow $\mathbf{A}_4 \to \mathbf{A}_5$: the unique orbit property holds in most cases, with an understood set of exceptions. We have essentially explained so far all the decorations in Figure 1, except for the purple circles!

1.4. Motivations and perspectives

As discussed throughout this introduction, the practical question of classifying integral Euclidean lattices of fixed dimension and determinant is a venerable one, whose historical developments have generally reflected theoretical and algorithmic advances in the study of Euclidean lattices. It is remarkable that the sizes of the genera classified in Sect. 1.1 and 1.2 are several orders of magnitude larger than those previously accessible. However, the motivations for studying

[8]This notion refines and generalizes that of exceptional vectors in odd unimodular lattices: see Remark 6.2.

such large genera might seem limited to the encyclopedic aspect and the computational challenge. Indeed, while the very first cases historically considered (say, in dimension at most 4) were linked to very simple and natural arithmetic questions, this can hardly be said to remain true in high dimensions.

Our main motivation for this work is actually quite different; it stems from the forthcoming companion paper [ChTb], in which we study automorphic forms and the rational cohomology of the group $\mathrm{GL}_n(\mathbb{Z})$, building upon a series of recent works by the authors [ChL19, ChR15, T17, ChT20]. As will be explained in that paper, new phenomena occur around dimension $n = 27$, but uncovering them requires the evaluation of certain local orbital integrals for orthogonal groups. While a direct computation of these integrals seems out of reach, they can be determined via a local-global method using the characteristic masses (in the sense of [Ch20]) of sufficiently many genera of even lattices of dimension ≤ 27 and prime (half-)determinant.

The specific genera needed for our computations in [ChTb] are the $\mathcal{G}_{n,p}$ for (n, p) in $\{(25, 3), (27, 3), (27, 5)\}$; they are circled in purple in Figure 1. This explains why, in the proofs of Theorems C and D in Sect. 7, we examine these genera in greater detail and sometimes provide alternative classification methods, given their primary importance for our applications. We finally mention that Theorem A and the methods above also enable the determination of $\mathcal{G}_{n,p}$ for many other pairs (n, p) (with $p \geq 11$ and $n \leq 27$), but we omit the discussion of these cases here.

Acknowledgments. Experiments presented in this work were carried out using the cluster cinaps of the LMO (Université Paris-Saclay) and the cluster "Cascade" of the PSMN (Pôle Scientifique de Modélisation Numérique, ENS de Lyon). We warmly thank the LMO and PSMN for sharing their machines.

Contents

2. General preliminaries on lattices

2.1. Basic notations, conventions, and terminology

If V is a Euclidean space, we usually denote by $x \cdot y$ its inner product. A *lattice* in V is a subgroup L generated by a basis of V. The *dual* of L is the lattice $L^\sharp := \{v \in V \mid w \cdot v \in \mathbb{Z},\ \forall w \in L\}$. We say that L is *integral* if we have $L \subset L^\sharp$. It is sometimes better to have a non embedded notion of an Euclidean integral lattice, and to think of them as a an abstract abelian group of finite rank L equipped with a positive definite symmetric $\mathbb{Z}$-bilinear form $L \times L \to \mathbb{Z}$, $(x, y) \mapsto x.y$, the ambient Euclidean space being then $V := L \otimes_{\mathbb{Z}} \mathbb{R}$. We freely use both points of view. **From now on, unless explicitly stated, the term** *lattice* **will always mean** *integral Euclidean lattice***.**

Let L be a lattice. The *norm* of a vector $v \in V$ is $v \cdot v$. The lattice L is called *even* if $v.v \in 2\mathbb{Z}$ for all $v \in L$, and *odd* otherwise. The *determinant* of L is the determinant of the *Gram matrix* $\mathrm{Gram}(e) = (e_i \cdot e_j)_{1 \le i,j \le n}$ of any $\mathbb{Z}$-basis $e = (e_1, \dots, e_n)$ of L. It is denoted $\det L$. We have $\det L \in \mathbb{Z}_{\ge 1}$, and we say that L is unimodular if $\det L = 1$.

For $i \ge 0$, the *configuration of vectors of L of norm $\le i$* (resp. of norm i) is the finite metric set

$$\mathrm{R}_{\le i}(L) = \{v \in L \mid v \cdot v \le i\} \text{ and } \mathrm{R}_i(L) = \{v \in L \mid v \cdot v = i\} \tag{2.1}$$

and we also set $\mathrm{r}_i(L) = |\mathrm{R}_i(L)|$. We denote by I_n the standard lattice $\mathbb{Z}^n$, with $x \cdot y = \sum_{i=1}^n x_i y_i$. Any lattice L may be uniquely written as $L = L_0 \perp L_1$ with $L_0 \simeq \mathrm{I}_m$ and $\mathrm{r}_1(L_1) = 0$, and we have $\mathrm{r}_1(L) = 2m$. A *root* in L is a vector of norm 2. The *root system* of L is $\mathrm{R}_2(L)$.

Example 2.2. (Root systems and lattices) *In this paper, a root system R, in an Euclidean space V, will always be assumed to have all its roots of norm* 2*, hence to be of* ADE *type* (*or simply laced*)*. We denote by* $\mathrm{Q}(R)$ *the even lattice it generates in* V (*root lattices*)*. We use bold fonts* $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_n$ *to denote isomorphism classes of root systems of those names, with the conventions* $\mathbf{A}_0 = \mathbf{D}_0 = \mathbf{D}_1 = \emptyset$ *and* $\mathbf{D}_2 = 2\mathbf{A}_1$*. We also denote by* A_n, D_n *and* E_n *the standard corresponding root lattices.*

Contrary to $\mathrm{R}_{\le 2}(L)$, there seems to be no known classification of the configurations of vectors of the form $\mathrm{R}_{\le 3}(L)$. We refer to [Ch25, §4.3] for a discussion of this problem, and to Sect. 2.14 below for some important invariants that we shall use.

The *isometry group* of the lattice L is $\mathrm{O}(L) = \{\gamma \in \mathrm{O}(V),\ \gamma(L) = L\}$ (a finite group). The *Weyl group* of L is the subgroup $\mathrm{W}(L) \subset \mathrm{O}(L)$ generated by the orthogonal reflections s_α about each $\alpha \in \mathrm{R}_2(L)$. This is a normal subgroup, isomorphic to the classical Weyl group $\mathrm{W}(R)$ of $R := \mathrm{R}_2(L)$. We define the *reduced isometry group* of L to be $\mathrm{O}(L)^{\mathrm{red}} := \mathrm{O}(L)/\mathrm{W}(L)$. Let ρ be the Weyl vector of a positive root system in $\mathrm{R}_2(L)$. The stabilizer $\mathrm{O}(L;\rho)$ of ρ in $\mathrm{O}(L)$ satisfies $\mathrm{O}(L) = \mathrm{W}(L) \rtimes \mathrm{O}(L;\rho)$, and in particular, is canonically isomorphic to $\mathrm{O}(L)^{\mathrm{red}}$. If $w \in \mathrm{W}(L)$ denotes the unique element satisfying $w(\rho) = -\rho$, then we have $-w \in \mathrm{O}(L;\rho)$. This element is nontrivial, hence has order 2, unless we have $-1 \in \mathrm{W}(L)$. We also denote by $\mathrm{W}(L)^{\pm}$ the subgroup of $\mathrm{O}(L)$ generated by -1 and $\mathrm{W}(L)$.

Lemma 2.3. *Let L be a lattice such that $-1 \in \mathrm{W}(L)$. Then $\mathrm{R}_2(L)$ has the same rank as L, and the abelian group $L^\sharp/L$ is killed by* 2*.*

Proof. The first assertion is clear. For the second, note that for $\alpha \in \mathrm{R}_2(L)$ and $x \in L^\sharp$, we have $\mathrm{s}_\alpha(x) = x - (\alpha \cdot x)\alpha \equiv x \bmod L$, so $\mathrm{W}(L)$ acts trivially on $L^\sharp/L$. □

Lemma 2.4. *Let U be a lattice and $V \subset U$ a sublattice. Let $\mathrm{O}(U,V)$ be the stabilizer of V in $\mathrm{O}(U)$, and $r : \mathrm{O}(U,V) \to \mathrm{O}(V)$ the restriction morphism. We have $\mathrm{W}(V) \subset r(\mathrm{W}(U) \cap \mathrm{O}(U,V))$ and $\mathrm{W}(V)^{\pm} \subset r(\mathrm{W}(U)^{\pm} \cap \mathrm{O}(U,V))$.*

Proof. For any root α of V (hence of U), we have $\mathrm{s}_\alpha \in \mathrm{W}(U) \cap \mathrm{O}(U,V)$ and $r(\mathrm{s}_\alpha) = \mathrm{s}_\alpha$. We conclude since $r(-\mathrm{id}_U) = -\mathrm{id}_V$. □

2.5. Groupoids and masses

The language of *groupoids* is particularly well-suited to the theory of lattices, providing both concise statements and useful points of view. Recall that a groupoid is a category $\mathcal{G}$ whose morphisms are all isomorphisms. Two groupoids are said to be *equivalent* if they are equivalent as categories. When the isomorphism classes of objects in a groupoid $\mathcal{G}$ form a set, we denote it by $\mathrm{Cl}(\mathcal{G})$. An equivalence of groupoids $\mathcal{G} \to \mathcal{G}'$ induces in particular a natural bijection $\mathrm{Cl}(\mathcal{G}) \overset{\sim}{\to} \mathrm{Cl}(\mathcal{G}')$, whenever defined.

If $\mathcal{C}$ is any collection of lattices, for instance any genus, then $\mathcal{C}$ may be viewed as a groupoid whose morphisms are the lattice isometries. As another example, the pairs (L,v) with L in $\mathcal{C}$ and $v \in L$, form a groupoid in a natural way: an isomorphism $(L,v) \to (L',v')$ is a lattice isometry $f : L \to L'$ with $f(v) = v'$. We will consider many variants of these examples in what follows, the groupoid structure being usually obvious from the context.

Let $\mathcal{G}$ be a groupoid with finitely many isomorphism classes of objects, and such that each object in $\mathcal{G}$ has a finite automorphism group. The *mass* of such a $\mathcal{G}$ is the rational number $\mathrm{mass}\,\mathcal{G} = \sum_{[x]\in \mathrm{Cl}(\mathcal{G})} \frac{1}{|\mathrm{Aut}_{\mathcal{G}}(x)|}$.

For instance, any collection $\mathcal{C}$ of lattices with bounded determinants has a mass, denoted by $\mathrm{mass}\,\mathcal{C}$. For such a $\mathcal{C}$ and any root system R, we also denote by $\mathcal{C}^R$ the full subgroupoid of lattices L in $\mathcal{C}$ with $\mathrm{r}_1(L) = 0$ and root system $\mathrm{R}_2(L) \simeq R$. The *reduced mass* of $\mathcal{C}^R$ is defined as $|\mathrm{W}(R)|\,\mathrm{mass}\,\mathcal{C}^R$. If $\mathcal{C}$ consists of the single lattice L, we simply write $\mathrm{mass}\,L = 1/|\mathrm{O}(L)|$, and the reduced mass of L is $|\mathrm{W}(R)|/|\mathrm{O}(L)| = 1/|\mathrm{O}(L)^{\mathrm{red}}|$ with $R = \mathrm{R}_2(L)$.

2.6. Residues and gluing constructions

Let L be a lattice.

(R1) The finite abelian group $\mathrm{res}\,L := L^\sharp/L$, sometimes called the *discriminant group* [Nik79], the *glue group* [CoS99] or the *residue*[9] [ChL19], is equipped with a non-degenerate $\mathbb{Q}/\mathbb{Z}$-valued symmetric bilinear form, defined by $(x,y) \mapsto x \cdot y \bmod \mathbb{Z}$. We have $|\mathrm{res}\,L| = \det L$. Any isometry $\sigma : L \to L'$ between lattices induces an isometry of finite bilinear spaces $\mathrm{res}\,\sigma : \mathrm{res}\ L \to \mathrm{res}\ L'$.

(R2) A subgroup $I \subset \mathrm{res}\,L$ is called *isotropic* if we have $x \cdot y \equiv 0$ for all $x,y \in I$, and a *Lagrangian* if we have furthermore $|I|^2 = |\mathrm{res}\,L|$. The map $\beta_L : M \mapsto M/L$ defines a bijection between the set of integral lattices containing L and the set of isotropic subgroups of res L. In this bijection we have $|M/L|^2 \det M = \det L$, in particular M/L is a Lagrangian of res L if and only if M is unimodular.

[9] We owe this pleasant notation and terminology to Jean Lannes. Not only does this choice avoid the overused term *discriminant*, but it also evokes the construction of Milnor's residue maps $\mathrm{W}(\mathbb{Q}) \to \mathrm{W}(\mathbb{F}_p)$ in the theory of Witt groups, as described in the appendix in [BLLV74].

(R3) Assume that L is even. Then the finite symmetric bilinear space res L may be equipped with a canonical quadratic form $\mathrm{q} : \mathrm{res}\, L \to \mathbb{Q}/\mathbb{Z}$ such that $\mathrm{q}(x+y) - \mathrm{q}(x) - \mathrm{q}(y) \equiv x \cdot y$, defined by $\mathrm{q}(x) = \frac{x \cdot x}{2} \bmod \mathbb{Z}$. When we view res L as equipped with such a structure, we rather denote it[10] by qres L, for *quadratic residue*. In the bijection β_L of (R2) above, the *even* lattices M correspond to the *quadratic isotropic* subgroups $I \subset \mathrm{qres}\, L$, *i.e.* satisfying $\mathrm{q}(I) = 0$.

We now state a form of the so-called *gluing construction*, which is essentially [Nik79, Prop. 1.5.1]. Recall that a subgroup A of a lattice L is called *saturated* if the abelian group L/A is torsion free, or equivalently, if A is a direct summand of L as $\mathbb{Z}$-module. If X is a bilinear or quadratic space, $-X$ denotes the same space but with the opposite bilinear or quadratic form.

Proposition 2.7. *Let A be a lattice and $H \subset \mathrm{res}\ A$ a subgroup equipped with the induced $\mathbb{Q}/\mathbb{Z}$-valued bilinear form. The following groupoids are naturally equivalent:*

(i) pairs (L, ι) with L a lattice and $\iota : A \to L$ an isometric embedding with saturated image such that the natural map $L \to \mathrm{res}\ A$ has image H.

(ii) pairs (B, η) with B a lattice and $\eta : H \to -\mathrm{res}\ B$ an isometric embedding.

In this equivalence, we have $B \simeq L \cap \iota(A)^{\perp}$ and $|H|^2 \det L = \det A \det B$. Assuming furthermore $H = \mathrm{res}\ A$, we also have $\mathrm{res}\, B \simeq -\mathrm{res}\ A \perp \mathrm{res}\ L$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the two respective groupoids in (i) and (ii). It is understood that a morphism $(L, \iota) \to (L', \iota')$ in $\mathcal{A}$ is an isometry $\sigma : L \xrightarrow{\sim} L'$ such that $\sigma \circ \iota = \iota'$, and similarly, that a morphism $(B, \eta) \to (B', \eta')$ in $\mathcal{B}$ is an isometry $\sigma : B \xrightarrow{\sim} B'$ with $\mathrm{res}\, \sigma \circ \eta = \eta'$. We also denote respectively by $\mathrm{O}(L, \iota)$ and $\mathrm{O}(B, \eta)$ the groups $\mathrm{Aut}_{\mathcal{A}}(L, \iota)$ and $\mathrm{Aut}_{\mathcal{B}}(B, \eta)$.

Proof. Fix (L, ι) in $\mathcal{A}$ and denote by B the orthogonal of $\iota(A)$ in L. By (R2), the subgroup $I := L/(\iota(A) \perp B)$ of res $\iota(A) \perp$ res B is totally isotropic. Both projections $\mathrm{pr}_A : I \to \mathrm{res}\ \iota(A)$ and $\mathrm{pr}_B : I \to \mathrm{res}\ B$ are injective as $\iota(A)$ and B are saturated in L. The image of pr_A is $(\mathrm{res}\ \iota)(H)$ by assumption. So the formula $\eta := \mathrm{pr}_B \circ \mathrm{pr}_A^{-1} \circ \mathrm{res}\ \iota$ defines an isometric embedding $H \to -\mathrm{res}\, B$, and we have $I = \mathrm{I}(\iota, \eta)$ with

$$\mathrm{I}(\iota, \eta) := \{(\mathrm{res}\ \iota)(h) + \eta(h) \mid h \in H\}. \tag{2.2}$$

This construction defines a natural functor $G : \mathcal{A} \to \mathcal{B}, (L, \iota) \mapsto (B, \eta)$. An automorphism of (L, ι) in $\mathcal{A}$ has the form $\mathrm{id}_{\iota(A)} \times g$ with $g \in \mathrm{O}(B)$, so the definition of $\mathrm{I}(\iota, \eta)$, and the injectivity of res ι, show that G is fully faithful. Conversely, fix (B, η) in $\mathcal{B}$. Define $\mathrm{I}(\mathrm{id}, \eta)$ as above in res $A \perp$ res B. This is an isotropic subspace, hence by (R2) of the form $L/(A \perp B)$ for some unique lattice L, which defines a natural object (L, id) in $\mathcal{A}$ whose image under G is (B, η). This shows that G is an equivalence.

The assertion $|H|^2 \det L = \det A \det B$ follows from $|I| = |H|$.

Assume finally $H = \mathrm{res}\, A$. The subspace $\eta(H) \subset \mathrm{res}\, B$ is isometric to $-\mathrm{res}\, A$, and in particular nondegenerate, so we may write $\mathrm{res}\, B = \eta(H) \perp S$. The subspace $\mathrm{I}(\iota, \eta) \subset \mathrm{res}\, \iota(A) \perp \eta(H)$ is a Lagrangian, so we deduce $\mathrm{I}(\iota, \eta)^{\perp} = \mathrm{I}(\iota, \eta) \perp S$, and then $\mathrm{res}\, L \simeq \mathrm{I}(\iota, \eta)^{\perp}/\mathrm{I}(\iota, \eta) \simeq S$. ☐

[10] This is important since on occasion we will really want to consider res L when L is even, and not qres L.

Remark 2.8. *(i) (unimodular case) In this construction, we have* $\det L = 1 \iff H = \operatorname{res} A$ *and* $\operatorname{res} A \simeq -\operatorname{res} B$.

(ii) (even variant) Assume A *is even. Pairs* (L, ι) *with* L *even correspond to pairs* (B, η) *with* B *even and* η *an isometric embedding* $H \to -\operatorname{qres} B$, *and if* $H = \operatorname{qres} A$ *we have* $\operatorname{qres} B \simeq -\operatorname{qres} A \perp \operatorname{qres} L$.

We now discuss the case $\operatorname{rk} A = 1$. Let L be a lattice and let $v \in L$ be nonzero. Recall that v is called *primitive* if $\mathbb{Z}v$ is saturated in L. Define the *modulus of* v *in* L to be the unique integer $\mathrm{m}(v) \geq 1$ satisfying

$$\{v \cdot x \mid x \in L\} = \mathrm{m}(v)\mathbb{Z}. \tag{2.3}$$

This is also the largest integer m with $v \in mL^\sharp$, *i.e.* such that $v \bmod mL$ is in the kernel of the natural $\mathbb{Z}/m$-valued bilinear form on L/mL. The integer $\mathrm{m}(v)$ always divides $v \cdot v$, as well as $\det L$ if v is primitive (consider the Gram matrix of a basis of L containing v).

Example 2.9. *Fix an integer* $d \geq 1$ *and a divisor* m *of* d. *The following natural groupoids are equivalent:*

(i) pairs (L, v) *with* L *even and* $v \in L$ *primitive with* $v \cdot v = d$ *and* $\mathrm{m}(v) = m$.

(ii) pairs (N, w) *with* N *even and* $w \in \operatorname{qres} N$ *of order* d/m *satisfying* $\mathrm{q}(w) \equiv -\frac{m^2}{2d} \bmod \mathbb{Z}$.

In this equivalence, we have $N \simeq L \cap v^\perp$ *and* $m^2 \det N = d \det L$.

For $d \geq 1$ we denote by $\langle d \rangle$ the rank 1 lattice $\mathbb{Z}e$ with $e \cdot e = d$. It is equivalent to give an isometric embedding $\iota : \langle d \rangle \to L$ (with saturated image) and a (primitive) norm d element of L, namely $\iota(e)$.

Proof. Set $A = \langle d \rangle$. For L a lattice and $\iota : A \to L$ an isometric embedding with $v = \iota(e)$ primitive, the image of the orthogonal (and surjective) projection $L \to A^\sharp$ is $mA^\sharp$ with $m = \mathrm{m}(v)$. The subgroup $H = m \operatorname{res} A$ of $\operatorname{res} A$ is cyclic of order d/m and generated by me/d, whose norm is m^2/d. It is thus equivalent to give an isometric embedding $\eta : H \to -\operatorname{qres} N$ and the element $\iota(e/d)$, which can be any element w of $\operatorname{qres} N$ of order d/m with $\mathrm{q}(w) = -\frac{m^2}{2d}$. The statement follows then from Proposition 2.7 in the even case (Remark 2.8 (ii)). ☐

Definition 2.10. *A primitive nonzero vector* v *in a lattice* L *is called* special *if it satisfies* $\mathrm{m}(v) = \det L$.

It is equivalent to require $v \in (\det L)L^\sharp$, as this implies $\det L \mid \mathrm{m}(v)$ and the opposite divisibility always holds. Special vectors will play a role in Sect. 7.

2.11. Characteristic vectors and even sublattices

If L is a lattice, a *characteristic vector* of L is an element $\xi \in L^\sharp$ such that for all $x \in L$ we have $x \cdot \xi \equiv x \cdot x \bmod 2$. Characteristic vectors always exist and form a unique class $\operatorname{Char} L$ in $L^\sharp/2L^\sharp$. For $\xi \in \operatorname{Char} L$, the lattice

$$L^{\mathrm{even}} = \{x \in L \mid x \cdot \xi \equiv 0 \bmod 2\} = \{x \in L \mid x \cdot x \equiv 0 \bmod 2\}$$

is the largest even sublattice of L, and called the *even part* of L; it has index 2 whenever L is odd.

Proposition 2.12. *Assume (L,ι) and (B,η) correspond to each other in the equivalence of Proposition 2.7, with $H = \operatorname{res} A$. Then B is even if and only if A contains a characteristic vector of L.*

Proof. We may view L as a sublattice of $(A \perp B)^\sharp = A^\sharp \perp B^\sharp$ (hence ι as an inclusion), and we denote by $\operatorname{pr}_A : L \to A^\sharp$ the orthogonal projection, with kernel $B^\sharp \cap L = B$. By assumption on H we have $\operatorname{pr}_A(L) = A^\sharp$, which implies

$$\operatorname{pr}_A^{-1}(2A^\sharp) = B + 2L. \tag{2.4}$$

We have a natural perfect pairing $L/2L \times L^\sharp/2L^\sharp \to \mathbb{Z}/2\mathbb{Z}$. For this pairing, the orthogonal of $\operatorname{Char} L$ is $L^{\mathrm{even}}/2L$, and that of $A + 2L^\sharp$ is the orthogonal mod 2 of A in $L/2L$, which is also $\operatorname{pr}_A^{-1}(2A^\sharp)/2L = (B + 2L)/2L$ by (2.4). So the image of A in $L^\sharp/2L^\sharp$ contains $\operatorname{Char} L$ if, and only if, $B \subset L^{\mathrm{even}}$. □

Example 2.13. *For every integer $d \geq 1$, there is an equivalence between the natural groupoids of:*

(i) pairs (L, v) with L an odd unimodular lattice and $v \in \operatorname{Char} L$ primitive of norm d,

(ii) pairs (N, w) with N an even lattice of determinant d, and $w \in \operatorname{qres} N$ of order d satisfying $\mathrm{q}(w) \equiv \frac{1}{2}(1 - \frac{1}{d}) \bmod \mathbb{Z}$.

In this equivalence, we have $N \simeq L \cap v^\perp$.

Proof. Apply Proposition 2.7 to $A = \langle d \rangle = \mathbb{Z}e$ and $H = \operatorname{res} A$. As in Example 2.9 isometric embeddings $\iota : A \to L$ with saturated image correspond bijectively to primitive elements of L of norm d, by mapping ι to $v = \iota(e)$. Any primitive element v of a unimodular lattice satisfies $\mathrm{m}(v) = 1$. As H is cyclic of order d and generated by e/d, it is the same to give an isometric embedding $\eta : \operatorname{res} A \to -\operatorname{res} N$ and an order d element $w \in \operatorname{res} N$ with $w \cdot w \equiv -1/d \bmod \mathbb{Z}$ (namely, $w = \eta(e/d)$). Assume (L, v) and (N, w) correspond to each other. We have $\det L = 1 \iff \det N = d$, and we assume that this condition holds. We may also assume $N = L \cap v^\perp$.

Assume that L is odd and $v \in \operatorname{Char} L$. By Proposition 2.12 we know that N is even. We either have $\mathrm{q}(w) \equiv -\frac{1}{2d} \bmod \mathbb{Z}$ or $\mathrm{q}(w) \equiv \frac{1}{2}(1 - \frac{1}{d}) \bmod \mathbb{Z}$. The first case is not possible as L would be even (Example 2.9).

Conversely assume that N is even and $\mathrm{q}(w) \equiv \frac{1}{2}(1 - \frac{1}{d}) \bmod \mathbb{Z}$. As H is cyclic generated by the class of e/d, the Lagrangian of $\operatorname{res} \iota(A) \perp \operatorname{res} N$ defining L is generated by the class of an element $f \in L$ of the form $f = \frac{v}{d} + n$, with $n \in N^\sharp$ and $n \equiv w \bmod N$. We have $f \cdot f = \frac{1}{d} + n \cdot n \in 1 + 2\mathbb{Z}$ and so L is odd. It then follows from Proposition 2.12 that v belongs to $\operatorname{Char} L$. □

2.14. Marked BV invariant of depth d

The BV invariant of a lattice L was introduced in [ACh25, §3]. It is a variant of some polynomial invariants defined by Bacher and Venkov in [BV01] (see Remark 3.3 in [ACh25]), and is especially simple to implement on a computer. This invariant will play a crucial role in the proof of Theorem 4.1, and we will now explain a generalization of it that we will use to study non-unimodular lattices in Sect. 7.

Fix a lattice A, as well as a $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_r$ of A. We are interested in finding invariants for the isometry classes of pairs (L, ι) with $\iota : A \to L$ an isometric embedding. Fix such a pair (L, ι) and choose some integer d (the *depth*). Consider the graph $\mathcal{G}_{\leq d}(L)$ whose set V of vertices consists of the unordered pairs $\{\pm v\}$ with $v \in L$ a nonzero vector of norm $\leq d$, and with an edge between $\{\pm v\}$ and $\{\pm v'\}$ if and only if $v \cdot v' \equiv 1 \bmod 2$. Let M be the adjacency $V \times V$-matrix of this graph. To each $\{\pm v\} \in V$, we associate the following two objects:

(i) The multiset $\mathrm{m}(v)$ of entries in the v-th column of M^2,

(ii) The set of r-tuples $\mathrm{a}(v) = \{\pm(a_1, \ldots, a_r)\} \subset \mathbb{Z}^r$ with $a_i = \iota(\alpha_i) \cdot v$.

In other words, $\mathrm{m}(v)$ is the collection of *numbers of length-2 paths in* $\mathcal{G}_{\leq d}(L)$ *starting at* v *and ending at each fixed vertex.*

Definition 2.15. *Fix a lattice A, a basis $\alpha = (\alpha_1, \ldots, \alpha_r)$ of A, and an integer $d \geq 1$. For a pair (L, ι) with L a lattice and $\iota : A \to L$ an isometric embedding, the* BV*-invariant of (L, ι) of depth d, and relative to α, is the multiset*

$$\mathrm{BV}_d^{\alpha}(L, \iota) = \{\{ \, (\mathrm{m}(v), \mathrm{a}(v)) \mid v \in V \, \}\}.$$

This is clearly an invariant of the isomorphism class of (L, ι). Changing α into $g(\alpha)$ with $g \in \mathrm{GL}_r(\mathbb{Z})$ amounts to applying g to each $\mathrm{a}(v)$. The exact choice of α is thus not essential, and we shall often simply write $\mathrm{BV}_d(L, \iota)$. The original, unmarked, BV-invariant, denoted $\mathrm{BV}(L)$, is the case $A = \{0\}$, $\alpha = ()$ and $d = 3$.

Remark 2.16. (Variants) There are several variants of BV which are both finer and more natural. For instance, we could rather have defined M as the $V \times V$-matrix $(|v \cdot w|)_{(\pm v, \pm w) \in V \times V}$ (*absolute variant*) or doubled the size of the set V of vertices of $\mathcal{G}_{\leq d}(L)$ by distinguishing a vector and its opposite (*signed variant*). The reason for our choices is purely practical. Indeed, the computation of BV requires[11] $\mathrm{O}(n^3)$ operations with $n = |\mathrm{R}_{\leq d}(L)|/2$, as the most time consuming part is to compute the square of the adjacency matrix M above, which is of size n. The $\pm$ tricks thus allows to divide the computation time by 8. Also, the fact that our M has only $\{0, 1\}$-coefficients allows to substantially speed up the computation of M^2 (see §2.17 (f)). Fortunately, these choices turned out to be harmless in practice, presumably because the graphs $\mathcal{G}_{\leq d}(L)$ we encountered are so random or complicated. See however Remark 7.22 for a case where the absolute variant is needed.

These invariants $\mathrm{BV}_d^{\alpha}(L, \iota)$ are typically efficient in practice when we are interested in pairs (L, ι) for which d is large enough so that $\mathrm{R}_{\leq d}(L)$ generates L over $\mathbb{Z}$. The tension is that when $\mathrm{R}_{\leq d}(L)$ is too large, $\mathrm{BV}_d^{\alpha}(L, \iota)$ is too long to compute. Our current implementation, discussed in Sect.2.17 (f), runs in about 150 `ms` for $n \approx 1300$, which is more than 10 times faster than the previous implementation of BV in [ACh25]. The most important open question about $\mathrm{BV}_d^{\alpha}(L, \iota)$ is to provide a conceptual explanation of why it is so sharp in the situations occurring in this paper. This is purely empirical so far.

[11] We ignore here the asymptotically faster algorithms for matrix multiplication, as the matrix sizes occurring in this paper do not warrant using them.

2.17. Lattice algorithms

In our computations, we make extensive use of several classical lattice algorithms. We used the open-source computer algebra system [PARI]. In order to perform our computations as efficiently as possible, we had to refine several of these algorithms (or their implementations in PARI/GP) and use specific variants tailored for the lattices we consider (such as unimodular lattices). This is actually an important aspect of our work, although we only briefly discuss it below, referring to [ChTa] for our scripts and a more furnished documentation. These algorithms have been mostly developed by the second author and will be the subject of an independent publication (see already [T24]).

(a) The Fincke-Pohst algorithm [FP85] is used to determine the short vectors in a lattice L, *i.e.* $\mathrm{R}_{\leq i}(L)$, from a Gram matrix g of L. This algorithm is implemented as `qfminim`(g, i) in PARI/GP. As this implementation uses floating-point numbers, hence approximations, the second author implemented an exact variant `eqfminim` of it.

(b) The root system of a lattice L, and its various attached objects (Weyl vector, simple roots, irreducible components, isomorphism class...) are easily determined from $\mathrm{R}_2(L)$: see e.g. [Ch25, Remark 4.2].

(c) The Plesken-Souvignier algorithm [PlS97] returns the order and generators of $\mathrm{O}(L)$ from a Gram matrix of L. It is possible to exploit our knowledge of root systems to improve this algorithm. Indeed, as explained in [Ch25, Remark 4.4], the extra bilinear forms allowed in Souvignier's code (PARI's `qfauto`) make it possible to directly compute the *reduced* isometry group of L, which is often much faster. As most of our lattices have a *"trivial"*[12] reduced isometry group, this is an important simplification. A much more efficient implementation of this idea was developed by Taïbi in [T24] and led to the function `qfautors` in [ChTa].

(d) For the Plesken-Souvignier algorithm, or its variant above, to be efficient, we must first find a $\mathbb{Z}$-basis of the lattice with shortest possible vectors. In [ACh25, §4], a simple probabilistic algorithm to find such bases is given. The function `goodbasis`, developed by the second author, takes a further step by combining this idea and an LLL reduction. All the Gram matrices given in [ChTa] have been found by this algorithm.

(e) The function `orbmod2` in [ChTa] takes as input a list of matrices $m_1, \ldots,$ m_r in $\mathrm{M}_n(\mathbb{Z})$ with odd determinant and returns representatives for the orbits of the subgroup $\langle m_1, \ldots, m_r \rangle \subset \mathrm{GL}_n(\mathbb{Z}/2)$ acting on $(\mathbb{Z}/2)^n$, as well as the cardinality of each orbit. This is a standard algorithm in computational group theory, adapted to the case at hand for greater efficiency ("population count" instructions allow for fast matrix multiplication over $\mathbb{Z}/2$).

(f) We implemented the invariant $\mathrm{BV}_d^{\alpha}(L, \iota)$ of Definition 2.15 as a function `fast_marked_HBV`, taking as input the Gram matrix of L and α. Of course

[12] By *"trivial"* here we mean $\mathrm{O}(L)^{\mathrm{red}} = \langle -w \rangle$, see Sect. 2.1.

it would be inefficient (in both space and time) to compute actual multisets, so we apply hash functions to them to obtain an integer between 0 and $2^N - 1$ (we chose a hash function with $N = 64$). After computing the set of vertices V using the Fincke-Pohst algorithm the main step consists of computing the square of the adjacency matrix M. We take advantage of the fact that M has coefficients in $\{0, 1\}$ to save space when storing M (using approximately $|V|^2$ bits) and to compute its square efficiently (again, using "population count" instructions). Note that it is not necessary to store M^2, as we only need the hash of the multiset of its columns.

3. Unimodular lattices having a pair of orthogonal roots

As explained in the introduction § 1.1, The starting point of our approach to Theorem A is the following general relation between rank $n + 2$ unimodular lattices having a pair of orthogonal roots and rank n unimodular lattices. We denote by Q_0 the root lattice $\mathrm{A}_1 \perp \mathrm{A}_1$.

Proposition 3.1. *For any integer $n \geq 1$, there are equivalences between the three following natural groupoids:*

(i) pairs (L, e) with L a unimodular lattice of rank n and e an element of $L/2L$ with $e \cdot e \equiv 2 \bmod 4$,

(ii) lattices M of rank n such that $\mathrm{res}\, M$ *is isomorphic to* $\mathrm{res}\, \mathrm{Q}_0$*,*

(iii) pairs $(U, \{\alpha, \beta\})$ with U a unimodular lattice of rank $n + 2$ and $\{\alpha, \beta\}$ a pair of orthogonal roots in U with $\frac{\alpha+\beta}{2} \notin U$.

If L is a unimodular lattice, its index 2 subgroups are the

$$\mathrm{M}_2(L; e) := \{v \in L \,|\, v \cdot e \equiv 0 \bmod 2\}, \tag{3.1}$$

with $e \in L/2L$ nonzero (and uniquely determined).

Lemma 3.2. *(i) For any lattice M satisfying* $\mathrm{res}\, M \simeq \mathrm{res}\, \mathrm{Q}_0$*, there is a unique unimodular lattice $M \subset L \subset M^\sharp$, and it satisfies $L/M \simeq \mathbb{Z}/2$.*

(ii) Let L be a unimodular lattice, e a nonzero element in $L/2L$, and set $M = \mathrm{M}_2(L; e)$. We have $\mathrm{res}\, M \simeq \mathrm{res}\, \mathrm{Q}_0$ *if and only if* $e \cdot e \equiv 2 \bmod 4$.

Proof. We have $\mathrm{res}\, \mathrm{Q}_0 = \mathbb{Z}/2\, x \perp \mathbb{Z}/2\, y$ with $x \cdot x \equiv y \cdot y \equiv 1/2$, and the third nonzero element $z = x + y$ satisfies $z \cdot z \equiv 0$ and is unique in $\mathrm{res}\, \mathrm{Q}_0$ with that property. This proves assertion (i) by §2.6 (R2).

We now prove (ii). By unimodularity of L, there is $f \in L$ with $e \cdot f \equiv 1 \bmod 2$. The three nonzero elements of $M^\sharp/M$ are thus the classes of f, $\frac{e}{2}$ and $\frac{e}{2} + f$. We conclude as we have $f \cdot f \equiv 0 \bmod \mathbb{Z}$, $\frac{e}{2} \cdot f \equiv 1/2 \bmod \mathbb{Z}$ and $(\frac{e}{2} + f) \cdot (\frac{e}{2} + f) \equiv \frac{e}{2} \cdot \frac{e}{2} \equiv \frac{e \cdot e}{4} \bmod \mathbb{Z}$. □

We now prove Proposition 3.1.

Proof. We respectively denote by $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$ the three groupoids defined in (i), (ii) and (iii) of the statement.

We first prove the equivalence between $\mathcal{B}_n$ and $\mathcal{C}_n$. Write $\mathrm{Q}_0 = \mathbb{Z}\alpha_0 \perp \mathbb{Z}\beta_0$. Note that the map $\iota \mapsto (\iota(\alpha_0), \iota(\beta_0))$ identifies the embeddings $\iota : \mathrm{Q}_0 \to U$ with the *ordered* pair (α, β) of orthogonal roots in U. Also, in this bijection, $\iota(\mathrm{Q}_0)$ is saturated in U if and only if we have $\frac{\alpha+\beta}{2} \notin U$. The equivalence between $\mathcal{B}_n$ and $\mathcal{C}_n$ is thus a variant of Proposition 2.7 in the case $\det L = 1$ and $A = \mathrm{Q}_0$ (see Remark 2.8 (i)). Fix $(U, \{\alpha, \beta\})$ in $\mathcal{C}_n$, we have seen that $D := \mathbb{Z}\alpha \perp \mathbb{Z}\beta$ is a saturated rank 2 sublattice of U. Following the analysis in the proof of Proposition 2.7, for $M := D^\perp \cap U$ then $U/(D \perp M) \subset \mathrm{res}\, D \perp \mathrm{res}\, M$ is the graph of an isometry $\mathrm{res}\, D \overset{\sim}{\to} -\mathrm{res}\, M$. As we have $D \simeq \mathrm{Q}_0$ and $-\mathrm{res}\, \mathrm{Q}_0 \simeq \mathrm{res}\, \mathrm{Q}_0$, we have a natural, essentially surjective, functor $\mathcal{C}_n \to \mathcal{B}_n, (U, \{\alpha, \beta\}) \mapsto M$. In order to prove that this is an equivalence, it only remains to show that the group morphism $\mathrm{O}(U, \{\alpha, \beta\}) \to \mathrm{O}(M), \sigma \mapsto \sigma_{|M}$, is bijective. But this follows from the fact that $\mathrm{O}(U, \{\alpha, \beta\})$ is the stabilizer of $U/(D \perp M)$ in $\mathrm{O}(D, \{\alpha, \beta\}) \times \mathrm{O}(M)$ and the fact that the natural map $\mathrm{O}(D, \{\alpha, \beta\}) \to \mathrm{O}(\mathrm{res}\, D)$ is surjective: the involution of D exchanging α and β induces the unique nontrivial element of $\mathrm{O}(\mathrm{res}\, D)$.

By Lemma 3.2 (ii), we have a well-defined functor $\mathcal{A}_n \to \mathcal{B}_n, (L, e) \mapsto \mathrm{M}_2(L; e)$. It is an equivalence by the same lemma, part (i). □

Although we shall not use it, it is easy to deduce from description (i) that the mass of these three groupoids, say for odd n, is the mass of the genus of rank n unimodular lattices times $\sum_{0 \leq i \leq (n-2)/4} \binom{n}{2+4i}$. Proposition 3.1 together with the proof of Lemma 3.2 (ii) show the:

Corollary 3.3. *Let L_n be a list of representatives of all rank n unimodular lattices. The following algorithm returns a list U_{n+2} containing representatives of all rank $n+2$ unimodular lattices having a pair of orthogonal roots. Start with an empty list U_{n+2} and then, for each $L \in \mathrm{L}_n$:*

(A1) Determine a set $\mathcal{E}(L)$ of representatives of the orbits of $\mathrm{O}(L)$ acting on $L/2L$, and only keep those $e \in \mathcal{E}(L)$ with $e \cdot e \equiv 2 \bmod 4$.

(A2) For each $e \in \mathcal{E}(L)$, choose $f \in L$ with $e \cdot f \equiv 1 \bmod 2$, and add to U_{n+2} the lattice

$$U(L, e) := (\mathrm{M}_2(L; e) \perp \mathrm{Q}_0) + \mathbb{Z}\,\frac{e + \alpha_0}{2} + \mathbb{Z}\frac{e + 2f + \beta_0}{2}$$

with $\mathrm{Q}_0 = \mathbb{Z}\alpha_0 \perp \mathbb{Z}\beta_0$.

We now discuss the redundancies in the list U_{n+2} produced by this algorithm. By construction, the isometry class of each U in U_{n+2} appears as many times as the number of $\mathrm{O}(U)$-orbits of pairs of orthogonal, and "saturated", roots in U. We now explain a simple way to minimize these redundancies. Let us totally order the isomorphism classes of irreducible ADE root systems, say

$$\mathbf{A}_1 \prec \mathbf{A}_2 \prec \cdots \prec \mathbf{A}_n \prec \cdots \prec \mathbf{D}_4 \prec \cdots \prec \mathbf{D}_n \prec \cdots \prec \mathbf{E}_6 \prec \mathbf{E}_7 \prec \mathbf{E}_8.$$

We extend this total ordering $\prec$ on all isotypic root systems $m\mathbf{X}_n$ using as well a lexicographic ordering on $(m, \mathbf{X}_n)$: *e.g.* $\mathbf{E}_6 \prec 2\mathbf{D}_5 \prec 3\mathbf{A}_1$. We denote by $\mathrm{m}_1(R)$ the number of irreducible components of R occurring with multiplicity 1.

Definition 3.4. *Let R be a root system and $\{\alpha,\beta\} \subset R$ a pair of orthogonal roots. We denote by $C_1 \prec C_2 \prec \cdots \prec C_r$ the isotypic components of R. We say that $\{\alpha,\beta\}$ is* relevant *if one of the following exclusive assertions*[13] *holds:*

(i) $\mathrm{m}_1(R) \geq 2$, *and $\{\alpha,\beta\}$ meets both C_1 and C_2.*

(ii) $\mathrm{m}_1(R) = 1$, *and $\{\alpha,\beta\} \subset C_1$.*

(iii) $\mathrm{m}_1(R) = 1$, *$C_1 \simeq \mathbf{A}_1$ or $\mathbf{A}_2$, and $\{\alpha,\beta\}$ meets both C_1 and C_2,*

(iv) $\mathrm{m}_1(R) = 0$, *and α and β are in distinct irreducible components of C_1.*

We have the following trivial fact:

Fact 3.5. *If a root system R has a pair of orthogonal roots, it also has a relevant such pair.*

For many R there are much less relevant pairs than arbitrary ones.

Example 3.6. *Assume (L,e) and $(U,\{\alpha,\beta\})$ correspond to each other as in Proposition 3.1, as well as $\mathrm{m}_1(R) \geq 2$ with $R = \mathrm{R}_2(U)$. Then R has a unique $\mathrm{W}(R)$-orbit of relevant pairs of orthogonal roots.*

Corollary 3.7. *Let L_n be as in Corollary 3.3. Then we produce a list U_{n+2} containing representatives of all rank $n+2$ unimodular lattices having a pair of orthogonal roots and no norm 1 vectors by modifying step (A2) into :*

(A2)' only add $U(L;e)$ to the list U_{n+2} if it has no norm 1 vector, and if $\{\alpha_0,\beta_0\}$ is relevant for the root system of $U(L;e)$.

Indeed, this follows from Fact 3.5 and from the fact that for any pair $\{\alpha,\beta\}$ of orthogonal roots the element $\frac{\alpha+\beta}{2}$ has norm 1. We end this discussion by observing that Proposition 3.1 also furnishes mass relations between L's and U's (hence useful ways to check computations). Here is an especially simple example:

Example 3.8. *In the situation of Example 3.6, assume $\{\alpha,\beta\}$ is relevant. Its orbit under $\mathrm{O}(U)$ is in bijection with $R_\alpha \times R_\beta$, where R_α, R_β are the irreducible components of R containing respectively α and β. Proposition 3.1 implies $|\mathrm{O}(U)| = \frac{|R_\alpha||R_\beta|}{\mathrm{n}(e)}|\mathrm{O}(L)|$, where $\mathrm{n}(e)$ is the size of the $\mathrm{O}(L)$-orbit of $e \in L/2L$.*

4. An application: the classification of rank 29 unimodular lattices

We now explain the details of our proof of Theorem A. By the classification of rank 28 unimodular lattices in [ACh25], it is enough to prove the following (see also Corollary 4.4 below).

Theorem 4.1. *There are* 38 592 290 *isometry classes of unimodular lattices of rank* 29 *with no norm* 1 *vectors. All these are distinguished by their* BV *invariant. A list of Gram matrices of representatives is given in* [ChTa].

[13] We could sharpen a bit this definition when R has an irreducible component with multiplicity 2, or in case (ii) with $C_1 \simeq \mathbf{D}_n$, but this would not affect significantly its applications.

The given list [14] of Gram matrices allows a simple proof of the theorem: check that they are all positive definite, integral, of determinant 1, with minimum > 1, have different BV invariant, and that the sum of their masses is the rational given by the mass formula, namely (see [CoS99, §16.2] and [Ch25, §6.4]):

$$96831378835988415227001493062183860198566 01/651885428274440745704591720448000000 00$$

As an indication, the computation of all the BV invariants takes about 65 days of CPU time on a single core (about 145 `ms` per lattice), and that of the reduced masses about 15 days (about 33 `ms` per lattice); the other checks are negligible. The fact that the BV invariant distinguishes all those lattices is quite miraculous and only follows from our whole computation. As in [ACh25], much remains to be explained about the apparent sharpness of this invariant.

We now explain how we found these Gram matrices. Recall from Sect. 1.1 that, for any root system R, we denote by X_n^R the set of isomorphism classes of rank n odd unimodular lattices L with no norm 1 vector and root system isomorphic to R, and by $\mathrm{m}_n(R)$ the reduced mass of this collection of lattices (see Sect. 2.5). Using [Ki03], we know $\mathrm{m}_{29}(R)$ for each R (see also [Ch25, §6.4]); it is nonzero for exactly 11 085 root systems R. The following Table 4.1 indicates the root systems contributing the most:

R	$7\mathbf{A}_1\mathbf{A}_2$	$7\mathbf{A}_1$	$8\mathbf{A}_1\mathbf{A}_2$	$6\mathbf{A}_1$	$6\mathbf{A}_1\mathbf{A}_2$	$8\mathbf{A}_1$	$7\mathbf{A}_1 2\mathbf{A}_2$	$5\mathbf{A}_1$	$9\mathbf{A}_1\mathbf{A}_2$
$2\,\mathrm{m}_{29}(R)$	1.64	1.54	1.52	1.45	1.42	1.39	1.21	1.15	1.13
$\lvert\mathrm{X}_{29}^R\rvert$	1.67	1.57	1.55	1.48	1.45	1.42	1.22	1.18	1.15
R	$6\mathbf{A}_1 2\mathbf{A}_2$	$9\mathbf{A}_1$	$8\mathbf{A}_1 2\mathbf{A}_2$	$5\mathbf{A}_1\mathbf{A}_2$	$4\mathbf{A}_1$	$5\mathbf{A}_1 2\mathbf{A}_2$	$6\mathbf{A}_1 3\mathbf{A}_2$	$7\mathbf{A}_1 3\mathbf{A}_2$	$9\mathbf{A}_1 2\mathbf{A}_2$
$2\,\mathrm{m}_{29}(R)$	1.06	1.05	1.04	1.01	0.77	0.71	0.68	0.68	0.67
$\lvert\mathrm{X}_{29}^R\rvert$	1.07	1.07	1.06	1.04	0.80	0.73	0.69	0.69	0.68

Table 4.1: Root systems R listed in the decreasing order of $|\mathrm{X}_{29}^R|$, with the values of $|\mathrm{X}_{29}^R|$ and $2\,\mathrm{m}_{29}(R)$ given in millions and rounded to 10^{-2}.

Remark 4.2. *For any unimodular lattice L of odd rank n, we claim that the reduced isometry group $\mathrm{O}(L)^{\mathrm{red}}$ always has even cardinality ≥ 2, as its canonical 2-torsion element $-w$ (see Sect. 2.1) is nontrivial. Indeed, if M denotes the index 2 even sublattice of L, we have $\mathrm{W}(M) = \mathrm{W}(L)$, and M is in the genus of D_n. So $\operatorname{res} M \simeq \mathbb{Z}/4$ as n is odd, and we conclude by Lemma 2.3.*

We first consider the root systems which do not contain any pair of orthogonal roots, namely $\emptyset$, $\mathbf{A}_1$ and $\mathbf{A}_2$. We have $\mathrm{m}_{29}(\emptyset) = 49612728929/11136000 \simeq 4455$, $\mathrm{m}_{29}(\mathbf{A}_1) = 18609637771/673920 \simeq 27614$ and $\mathrm{m}_{29}(\mathbf{A}_2) = 113241/80 \simeq 1415$.

Proposition 4.3. *We have $|\mathrm{X}_{29}^{\emptyset}| = 10\,092$, $|\mathrm{X}_{29}^{\mathbf{A}_1}| = 59\,105$ and $|\mathrm{X}_{29}^{\mathbf{A}_2}| = 3\,714$, and neighbor forms for representatives of those lattices are given in* [ChTa].

Proof. The case $\mathrm{X}_{29}^{\emptyset}$ was done in [ACh25], with a method lengthily discussed in Sect. 6.7 there. A similar method works in the other cases, starting with running the `BNE` algorithm *loc. cit.* for the *visible* root systems $\mathbf{A}_1$ and $\mathbf{A}_2$. □

[14] Actually, each lattice in our lists is given together with its root system, its reduced mass and its hashed BV invariant.

Of course, we deal with the 11 082 remaining root systems using the algorithms described in Corollaries 3.3 and 3.7. In order to have an idea *a priori* of the size of the list U_{29} provided by Corollary 3.3, let us denote by[15] $\mathrm{np}(R)$ the number of $\mathrm{W}(R)$-orbits of pairs of orthogonal roots in the root system R. Under the heuristic, confirmed by the final computation, that almost all lattices U in X_{29}^{R} satisfy $|\mathrm{O}(U)^{\mathrm{red}}| = 2$, in which case $\mathrm{O}(U)$-orbits of pairs of orthogonal roots coincide with their $\mathrm{W}(U)$-orbits by Remark 4.2, the redundancy of the isomorphism class of U in the list U_{29} should be $\simeq \mathrm{np}(\mathrm{R}_2(U))$, so we expect

$$(4.1) \qquad |\mathrm{U}_{29}| \;\simeq\; 2\sum_{R} \mathrm{m}_{29}(R)\,\mathrm{np}(R) \;\simeq\; 1,28\cdot 10^{9}.$$

This fits our computations: we find about 1.30 billions[16] isomorphism classes of pairs (L, e), using the list L_{27} (with 17 059 elements) given in [Ch25] and the algorithm `orbmod2` (see Sect. 2.17). This computation takes about 28 days of CPU time on a single core: about 2 days to compute generators of $\mathrm{O}(L)$ for each $L \in \mathrm{L}_{27}$, and then about 1.72 `ms` per orbit.

If we replace $\mathrm{np}(R)$ in the sum (4.1) by the number $\mathrm{npr}(R)$ of *relevant* pairs in R as in Definition 3.4, we rather find about 309 millions of orbits, that is 4 times less! As it is faster to check if a pair of roots is relevant than computing a BV invariant,[17] this shows that it is worth excluding first all pairs (L, e) such that $\{\alpha_0, \beta_0\}$ is not relevant for $U(L; e)$, as in Corollary 3.7, before computing the invariant. The final list U_{29} obtained this way has about 312 millions of elements, very close to the estimation above. As an indication, Table 4.2 below indicates the root systems most affected by the exclusion of non relevant pairs, and Table 4.3 those contributing the most to the final list.

R	$8\mathbf{A}_1 2\mathbf{A}_2$	$8\mathbf{A}_1\mathbf{A}_2$	$7\mathbf{A}_1 2\mathbf{A}_2$	$9\mathbf{A}_1\mathbf{A}_2$	$9\mathbf{A}_1 2\mathbf{A}_2$	$7\mathbf{A}_1\mathbf{A}_2$	$10\mathbf{A}_1\mathbf{A}_2$	$6\mathbf{A}_1 2\mathbf{A}_2$
$\mathrm{exc}(R)$	45.9	42.6	42.2	40.6	36.2	34.4	29.3	28.6
R	$7\mathbf{A}_1 3\mathbf{A}_2$	$8\mathbf{A}_1 3\mathbf{A}_2$	$6\mathbf{A}_1 3\mathbf{A}_2$	$6\mathbf{A}_1\mathbf{A}_2$	$10\mathbf{A}_1 2\mathbf{A}_2$	$7\mathbf{A}_1 2\mathbf{A}_2\mathbf{A}_3$	$11\mathbf{A}_1\mathbf{A}_2$	$9\mathbf{A}_1 3\mathbf{A}_2$
$\mathrm{exc}(R)$	28.5	24.7	22.4	21.3	20.4	16.1	15.7	14.5

Table 4.2: Root systems R in the decreasing order of the quantity $\mathrm{exc}(R) = 2\,\mathrm{m}_{29}(R)(\mathrm{np}(R) - \mathrm{npr}(R))$ (counted in millions).

[15] For R irreducible we have $\mathrm{np}(R) = 1$, except $\mathrm{np}(\mathbf{A}_1) = \mathrm{np}(\mathbf{A}_2) = 0$ and $\mathrm{np}(\mathbf{D}_m) = 2$ for $m \geq 4$. For $R = \coprod_{i=1}^{h} R_i$ with R_i irreducible, we have $\mathrm{np}(R) = \frac{h(h-1)}{2} + \sum_{i=1}^{h} \mathrm{np}(R_i)$.

[16] To be more precise, as we are only interested in U with $\mathrm{r}_1(U) = 0$, we only enumerated pairs (L, e) with $\mathrm{R}_1(L) \cap \mathrm{M}_2(L; e) = \emptyset$, but this reduction is innocent.

[17] Checking whether a pair or orthogonal roots $\{\alpha, \beta\}$ is relevant in $\mathrm{R}_2(L)$ takes about 9 `ms`, which is more than 15 times faster than computing the BV invariant of L. The selection of all relevant pairs took about 4.5 months of CPU time.

R	$8\mathbf{A}_1$	$9\mathbf{A}_1$	$7\mathbf{A}_1$	$10\mathbf{A}_1$	$6\mathbf{A}_1$	$11\mathbf{A}_1$	$8\mathbf{A}_1\mathbf{A}_2$	$5\mathbf{A}_1$	$7\mathbf{A}_1\mathbf{A}_2$
$\mathrm{npr}(R)$	28	36	21	45	15	55	8	10	7
$\mathrm{red}(R)$	38.9	37.9	32.4	29.8	21.7	18.5	12.2	11.5	11.5
R	$9\mathbf{A}_1\mathbf{A}_2$	$12\mathbf{A}_1$	$6\mathbf{A}_1\mathbf{A}_2$	$10\mathbf{A}_1\mathbf{A}_2$	$5\mathbf{A}_1\mathbf{A}_2$	$4\mathbf{A}_1$	$13\mathbf{A}_1$	$11\mathbf{A}_1\mathbf{A}_2$	$4\mathbf{A}_1\mathbf{A}_2$
$\mathrm{npr}(R)$	9	66	6	10	5	6	78	11	4
$\mathrm{red}(R)$	10.1	8.98	8.54	6.51	5.04	4.64	3.30	3.15	2.32

Table 4.3: Root systems R, in the decreasing order of their redundancy $\mathrm{red}(R) = 2\,\mathrm{m}_{29}(R)\,\mathrm{npr}(R)$ (counted in millions).

Observe the many root systems R of type $m\mathbf{A}_1$ in Table 4.3. For those we have $\mathrm{npr}(R) = \frac{m(m-1)}{2}$ (all pairs are relevant). It is a bit unfortunate that such root systems also tend to have a large reduced mass $\mathrm{m}_{29}(R)$ (see also [Ki03, Rem. 8 §3]). The final computation of the BV invariants of the elements in U_{29} takes about 1 year and a half of CPU time on a single core (but of course this step is straightforward to parallelize). This is still much less than the computation of X_{28} in [ACh25], proving the clear superiority of this method over the neighbor computations. We do find 38 592 290 different BV invariants, and then select arbitrarily one lattice for each. We computed Gram matrices for these lattices using the `goodbases` algorithm (see Sect. 2.17), which takes about 220 `h` of CPU time on a single core (about 21 `ms` per lattice). □

We end this section by providing a few additional information about the rank 29 unimodular lattices L with no norm 1 vectors that follow from our computation:

(i) The heuristic average number of roots of L, namely $\frac{\sum_R \mathrm{m}_{29}(R)|R|}{\sum_R \mathrm{m}_{29}(R)} \simeq 27.1$, is confirmed. Moreover, we have $\mathrm{r}_3(L) = 1856 - 128\,|\mathrm{Exc}\,L| + 10\,\mathrm{r}_2(L)$ by a theta series argument as in [BV01]. Here $\mathrm{Exc}\,L$ is the set of exceptional vectors in L: see Sect. 6. We will see in Thm. 6.10 that $|\mathrm{Exc}\,L|$ is 0 for 92.9% of the lattices, and 0.159 on average. So the average value of $\mathrm{r}_3(L)$ is about 2107, and the graph appearing in the computation of BV has $\frac{1}{2}(\mathrm{r}_3(L) + \mathrm{r}_2(L)) \simeq 1067$ vertices on average.

(ii) Let $\mathrm{d}(L)$ be the smallest integer $d \geq 2$ such that L is generated over $\mathbb{Z}$ by $\mathrm{R}_{\leq d}(L)$. Then L always admits a $\mathbb{Z}$-basis in $\mathrm{R}_{\leq d}(L)$ for $d = \mathrm{d}(L)$. Moreover, we always have $\mathrm{d}(L) = 3, 4$ or 5, and $\mathrm{d}(L) = 3$ for 38 590 862 lattices, $\mathrm{d}(L) = 4$ for 1421 lattices, and $\mathrm{d}(L) = 5$ for 7 lattices.

(iii) Some statistics for the order of reduced isometry groups are given in Table 4.4. It is 2 (resp. 4) for 95.2% (resp. 4.18%) of the lattices.

`ord`	2	4	6	8	10	12	14	16	18	20	24	28	32	36	40
#	36 741 838	1 613 885	7 942	165 479	22	12 147	2	28 136	19	154	5 648	13	7 797	108	239
`ord`	42	48	56	60	64	72	80	84	96	104	108	120	128	144	160
#	8	2204	3	11	2070	293	94	24	1045	3	3	30	675	213	30
`ord`	192	216	224	232	240	252	256	288	320	336	384	400	432	480	512
#	383	32	1	1	82	3	266	178	9	14	181	1	21	62	90

Table 4.4: Number # of lattices with reduced isometry group of order `ord` ≤ 512.

(iv) An analysis of the Jordan-Hölder factors using GAP and similar to that in [Ch25, §12] shows that only 454 lattices have a non-solvable reduced isometry group. Exactly 4 of those groups have a Jordan-Hölder factor not appearing for unimodular lattices of smaller rank: see Table 4.5.

$\mathrm{R}_2(L)$	$\mathbf{A}_1\mathbf{A}_4$	$\mathbf{A}_1\mathbf{A}_5$	$\mathbf{D}_6$	$\mathbf{A}_5$
$\lvert \mathrm{O}(L)^{\mathrm{red}} \rvert$	20 401 920	26 127 360	52 254 720	3 592 512 000
$\mathrm{O}(L)^{\mathrm{red}}$	$\mathbb{Z}/2 \times \mathrm{M}_{23}$	$\mathbb{Z}/2 \times (\mathrm{U}_4(3) : \mathbb{Z}/2 \times \mathbb{Z}/2)$	$\mathbb{Z}/2 \times (\mathrm{U}_4(3) : \mathrm{D}_8)$	$\mathbb{Z}/2 \times (\mathrm{McL} : \mathbb{Z}/2)$

Table 4.5: The reduced isometry groups $\mathrm{O}(L)^{\mathrm{red}}$ of the 4 rank 29 unimodular lattices L with a Jordan-Hölder factor not appearing in smaller rank, using GAP's notation.

These 4 lattices could presumably be directly constructed, and analysed, using the Leech lattice.

A computer calculation shows that the assertion about BV in Theorem 4.1 extends to all unimodular lattices, without any restriction on norm 1 vectors. It is thus not even necessary to consider norm 1 vectors separately.

Corollary 4.4. *Let L and L' be two unimodular lattices of same rank ≤ 29. Then L and L' are isometric if, and only if, we have* $\mathrm{BV}(L) = \mathrm{BV}(L')$.

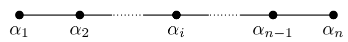
5. Fertile weights, extensions of root systems and gluing

If M is an even lattice, its *Venkov map* is the map $\nu : \mathrm{qres}\, M \to \mathbb{Q}_{\geq 0}$ defined by $\nu(\overline{x}) = \min_{y \in x+M} \; y \cdot y$. For any $x \in M^\sharp$ we have $\nu(\overline{x}) \equiv x \cdot x \bmod 2\mathbb{Z}$.

Assume now R is a root system. The elements of $\mathrm{Q}(R)^\sharp$ are usually called *weights*. A weight $\xi \in \mathrm{Q}(R)^\sharp$ is said *minuscule* if it satisfies $|\xi \cdot \alpha| \leq 1$ for all $\alpha \in R$. (Some authors add the condition $\xi \neq 0$, which we do not impose.) For each $\xi \in \mathrm{Q}(R)^\sharp$ the minimal vectors of $\xi + \mathrm{Q}(R)$ coincide with its minuscule weights, and form a single $\mathrm{W}(R)$-orbit (see [Bou75, Ch. VIII §7.3]). In particular, the Venkov map of $\mathrm{Q}(R)$ is the norm of the minuscule lifts.

The study of minuscule weights reduces to the case where R is irreducible, as $\xi \in \mathrm{Q}(R)^\sharp$ is minuscule if, and only if, so are its orthogonal projections to each of the irreducible components of R. For R irreducible, if we choose a positive system for R then the non-zero dominant minuscule ξ are the fundamental weights ϖ_i associated to the simple roots α_i having coefficient 1 in the highest root $\tilde{\alpha}$. From the *"Planches"* in [Bou81] we easily deduce the values of $\varpi_i \cdot \varpi_i$.

Example 5.1. *When R has type $\mathbf{A}_n$, all the fundamental weights are minuscule and their norms are as follows:*



with $\quad \varpi_i \cdot \varpi_i = \frac{i(n+1-i)}{n+1}$.

Also, there is a group isomorphism $\mathbb{Z}/(n+1) \xrightarrow{\sim} \mathrm{qres}\,\mathrm{Q}(R)$ sending, for all $1 \leq i \leq n$, the class of i mod $n+1$ to that of ϖ_i mod $\mathrm{Q}(R)$.

Definition 5.2. *Let $\xi \in \mathrm{Q}(R)^\sharp$. We say that ξ is* fertile *if we have $\xi \cdot \xi < 2$. Similarly, we say that a class $c \in \mathrm{qres}\ \mathrm{Q}(R)$ is fertile if we have $\nu(c) < 2$.*

Fertile weights are preserved under $\mathrm{O}(\mathrm{Q}(R))$ and are trivially minuscule. As an example, the fertile dominant weights in type $\mathbf{A}_n$ are ϖ_1, ϖ_n, as well as ϖ_2, ϖ_{n-1} for $n \geq 3$, and ϖ_3, ϖ_{n-3} for $5 \leq n \leq 6$. Fertile weights will serve in the following simple construction. Fix a fertile $\varpi \in \mathrm{Q}(R)^\sharp$. In the orthogonal direct sum $\mathrm{Q}(R)^\sharp \perp \mathbb{Z}e_0$ with $e_0 \cdot e_0 = 2 - \varpi \cdot \varpi$ (positive!), consider the lattice

$$\mathrm{Q}(R_\varpi) \,=\, \mathrm{Q}(R) \,+\, \mathbb{Z}\,(e_0 - \varpi). \tag{5.1}$$

It only depends on the (fertile) class c of ϖ in $\mathrm{qres}\,\mathrm{Q}(R)$. This is a root lattice as $e_0 - \varpi$ is a root. We denote by R_ϖ or R_c its root system, hence the notation above. By construction, we have a natural embedding $\upsilon_c : \mathrm{Q}(R) \to \mathrm{Q}(R_c)$.

Remark 5.3. (Dynkin diagram of R_c) *Fix $B = \{\alpha_1, \dots, \alpha_n\}$ a set of simple roots for R and ϖ a fertile weight dominant for B. Let $I \subset \{1, \dots, n\}$ be the subset of i such that $\varpi \cdot \alpha_i = 1$, then $B \cup \{e_0 - \varpi\}$ is a basis of R_c, whose Dynkin diagram is obtained from that of B by adding a single node for $e_0 - \varpi$ and by connecting this node to each α_i with $i \in I$. Note that this Dynkin diagram has a marked node, namely at $e_0 - \varpi$. Any Dynkin diagram with a marked node can be obtained by this construction, in a unique way.*[18]

Example 5.4. *(i) Assume $R \simeq \mathbf{A}_n$. We have $R_{\varpi_1} \simeq R_{\varpi_n} \simeq \mathbf{A}_{n+1}$, $R_{\varpi_2} \simeq R_{\varpi_{n-1}} \simeq \mathbf{D}_{n+1}$ for $n \geq 3$, and $R_{\varpi_3} \simeq R_{\varpi_{n-2}} \simeq \mathbf{E}_{n+1}$ for $n = 5, 6, 7$.*

(ii) Assume $R \simeq 3\mathbf{A}_1$. Then $\varpi = \varpi_1 + \varpi_2 + \varpi_3$ has norm $3/2 < 2$, hence is fertile, and we have $R_\varpi \simeq \mathbf{D}_4$.

We now explain why these apparently irrelevant notions appear in this work. We first set a few definitions concerning the gluing construction of even unimodular lattices from $\mathrm{Q}(R)$ (see § 2.6).

Definition 5.5. *For R a root system, we denote by:*

(a) *$\mathcal{M}^R$ the groupoid of pairs (M, η) with M an even lattice and $\eta : \mathrm{qres}\,M \overset{\sim}{\to} -\mathrm{qres}\,\mathrm{Q}(R)$ an isometry,*

(b) *$\mathcal{U}_R$ the groupoid of pairs (U, ι) with U an even unimodular lattice and $\iota : \mathrm{Q}(R) \to U$ an isometric embedding with saturated image,*

(c) *$F_R : \mathcal{M}^R \to \mathcal{U}_R$ the natural equivalence given by Proposition 2.7 and Remark 2.8 (ii).*[19]

We also write $\mathcal{M}^{\mathrm{Q}(R)}$, $\mathcal{U}_{\mathrm{Q}(R)}$ and $F_{\mathrm{Q}(R)}$ for $\mathcal{M}^R$, $\mathcal{U}_R$ and F_R.

If A, B, L are lattices, and if we have two isometric embeddings $\iota : A \to L$ and $\upsilon : A \to B$, an *extension of ι to B via υ* is an isometric embedding $\iota' : B \to L$ such that $\iota' \circ \upsilon = \iota$. We denote by $\mathrm{Ext}(\iota, \upsilon)$ the set of those extensions ι'. It has a natural action of $\mathrm{O}(L, \iota)$ given by $(g, \iota') \mapsto g \circ \iota'$. The main observation in this section is the following:

[18] The groupoid $\mathcal{A}$ of triples (R, B, ϖ) with R a root system, B a basis and ϖ a dominant fertile weight, is equivalent to that $\mathcal{B}$ of (R', B', α) with R' a root system, B' a basis of R' and $\alpha \in B'$. Indeed, we have defined $\mathcal{A} \to \mathcal{B}, (R, B, \varpi) \mapsto (R_c, B \cup \{\alpha\}, \alpha)$ with $\alpha = e_0 - \varpi$. But we also have $\mathcal{B} \to \mathcal{A}, (R', B', \alpha) \to (R, B, \varpi)$ with $B = B' \smallsetminus \{\alpha\}$ and ϖ the orthogonal projection of $-\alpha$ to the space generated by B. These are trivially inverse equivalences.

[19] Recall that for (M, η) in $\mathcal{M}^R$, and $(U, \iota) = F_R(M, \eta)$, then $M \perp \mathrm{Q}(R) \subset U \subset M^\sharp \perp \mathrm{Q}(R)^\sharp$ is defined by the Lagrangian $\mathrm{I}(\iota, \eta)$ of (2.2), with ι the natural inclusion $\mathrm{Q}(R) \to U$.

Proposition 5.6. *Let (M,η) in $\mathcal{M}^R$ and denote by (U,ι) in $\mathcal{U}_R$ its image under F_R. Fix a fertile class $c \in \operatorname{qres} \mathrm{Q}(R)$ and denote by $\operatorname{Exc}_{c,\eta} M$ the set of elements $e \in M^\sharp$ with $e \cdot e = 2 - \nu(c)$ and $\eta(e) \equiv -c$. For any fertile ϖ in c, the map*

$$f : \operatorname{Ext}(\iota, \upsilon_c) \longrightarrow \operatorname{Exc}_{c,\eta} M, \ \ \iota' \mapsto \iota'(e_0 - \varpi) + \iota(\varpi), \tag{5.2}$$

is well-defined, bijective and independent of the choice of ϖ in c. Moreover:

(a) for $\iota' \in \operatorname{Ext}(\iota,\upsilon_c)$ and $e = f(\iota')$, then the subgroup $\iota'(\mathrm{Q}(R_c))$ is saturated in U if, and only if, the element e is primitive in $M^\sharp$.

(b) the map f is equivariant with respect to the natural action of $\mathrm{O}(U,\iota)$ on $\operatorname{Ext}(\iota,\upsilon_c)$, that of $\mathrm{O}(M,\eta)$ on $\operatorname{Exc}_{c,\eta} M$, and the natural group isomorphism $\mathrm{O}(M,\eta) \overset{\sim}{\to} \mathrm{O}(U,\iota)$ given by F_R.

Proof. We have $M \perp \mathrm{Q}(R) \subset U \subset M^\sharp \perp \mathrm{Q}(R)^\sharp$ and ι is the natural inclusion. Choose a fertile ϖ in c. As $\mathrm{Q}(R_c)$ is generated by $e_0 - \varpi$ and R, it is equivalent to give $\iota' \in \operatorname{Ext}(\iota,\upsilon_c)$ and the element $\iota'(e_0 - \varpi)$, which can be any element $\beta \in U$ satisfying $\beta \cdot \beta = 2$ and $\beta \cdot x = -\varpi \cdot x$ for all $x \in \mathrm{Q}(R)$. An element $\beta = e + q$, with $e \in M^\sharp$ and $q \in \mathrm{Q}(R)^\sharp$, has these properties if, and only if, we have $\eta(e) \equiv q$, $e \cdot e + q \cdot q = 2$ and $q \cdot x = -\varpi \cdot x$ for all $x \in \mathrm{Q}(R)$. But this is equivalent to $q = -\varpi$ and $e \in \operatorname{Exc}_{c,\eta} M$. This shows that f is well-defined and bijective. It is obvious that it does not depend on the choice of ϖ.

For assertion (a), fix $\iota' \in \operatorname{Ext}(\iota,\upsilon_c)$ and set $e = f(\iota')$. As $\iota(\mathrm{Q}(R))$ is saturated in U, and as we have $\mathrm{Q}(R_c) = \mathbb{Z}(e_0 - \varpi) \oplus \mathrm{Q}(R)$, the subgroup $\iota'(\mathrm{Q}(R_c))$ is saturated in U if, and only if, the element $\iota'(e_0 - \varpi) = e - \varpi$ is primitive in $U/\mathrm{Q}(R)$. But the orthogonal projection $U \to M^\sharp$ induces a $\mathbb{Z}$-linear isomorphism $U/\mathrm{Q}(R) \overset{\sim}{\to} M^\sharp$ sending $e - \varpi$ to e, concluding the proof of (a).

We finally check the (trivial!) assertion (b). Let $\mathrm{O}(M,\eta) \overset{\sim}{\to} \mathrm{O}(U,\iota), g \mapsto \tilde{g}$, be the isomorphism induced by F_R. By definition, $\tilde{g}$ is the unique element h in $\mathrm{O}(U,\iota)$ with $h_{|M} = g$. For ι' in $\operatorname{Ext}(\iota,\upsilon_c)$ and g in $\mathrm{O}(M,\eta)$ we have $\tilde{g}(\iota(\varpi)) = \iota(\varpi)$, hence $g(f(\iota')) = \tilde{g}(f(\iota')) = \tilde{g}(\iota'(e_0 - \varpi)) + \iota(\varpi) = f(\tilde{g} \circ \iota')$. □

Corollary 5.7. *Fix a root system R and $c \in \operatorname{qres} \mathrm{Q}(R)$ a fertile class. Let $\mathcal{M}^{R,c}$ be the groupoid of triples (M,η,e) with (M,η) in $\mathcal{M}^R$ and $e \in M^\sharp$ primitive such that $e \cdot e = 2 - \nu(c)$ and $\eta(e) \equiv -c$. Then there is an equivalence of groupoids $F_{R,c} : \mathcal{M}^{R,c} \to \mathcal{U}_{R_c}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}^{R,c} & \xrightarrow{F_{R,c}} & \mathcal{U}_{R_c} \\ \downarrow{\scriptstyle O} & & \downarrow{\scriptstyle O'} \\ \mathcal{M}^{R} & \xrightarrow{F_{R}} & \mathcal{U}_{R} \end{array}$$

where $O : \mathcal{M}^{R,c} \to \mathcal{M}$ and $O' : \mathcal{U}_{R_c} \to \mathcal{U}_R$ are the respective forgetful functors $(M,\eta,e) \mapsto (M,\eta)$ and $(U,\iota') \mapsto (U, \iota' \circ \upsilon_c)$.

Proof. The functor O' is well-defined as $\mathrm{Q}(R)$ is saturated in $\mathrm{Q}(R_c)$. Fix (M,η,e) in $\mathcal{M}^{R,c}$ and set $(U,\iota) = F_R(M,\eta)$. By Proposition 5.6, there is a unique $\iota' \in \operatorname{Ext}(\iota,\upsilon_c)$ with saturated image and $f(\iota') = e$: define $F_{R,c}(M,\eta,e)$

as the object (U, ι') in $\mathcal{U}_{R_c}$. This naturally extends to a functor $F_{R,c} : \mathcal{M}^{R,c} \to \mathcal{U}_{R_c}$. By the proposition again part (b), $F_{R,c}$ is fully faithful. Also, it trivially satisfies $O' \circ F_{R,c} = F_R \circ O$ (and $\mathcal{M}^{R,c} = \mathcal{M}^R \times_{\mathcal{U}_R} \mathcal{U}_{R,c}$). To check the essential surjectivity, choose (U, ι') in $\mathcal{U}_{R_c}$. As F_R is essentially surjective, we may assume $(U, \iota' \circ \upsilon_c) = F_R(M, \eta)$ for some (M, η) in $\mathcal{M}^R$. By surjectivity of the map (5.2), there is $e \in M^\sharp$ such that $F_{R,c}(M, \eta, e) = (U, \iota')$, and we are done. □

By composing $F_{R,c}$ with the natural inverse of F_{R_c} (Sect. 2.6), we deduce:

Corollary 5.8. *We have a natural equivalence $\mathcal{M}^{R,c} \to \mathcal{M}^{R_c}$, sending any (M, η, e) in $\mathcal{M}^{R,c}$ to some (M', η') in $\mathcal{M}^{R_c}$ with $M' = M \cap e^\perp$.*

As an example we consider the case $R \simeq \mathbf{A}_n$ and $c \equiv \varpi_1$. In this case we have seen $R_c \simeq \mathbf{A}_{n+1}$, $\nu(c) = \frac{n}{n+1}$ and so $2 - \nu(c) = \frac{n+2}{n+1}$.

Corollary 5.9. *$\mathcal{M}^{\mathrm{A}_n, \varpi_1}$ is naturally equivalent to the following groupoids:*

(i) pairs (M, e) with M an even lattice such that $\mathrm{qres}\, M \simeq -\mathrm{qres}\, \mathrm{A}_n$, and $e \in M^\sharp$ a primitive element with $e \cdot e = \frac{n+2}{n+1}$,

(ii) pairs (M', η') with M' an even lattice and $\eta' : \mathrm{qres}\, M' \to -\mathrm{qres}\, \mathrm{A}_{n+1}$ an isometry.

In this correspondence, we have $M' = M \cap e^\perp$, hence $\mathrm{rk}\, M' = \mathrm{rk} M - 1$.

As will follow from the proof, we may also replace the condition $\mathrm{qres}\, M \simeq -\mathrm{qres}\, \mathrm{A}_n$ in (i) by the (only apparently) weaker condition $\det M = n + 1$.

Proof. The equivalence with (ii) is Corollary 5.8. For that with (i), we claim that for (M, e) as in (i) there is a unique isometry $\eta : \mathrm{qres}\, M \overset{\sim}{\to} -\mathrm{qres}\, \mathrm{A}_n$ with $\eta(e) \equiv -\varpi_1$. The uniqueness is obvious as ϖ_1 generates $\mathrm{qres}\, \mathrm{A}_n$. For the existence, we have a unique group morphism $-\mathrm{qres}\, \mathrm{A}_n \to \mathrm{qres}\, M$ mapping the generator $-\varpi_1$ to e. It is clearly an isometry, in particular it is injective as $\mathrm{qres}\, \mathrm{A}_n$ is nondegenerate. As source and target have the same cardinality, it is bijective. This constructs η^{-1}. □

Observe that even unimodular lattices entirely disappeared in the statement of Corollary 5.8! Also, the statement of Corollary 5.9 establishes a rather surprising equivalence between the two genera (i) and (ii), and we may wish to have a more direct proof of it. This led us to discover the following result, which does include that equivalence and generalizes it in another direction. Let us stress however that the use of the approach above, and more precisely Proposition 5.6, will still be crucial to prove some subsequent results, such as Theorem 6.5 and Proposition 7.7. See Definition 2.10 for the notion of special vector.

Proposition 5.10. *Let n, d_1, d_2 be integers ≥ 1 with $\gcd(d_1, d_2) = 1$ and $d_1 d_2$ even. The following natural groupoids are equivalent:*

(i) pairs (L, u) where L is a rank n even lattice of determinant d_1 and u is a primitive special vector in L with $u \cdot u = d_1 d_2$,

(ii) pairs (L, v) where L is a rank n even lattice of determinant d_1 and v is a primitive vector in $L^\sharp$ with $v \cdot v = d_2 / d_1$,

(iii) pairs (N, w) where N is a rank $n-1$ even lattice of determinant d_2 and w a generator of $\mathrm{qres}\, N$ *with* $\mathrm{q}(w) \equiv -\frac{d_1}{2d_2} \bmod \mathbb{Z}$.

In these correspondences, we have $u = d_1 v$ and $N = L \cap u^\perp = L \cap v^\perp$.

As we shall see during the proof, for all pairs (L, v) as in (ii), the group $\mathrm{qres}\, L$ is cyclic of order d, generated by the class of v, which satisfies by definition $\mathrm{q}(v) \equiv \frac{d_2}{2d_1} \bmod \mathbb{Z}$. This shows that the genus of L is uniquely determined, and of course, that of N is also determined by (iii).[20]

Proof. Let L be an even lattice with $\det L = d_1$. We have $d_1 L^\sharp \subset L$ and the map $v \mapsto d_1 v$ defines a bijection between the set of $v \in L^\sharp$ with $v \cdot v = d_2/d_1$, and that of $u \in d_1 L^\sharp$ with $u \cdot u = d_1 d_2$. Assume we have $v \in L^\sharp$ with $v \cdot v = d_2/d_1$ and $\gcd(d_1, d_2) = 1$. Then $L^\sharp/L$ is cyclic of order d_1, generated by the class $\overline{v}$ of v. Indeed, for $k \in \mathbb{Z}$ with $kv \in L$ we have $kv \cdot v \in \mathbb{Z}$ hence $k \equiv 0 \bmod d_1$. In particular, the group $d_1 L^\sharp / d_1 L \simeq \mathbb{Z}/d_1$ is generated by $d_1 v$. It is clear that if $d_1 v$ is primitive in L then v is primitive in $L^\sharp$. Let us check the converse. Assume that v is primitive in $L^\sharp$, denote $u = d_1 v$ and assume $u = mu'$ with $m \in \mathbb{Z}$ and $u' \in L$. Since the image of u generates $d_1 L^\sharp / d_1 L$ we have that m is coprime to d_1. Writing $1 = am + bd_1$ with $a, b \in \mathbb{Z}$ we deduce $v = mu'/d_1$ with $u'/d_1 = av + bu' \in L^\sharp$ and so $m = \pm 1$. We have shown that $(L, v) \mapsto (L, d_1 v)$ is an equivalence between the groupoids in (i) and (ii).

The equivalence between (i) and (iii) is a consequence of Example 2.9 in the special case $\det L = d_1$, $d = d_1 d_2$ and $\mathrm{m}(v) = d_1$ (v is a primitive special vector of L), and $\det N = d_2$. ☐

Remark 5.11. *The statement also holds if we remove all four occurrences of the word* even *in it, and replace* $\mathrm{qres}\, N$ *by* $\mathrm{res}\, N$, *and* $\mathrm{q}(w) \equiv -\frac{d_1}{2d_2}$ *by* $w \cdot w \equiv -\frac{d_1}{d_2}$.

6. Exceptional vectors in odd unimodular lattices

Let L be a unimodular lattice of rank n. Recall that if L is odd, then L is in the genus of I_n, so the norm of any characteristic vector of L is[21] $\equiv n \bmod 8$. We denote by $\mathrm{Exc}\, L$ the set of characteristic vectors of L with norm < 8, and following [BV01] we introduce the following definition:

Definition 6.1. *A unimodular lattice L is called exceptional if* $\mathrm{Exc}\, L \neq \emptyset$.

If L is even, then we have $\mathrm{Exc}\, L = \{0\}$, so L is exceptional and we now focus on the odd case. We are interested here in the set $\mathrm{Exc}\, L$, together with its natural action of $\mathrm{O}(L)$.

Remark 6.2. *The sets* $\mathrm{Exc}_{c,\eta}\, M$ *introduced in Section 5 (and in Proposition 5.6) are related to the sets* $\mathrm{Exc} L$ *as follows, which justifies our terminology for the former. Indeed, let L be an odd unimodular lattice of rank* $n \not\equiv 0 \bmod 8$ *and let k be the integer satisfying* $0 < k < 8$ *and* $n + k \equiv 0 \bmod 8$. *Let* $M = L^{\mathrm{even}}$, *so that M is in the genus of* D_n. *Realize* D_k *as* $\mathrm{I}_k^{\mathrm{even}}$. *There are exactly two isometries* $\eta_1, \eta_2 : \mathrm{qres}\, M \to -\mathrm{qres}\, \mathrm{D}_k$ *mapping* L/M *to* $\mathrm{I}_k/\mathrm{D}_k$. *Set* $c =$

[20] It would be possible to give the precise conditions on (n, d_1, d_2) such that those genera are non empty, but we shall not do it.

[21] The characteristic vectors of I_n are the $(x_i) \in \mathbb{Z}^n$ with $x_i \equiv 1 \bmod 2$.

$\frac{1}{2}(1,\ldots,1) \in \mathrm{qres}\,\mathrm{D}_k$. *Then* $\frac{1}{2}\mathrm{Exc}\,L$ *is equal to* $\mathrm{Exc}_{c,\eta_1} M \sqcup \mathrm{Exc}_{c,\eta_2} M$*: for* $e \in \mathrm{Exc}\,L$ *we have* $e/2 \in M^\sharp \smallsetminus L$ *and exactly one of* η_1*,* η_2 *maps* $e/2 + M$ *to* $-c$*.*

Using $\mathrm{Char}(A \perp B) = \{a + b \,|\, a \in \mathrm{Char}(A), b \in \mathrm{Char}(B)\}$, the study of exceptional unimodular lattices is easily reduced to that of those with no norm 1 vectors (see e.g. [Ch25, Prop. 9.2]):

Lemma 6.3. *Let* L *be a unimodular lattice of rank* $8k + r$ *with* $\mathrm{r}_1(L) = 0$ *and* $0 \leq r < 8$*. Then* $L \perp \mathrm{I}_s$ *is exceptional if, and only if, so is* L *and* $r + s < 8$*.*

Assume L is odd exceptional with $\mathrm{r}_1(L) = 0$. This clearly forces $n \not\equiv 0, 1 \bmod 8$. We first recall some results in [Ch25, §9.1].

(i) For $n \equiv 2, 3 \bmod 8$, we have $|\mathrm{Exc}\,L| = 2$.

(ii) For $n \equiv 4 \bmod 8$, we have $2 \leq |\mathrm{Exc}\,L| \leq 2n$.

The situation in case (ii) is more interesting. Indeed, setting $M = L^{\mathrm{even}}$, there are exactly 3 unimodular lattices in $M^\sharp$ containing M, namely L and two others called the *companions* of L. Those two companions are odd, so have even part M as well. If L is exceptional with $\mathrm{r}_1(L) = 0$, exactly one of its two companions L' satisfies $\mathrm{r}_1(L') \neq 0$. It is non exceptional, satisfies $\mathrm{O}(L') = \mathrm{O}(M)$ and the map $v \mapsto v/2$ induces a natural $\mathrm{O}(M)$-equivariant bijection $\mathrm{Exc}\,L \overset{\sim}{\to} \mathrm{R}_1(L')$ (in particular, $\mathrm{O}(M)$ preserves $\mathrm{Exc}\,L$). As we have $\mathrm{W}(L) = \mathrm{W}(L') = \mathrm{W}(M)$, and $\mathrm{W}(D_r)^\pm$ trivially acts transitively on $\mathrm{R}_1(\mathrm{I}_r)$, these descriptions show:

Proposition 6.4. *Assume* L *is an exceptional odd unimodular lattice of rank* n *with* $n \equiv 2, 3, 4 \bmod 8$ *and* $\mathrm{r}_1(L) = 0$*. Then* $\mathrm{W}(L)^\pm$ *acts transitively on* $\mathrm{Exc}\,L$*.*

Our first aim now is to pursue this study in the quite more subtle case $n \equiv 5 \bmod 8$, which is the situation of interest for $n = 29$.

Theorem 6.5. *Assume* L *is an exceptional odd unimodular lattice of rank* n *with* $n \equiv 5 \bmod 8$ *and* $\mathrm{r}_1(L) = 0$*. Then we have* $|\mathrm{Exc}\,L| = 2m$ *for* $1 \leq m \leq n$*, and* $\mathrm{W}(L)^\pm$ *acts transitively on* $\mathrm{Exc}\,L$*.*

An immediate consequence of this theorem is the following result, that was empirically observed for $n = 29$ in [ACh25].

Corollary 6.6. *If* L *is an exceptional unimodular lattice of rank* $n \equiv 5 \bmod 8$ *and with no nonzero vector of norm* ≤ 2*, then we have* $|\mathrm{Exc}\,L| = 2$*.*

Note that Lemma 6.3, together with Proposition 6.4 and Theorem 6.5, imply Theorem E of the introduction. As a preliminary, we shall consider certain embeddings of root systems.

Lemma 6.7. *Let* Q *be an irreducible root lattice. Then there is a unique* $\mathrm{O}(Q)$*-orbit of isometric embeddings* $f : \mathrm{A}_3 \to Q$*, unless we have* $Q \simeq \mathrm{D}_m$ *with* $m \geq 5$ *and in which case there are two.*

Proof. See Table 4 in [Ki03] (note however that the line $S = \mathrm{A}_3$, $T = \mathrm{D}_j$ of that table is incorrect in the case $j = 4$, as there is a unique orbit of size 12) or the companion paper [ChTc]. □

We use the standard description of the A_n lattice, with simple roots $\alpha_i = \epsilon_{i+1} - \epsilon_i$ for $1 \le i \le n$. Here are examples of both types of embeddings $\mathrm{A}_3 \to \mathrm{D}_m$:

The orthogonal of such an A_3 in D_m is isometric to D_{m-3} in Type I, and to $\mathrm{D}_1 \perp \mathrm{D}_{m-4}$ in type II (setting $\mathrm{D}_n = \mathrm{I}_n^{\mathrm{even}}$ as well for $n = 1, 2, 3$).

Lemma 6.8. *Fix embeddings $f : \mathrm{A}_3 \to Q$, with Q an irreducible root lattice, $\mu : \mathrm{A}_3 \to \mathrm{A}_4$ and $\nu : \mathrm{A}_3 \to \mathrm{D}_4$. Let Φ denote the set of embeddings $f' : \mathrm{A}_4 \to Q$* extending *f* via *μ, that is, with $f' \circ \mu = f$. Then Φ has a natural action of the subgroup W of $\mathrm{W}(Q)$ fixing pointwise $f(\mathrm{A}_3)$, and*

(i) either there is an embedding $f'' : \mathrm{D}_4 \to Q$ extending f via ν,

(ii) or we have $Q \simeq \mathrm{A}_m$, $|\Phi| = m - 3$ and W acts transitively on Φ.

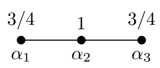
Proof. Observe that neither the hypotheses, nor the conclusions, are affected by replacing f, μ, ν, by $g \circ f$, $h \circ \mu$ and $k \circ \nu$, with $g \in \mathrm{O}(Q)$, $h \in \mathrm{O}(\mathrm{A}_4)$ and $k \in \mathrm{O}(\mathrm{D}_4)$: use the bijections $f' \mapsto g \circ f' \circ h^{-1}$ and $f'' \mapsto g \circ f'' \circ k^{-1}$.

Assume $Q \simeq \mathrm{A}_m$. By this observation and Lemma 6.7, we may assume $Q = \mathrm{A}_m$ and that f and μ send α_i to α_i for $1 \le i \le 3$. It is the same to give $f' \in \Phi$ and $f'(\alpha_4)$, which is any root $\alpha = \epsilon_i - \epsilon_j$ in A_m with $\alpha \cdot \alpha_1 = \alpha \cdot \alpha_2 = 0$ and $\alpha \cdot \alpha_3 = -1$. This is equivalent to $i = 4$ and $5 \le j \le m + 1$. This shows (ii).

Assume $Q \simeq \mathrm{D}_m$ with $m \ge 5$. Again we may assume $Q = \mathrm{D}_m$. By Lemma 6.7 we may assume that f is as in the pictured examples above, and we see that (i) holds for both types I and II of f. Assume finally Q has type $\mathbf{E}$ or $\mathbf{D}_4$. Choose any embedding $f'' : \mathrm{D}_4 \to Q$. Then $f'' \circ \nu$ and f are in the same $\mathrm{O}(Q)$-orbit by Lemma 6.7, so (i) holds again. $\square$

We are finally able to prove Theorem 6.5.

Proof. Let L be a unimodular lattice with odd rank $n \equiv 5 \bmod 8$. Then L is the unique proper overlattice of its even part $M := L^{\mathrm{even}}$, and M is in the genus of D_n. As we have $\mathrm{qres}\,\mathrm{D}_n \simeq -\mathrm{qres}\,\mathrm{A}_3$, we may choose an isometry $\eta : \mathrm{qres}\, M \to -\mathrm{qres}\,\mathrm{A}_3$. In the notations of Sect. 5, we have $(M, \eta) \in \mathcal{M}^{\mathrm{A}_3}$, and we may consider the associated $(U, \iota) = F_{\mathrm{A}_3}(M, \eta)$, with U a rank $n + 3$ even unimodular lattice and $\iota : \mathrm{A}_3 \to U$ an embedding with saturated image. The Dynkin diagram of $R := \mathrm{R}_2(\mathrm{A}_3)$, labelled by norms of fundamental weights, is



We are going to apply Proposition 5.6 to both the fertile classes of ϖ_1 and ϖ_2, which satisfy $R_{\varpi_1} \simeq \mathbf{A}_4$ and $R_{\varpi_2} \simeq \mathbf{D}_4$.

(a) We have $2 - \varpi_2 \cdot \varpi_2 = 1$. As any element $e \in M^\sharp \smallsetminus L$ satisfies $e \cdot e \equiv \frac{1}{4} \bmod \mathbb{Z}$, each norm 1 vector $e \in M^\sharp$ belongs to L and satisfies $\eta(e) \equiv -\varpi_2$. By Proposition 5.6, the extensions of ι to $\mathrm{Q}(R_{\varpi_2}) \simeq \mathrm{D}_4$ via υ_{ϖ_2} are thus in bijection with norm 1 vectors in L. As we have $\mathrm{r}_1(L) = 0$ by assumption on L, there are no such extensions.

(b) We have $2 - \varpi_1 \cdot \varpi_1 = 5/4$. Let E denote the set of norm $5/4$ vectors in $M^\sharp$. Observe that the $\mathrm{O}(M)$-equivariant map $E \to L$, $e \mapsto 2e$, induces a bijection $E \overset{\sim}{\to} \mathrm{Exc}\, L$. For $e \in E$ we have either $\eta(e) \equiv -\varpi_1$ or $\eta(e) \equiv -\varpi_3$, and we denote by $E = E_1 \sqcup E_3$ the corresponding partition of E. It satisfies $E_1 = -E_3$. By Proposition 5.6, the set E_1 is in natural $\mathrm{O}(M, \eta)$-equivariant bijection with the set Ψ of extensions of ι to $R_{\varpi_1} \simeq \mathbf{A}_4$ via υ_{ϖ_1}.

Let Q be the lattice generated by the unique irreducible component of U containing $\iota(\mathrm{A}_3)$. We apply Lemma 6.8 to the embedding $f : \mathrm{A}_3 \to Q$ defined by ι, as well as $\mu = \upsilon_{\varpi_1}$ and $\nu = \upsilon_{\varpi_2}$. We have $\Psi = \Phi$. By (a) above, we are not in case (i) of this Lemma, so its assertion (ii) is satisfied. But the subgroup of $\mathrm{W}(U)$ fixing pointwise $\iota(\mathrm{A}_3)$ coincides with $\mathrm{W}(L)$ by [Bou81, Ch. V, §3.3, Prop. 2] and the equalities $M = U \cap \iota(\mathrm{A}_3)^\perp$ and $\mathrm{W}(M) = \mathrm{W}(L)$. So E_1 consists of a single $\mathrm{W}(L)$-orbit, and we have $Q \simeq \mathrm{A}_{m+3}$ with $m = |E_1| = |E_3|$. ☐

Corollary 6.9. *Let $m, n \geq 0$ be integers with $n \equiv 5 \bmod 8$. The set of isometry classes of rank n unimodular lattices satisfying $\mathrm{r}_1(L) = 0$ and $|\mathrm{Exc}\, L| = 2m$ is in natural bijection with that of pairs (U, C) with U a rank $n + 3$ even unimodular lattice and C an irreducible component of $\mathrm{R}_2(U)$ of type $\mathbf{A}_{m+3}$.*

Proof. Given the proof of Theorem 6.5, this follows from the equivalence $\mathrm{F}_{\mathrm{A}_3}$ and the two following facts. First, we have $\mathrm{O}(\mathrm{qres}\, \mathrm{A}_3) = \langle -\mathrm{id} \rangle$ and for (M, η) in $\mathcal{M}_3^{\mathrm{A}}$ we always have $(M, \eta) \simeq (M, -\eta)$. Second, if we have (U, i_1) and (U, i_2) in $\mathcal{U}_{\mathrm{A}_3}$ such that $i_1(\mathrm{A}_3)$ and $i_2(\mathrm{A}_3)$ lie in a same irreducible root lattice $Q \subset U$ of type A_{m+3}, there is $g \in \mathrm{O}(U)$ with $g \circ i_1 = i_2$. Indeed, there is $h \in \mathrm{O}(Q)$ with $h \circ i_1 = i_2$ by Lemma 6.7, and we conclude as $\mathrm{O}(Q) = \mathrm{W}(Q)^\pm$ and the restriction map $\mathrm{W}(L)^\pm \to \mathrm{W}(Q)^\pm$ is trivially surjective. ☐

For example, we easily deduce from this and the classification of Niemeier lattices[22], the number of exceptional vectors of the 12 rank 21 unimodular lattices with no norm 1 vectors. We finally consider the rank $n = 29$. It is trivial to compute $|\mathrm{Exc}\, L|$ for each lattice L in the list given by Thm. 4.1. We find:

Theorem 6.10. *There are 2 721 152 exceptional unimodular lattices of rank 29 with no norm 1 vectors. The number # of those lattices having `exc` exceptional vectors is given in Table 6.1.*

`exc`	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	40	42	46	56	other
#	2439727	237232	33400	7509	1966	734	257	165	58	40	18	19	7	7	2	3	2	1	2	1	1	1	0

Table 6.1: Number # of exceptional unimodular lattices of rank 29 having `exc` exceptional vectors.

This table is coherent with the computation in [Ki03] of the mass $\mathrm{m}_{32}^{\mathrm{II}}(R)$ of the rank 32 even unimodular lattices with root system $\simeq R$. Indeed, if $\mathrm{N}_m(R)$ denotes the number of irreducible components of R of type $\mathbf{A}_m$, Corollary 6.9

[22] Recall that the isometry group of a Niemeier lattice N permutes transitively the irreducible components of $\mathrm{R}_2(N)$ of same type.

shows that $f_{2m} = 2\sum_R \mathrm{N}_{m+3}(R)\,|\mathrm{W}(R)|\,\mathrm{m}^{\mathrm{II}}_{32}(R)$ is a (heuristically close) lower bound for the number of isomorphism classes of L with $\mathrm{r}_1(L) = 0$ and $|\mathrm{Exc}\,L| = 2m > 0$. We numerically find $\lceil f_0 \rceil = 35\,026\,757$, and for $m \geq 1$:

$2m$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	40	42	46	56	other
$\lceil f_{2m} \rceil$	2323793	211670	28918	5847	1558	551	210	120	46	32	16	16	6	7	2	3	2	1	2	1	1	1	0

This is consistent with the table of Theorem 6.10 and explains all 0 values there.

Remark 6.11. (Case $n \equiv 6 \bmod 8$) *Assume L is a unimodular lattice of rank $n \equiv 6 \bmod 8$ with $\mathrm{r}_1(L) = 0$. Then $\mathrm{Exc}\,L$ can be studied using similar ideas. We can prove that each of the two fibers of the natural map $f : \mathrm{Exc}\,L \to \mathrm{Char}(L)/2L^{\mathrm{even}}$ is a single $\mathrm{W}(L)^{\pm}$-orbit, or empty. To be slightly more precise, let us fix an isometry $\eta : \mathrm{qres}\,L^{\mathrm{even}} \to -\mathrm{qres}\,(\mathrm{A}_1 \perp \mathrm{A}_1)$ and denote by (U, β_1, β_2) the rank $n+2$ even unimodular lattice equipped with two orthogonal roots associated to $(L^{\mathrm{even}}, \eta)$. Denote by C_i the irreducible component of $\mathrm{R}_2(U)$ containing β_i. We can show $C_1 \neq C_2$, and that the fibers of f are naturally indexed by $\{1, 2\}$ and have size $2\mathrm{h}(C_i) - 4$, where $\mathrm{h}(R)$ denotes the Coxeter number of R.*

7. Even lattices of prime (half-)determinant

Our main aim in this section is to prove Theorems C and D of the introduction. We start with some information about the genera $\mathcal{G}_{n,p}$ briefly introduced in Sect. 1.2.

7.1. The genera $\mathcal{G}_{n,p}$

Let $n \geq 1$ be an integer and p an odd prime. We are interested in this section in the even lattices of rank n and determinant $d = p$ or $2p$. A mod 2 inspection shows that the $d = p$ (resp. $d = 2p$) case is only possible for n even (resp. odd). Moreover, as is well-known, these lattices form a single genus that we will denote by $\mathcal{G}_{n,p}$, and which satisfies the following properties:

– For n even, we have $\mathcal{G}_{n,p} \neq \emptyset \iff n + p \equiv 1 \bmod 4$. For $L \in \mathcal{G}_{n,p}$, the nonzero values of $x \mapsto x \cdot x \bmod \mathbb{Z}$ on $\mathrm{qres}\,L \simeq \mathbb{Z}/p$ are the $\frac{a}{p}$ such that the Legendre symbol $(\frac{a}{p})$ is $(-1)^{(n+p-1)/4}$. We denote below by $\mathrm{R}_{n,p}$ the isometry class of this quadratic space. Note that $(-1)^{(n+p-1)/4}(\frac{2}{p})$ only depends on $p \bmod 4$.

– For n odd, we always have $\mathcal{G}_{n,p} \neq \emptyset$, and for $L \in \mathcal{G}_{n,p}$, we have $\mathrm{qres}\,L \simeq \mathrm{R}_{n-1,p} \perp \mathrm{qres}\,\mathrm{A}_1$ for $n + p \equiv 2 \bmod 4$, and $\mathrm{qres}\,L \simeq \mathrm{R}_{n+1,p} \perp -\mathrm{qres}\,\mathrm{A}_1$ otherwise.

This information is gathered in Table 7.1 below: the sign $\pm$ is the Legendre symbol $(\frac{a}{p})$ if the residual *quadratic* form $\mathrm{q}(x) = \frac{x \cdot x}{2} \bmod \mathbb{Z}$ takes the nonzero value $a/p \bmod \mathbb{Z}$ on $\mathrm{qres}\,L$, and we write it in blue if we have n odd and if q takes the value $-1/4$ (rather than $1/4$).

$p \bmod 4 \setminus n \bmod 8$	0	1	2	3	4	5	6	7
1	+	+		−	−	−		+
3		+	+	+		−	−	−

Table 7.1: The genera $\mathcal{G}_{n,p}$.

In the important cases $p \leq 7$ for us, Table 7.2 gives an example of a lattice in $\mathcal{G}_{n,p}$, up to adding copies of E_8. In this table, L' denotes the orthogonal of a root in L, F_8 denotes the unique even lattice of determinant 5 containing $\mathrm{E}_7 \perp \langle 10 \rangle$ and S_2 denotes a rank 2 lattice with Gram matrix $\left[\begin{smallmatrix} 2 & -1 \\ -1 & 4 \end{smallmatrix}\right]$.

$p \setminus n$	1	2	3	4	5	6	7	8
3	$\langle 6 \rangle$	A_2	$\mathrm{A}_1 \perp \mathrm{A}_2$		A_5	E_6	$\mathrm{A}_1 \perp \mathrm{E}_6$	
5	$\langle 10 \rangle$		A_4'	A_4	$\mathrm{A}_1 \perp \mathrm{A}_4$		F_8'	F_8
7	$\langle 14 \rangle$	S_2	$\mathrm{A}_1 \perp \mathrm{S}_2$		A_6'	A_6	$\mathrm{A}_1 \perp \mathrm{A}_6$	

Table 7.2: Some lattices in $\mathcal{G}_{n,p}$, for $n \leq 8$ and $p \leq 7$.

We conclude with a table relevant for the application of Proposition 5.10. In this table, it is convenient to include the genus $\mathcal{G}_{n,1}$ of rank n even lattices with (half-)determinant 1.

$p \setminus n \bmod 8$	1	2	3	4	5	6	7	8
1	$\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}$						$\frac{3}{2}, \frac{7}{2}, \frac{11}{2}$	
3	$\frac{1}{6}, \frac{13}{6}$	$\frac{2}{3}, \frac{8}{3}, \frac{14}{3}$	$\frac{7}{6}$		$\frac{5}{6}$	$\frac{4}{3}, \frac{10}{3}$	$\frac{11}{6}$	
5	$\frac{1}{10}, \frac{9}{10}$		$\frac{3}{10}, \frac{7}{10}$	$\frac{4}{5}, \frac{6}{5}, \frac{14}{5}$	$\frac{13}{10}$		$\frac{11}{10}$	$\frac{2}{5}, \frac{8}{5}, \frac{12}{5}$
7	$\frac{1}{14}, \frac{9}{14}$	$\frac{2}{7}, \frac{4}{7}, \frac{8}{7}$	$\frac{11}{14}$		$\frac{5}{14}, \frac{13}{14}$	$\frac{6}{7}, \frac{10}{7}, \frac{12}{7}$	$\frac{3}{14}$	

Table 7.3: Rationals $\frac{m}{d}$ with $m \leq 14$ and $\gcd(m,d) = 1$ satisfying $\lambda \equiv x.x \bmod 2\mathbb{Z}$ for some nonzero x in the residue of the lattices in $\mathcal{G}_{n,p}$, with $d = p$ or $2p$.

7.2. Proof of Theorem C: the orbit method

We focus on the most important cases $n \geq 23$, referring the reader to [ChTa] for the easier cases $n \leq 22$. As already explained in Sect. 1.2, our strategy is to deduce everything from the classification of unimodular lattices of rank ≤ 29, following the diagram in Figure 1 that we now decipher.

(a) Each disc in Figure 1 represents the genus $\mathcal{G}$ of lattices of the given dimension, determinant and parity, where even genera are blue, and odd in yellow. Each blue genus is either of the form $\mathcal{G}_{n,p}$, or equal to the (already known) genus[23] of even Euclidean lattices of determinant 1 (case n even) or 2 (case n odd).

(b) Each arrow $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ in Figure 1 means that we shall deduce, from a set $\mathcal{L}$ of representatives for the isometry classes in $\mathcal{G}$, a set $\mathcal{L}'$ of representatives of that in $\mathcal{G}'$, using the vectors in each $L \in \mathcal{L}$ of a certain type determined by the label t.

(c) Each label t as in (b) consists of an integer ν and an attribute χ which is either `char`, `exc`, `sp` or empty. A vector of *type* t in a lattice L in $\mathcal{G}$ is then

[23] We may think of $\mathcal{G}_n$ as $\mathcal{G}_{n,1}$.

defined as a primitive[24] vector of L of norm ν, which is respectively characteristic (§2.11), exceptional (Definition 6.1), special (Definition 2.10), or with no extra property. We also write $t = (\nu, \chi)$.

The general algorithm is as follows. Start with an arrow $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ in Figure 1 and a subset $\mathcal{L}$ of $\mathcal{G}$ as in (b). Set $\mathcal{L}' = \emptyset$, and for each $L \in \mathcal{L}$, do the following:

A1. Compute the set S of all vectors of type t in L,

A2. Compute a set of generators of $\mathrm{O}(L)$,

A3. Determine representatives $v_1, \ldots, v_n$ for the action of $\mathrm{O}(L)$ on S,

A4. For $i = 1, \ldots, n$, compute a Gram matrix for $v_i^\perp \cap L$, and add it to $\mathcal{L}'$.

Proposition 7.3. *For each $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ and $\mathcal{L}$ as above, the algorithm above returns a set $\mathcal{L}'$ of representatives for the isometry classes in $\mathcal{G}'$.*

Proof. In order to prove this Proposition, we apply:

(i) Example 2.9 with $(d, m) = (\nu, 1)$ in the case $t = (\nu, \emptyset)$ (in all cases we have that $\det \mathcal{G}$ and ν are coprime so the modulus condition is automatically satisfied),

(ii) Example 2.13 with $d = \nu$ for $t = (\nu, \texttt{char})$ or $t = (\nu, \texttt{exc})$,

(iii) Proposition 5.10 with $d_1 = \det \mathcal{G}$ and $d_2 = \det \mathcal{G}'$ in the case $t = (d_1 d_2, \texttt{sp})$.

Here we have denoted by $\det \mathcal{G}$ the common determinant of any L in $\mathcal{G}$. Each of these statements provides an equivalence between the groupoids $\mathcal{A}$ of pairs (L, v) with L in $\mathcal{G}$ and v in L of type t, and that $\mathcal{B}$ of pairs (N, w) with $N \in \mathcal{G}'$ and $w \in \mathrm{qres}\, N$ having a certain order $o > 1$ and with a certain $q = \mathrm{q}(w) \in \mathbb{Q}/\mathbb{Z}$ determined by t. Those o and q are as follows, with $d_1 = \det \mathcal{G}$ and $d_2 = \det \mathcal{G}'$:

case		(i)		(ii)		(iii)	
o	q	ν	$-\frac{1}{2\nu}$	d_2	$\frac{1}{2}\frac{d_2-1}{d_2}$	d_2	$-\frac{d_1}{2d_2}$

Table 7.4: Values of o and q in each case

We claim that for any given $N \in \mathcal{G}'$, there is a $w \in \mathrm{qres}\, N$, unique up to sign, such that $(N, w) \in \mathcal{B}$. As the isometry -1 of N induces an isomorphism $(N, w) \xrightarrow{\sim} (N, -w)$, it shows that the forgetful functor $\mathcal{B} \to \mathcal{G}', (N, w) \mapsto N$, induces a bijection on isomorphism classes, concluding the proof.

Let us now check the claim. The existence part would follow for instance case-by-case by comparing Tables 7.3 and 7.4. This is a bit tedious, and actually unnecessary, since the fact that we did find in each case at least one (L, v) in $\mathcal{A}$, hence one (N, w) in $\mathcal{B}$, concludes, as all elements in $\mathcal{G}'$ have isomorphic quadratic residues. For uniqueness, assume $w, w' \in \mathrm{qres}\, N$ have the same order o and satisfy $\mathrm{q}(w) = \mathrm{q}(w')$. As $\mathrm{res}\, N$ is cyclic we must have $w' = \lambda w$ for some $\lambda \in \mathbb{Z}$, hence $(1 - \lambda^2)\mathrm{q}(w) \equiv 0 \bmod \mathbb{Z}$. In all cases we have $\mathrm{q}(w) \equiv \frac{x}{2o} \bmod \mathbb{Z}$ for some $x \in \mathbb{Z}$ with $\gcd(x, 2o) = 1$, hence $\lambda^2 \equiv 1 \bmod 2o$, which forces $\lambda \equiv \pm 1 \bmod o$ as we have either $o = 2$ or $2p$ with p prime, hence $w' = \pm w$. □

[24] As ν is always square free, the primitive condition is automatic.

Remark 7.4. (*On Step* A1 *of the algorithm*)

- (Case (ii)) *In order to enumerate the characteristic vectors of norm* ν *in a unimodular lattice* L*, we may choose* $\xi \in \mathrm{Char}\, L$ *and enumerate the norm* ν *vectors in the lattice* $\mathbb{Z}\xi + 2L$ *not belonging to* $2L$*. This is especially simple when* $\nu < 8$ *(exceptional case) and* $\mathrm{r}_1(L) = 0$*, since all nonzero vectors of* $2L$ *have norm* ≥ 8.
- (Case (iii)) *We may enumerate the special vectors by enumerating first the vectors of norm* d_2/d_1 *in* $L^\sharp$*, and apply* $x \mapsto d_1 x$ *(see Proposition 5.10).*

In order to perform Step A3, we may use standard orbit algorithms, such as the one implemented as `qforbits` in `PARI/GP`. However, we claim that *this is unnecessary for all green arrows in Figure 1 !* More precisely:

Proposition 7.5. *For each green arrow* $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ *in Figure 1, and each* $L \in \mathcal{G}$*, there is at most one* $\mathrm{O}(L)$*-orbit of type* t *vectors of* L.

We stress that this proposition, although interesting in itself (see §1.3), is not essential to the proof of Theorem C. Indeed, as we shall see below, in those green situations the number of type t vectors does not exceed a few thousands, so that Step A3 is actually straightforward for the computer. The proof of Proposition 7.5 will use the methods developed in section 5.

Example 7.6. (*Determination of* $\mathcal{G}_{28,5}$) The numbers of exceptional unimodular lattices with no norm 1 vectors of rank 24, 25, 26, 27, 28 and 29 are respectively 24, 0, 97, 557 (Borcherds, Mégarbané, [CH25, §9]), 16381 ([ACH25]) and 2 721 152 (Theorem 6.10). By Lemma 6.3, there are thus exactly

$$24 + 97 + 557 + 16381 + 2721152 = 2738211$$

exceptional unimodular lattices of rank 29, hence as many classes in $\mathcal{G}_{28,5}$ by Proposition 7.5, providing a satisfactory explanation of this large number in Table 1.2. Similarly, $\mathcal{G}_{26,3}$ has $557 + 97 + 24 = 678$ isometry classes, and $\mathcal{G}_{25,2}$ has $24 + 97 = 121$. It is easy to see[25] that for $p = 5, 3, 2$ and a given M in $\mathcal{G}_{23+p,p}$, the number of norm 1 vectors of the exceptional lattice L from which M comes coincides with the number of norm $\frac{p-1}{p}$ vectors in $M^\sharp$, multiplied by 2 for $p = 2$.

Proof. (Of Proposition 7.5) For the three arrows in exceptional cases, the proposition follows from Theorem E. For the two arrows $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ in Figure 1 labelled by $\mathrm{A}_n \to \mathrm{A}_{n+1}$, $\mathcal{G}$ is the genus of even lattices M with $\mathrm{qres}\, M \simeq -\mathrm{qres}\, \mathrm{A}_n$, and we are in the situation of Corollary 5.9. The result follows in this case from Proposition 7.7 below. The green arrow $\mathrm{A}_5 \to \mathrm{A}_6$ is the special case $n = 5$ (odd). The arrow $\mathrm{E}_6 \to \mathrm{E}_7$ could also be dealt with as a special case of Corollary 5.8 for $R \simeq \mathbf{E}_6$ and $c \equiv \varpi_1$ (but this case is not new). □

[25] Let L be an odd exceptional unimodular lattice, $v \in \mathrm{Exc}\, L$ with norm r and set $M = L \cap v^\perp$. The orthogonal projection $L \to M^\sharp$ induces a map $\mathrm{R}_1(L) \to \mathrm{R}_{\frac{r-1}{r}}(M^\sharp)$ which is bijective for $r > 2$, and a $2:1$ surjection for $r = 2$.

Proposition 7.7. *Let M be an even lattice with* $\mathrm{qres}\, M \simeq -\mathrm{qres}\, \mathrm{A}_n$ *for* $n \geq 4$. *Assume that the natural map* $\mathrm{O}(M) \to \mathrm{O}(\mathrm{qres}\, M)$ *is surjective.*[26] *Let $\mathcal{E}$ be the set of primitive $e \in M^\sharp$ with $e \cdot e = \frac{n+2}{n+1}$. Then $\mathrm{O}(M)$ has at most two orbits on $\mathcal{E}$, and at most one if either n is odd or $M^\sharp$ has no vector of norm $\frac{4}{n+1}$.*

Proof. Fix a positive system on A_n, consider the associated (fertile) fundamental weight ϖ_1 and choose an isometry $\eta : \mathrm{qres}\, M \to -\mathrm{qres}\, \mathrm{A}_n$. The isometry group of $\mathrm{qres}\, \mathrm{A}_n$ acts transitively on the vectors $w \in \mathrm{qres}\, \mathrm{A}_n$ with $w \cdot w \equiv \varpi_1 \cdot \varpi_1 \equiv \frac{n}{n+1} \bmod 2\mathbb{Z}$. By our assumption on M, we deduce $\mathcal{E} = \mathrm{O}(M) \cdot \mathrm{Exc}_{\varpi_1,\eta} M$. It is thus enough to determine the number r of $\mathrm{O}(M,\eta)$-orbits on $\mathrm{Exc}_{\varpi_1,\eta} M$.

Let (U,ι) be the pair corresponding to (M,η) under the gluing construction; so U is an even (unimodular) lattice and $\iota : \mathrm{A}_n \to U$ is a saturated embedding. Let υ be the embedding $\upsilon_{\varpi_1} : \mathrm{A}_n \to \mathrm{A}_{n+1}$, let $\mathcal{E}'$ be the set of saturated embeddings $\iota' : \mathrm{A}_{n+1} \to U$ with $\iota' \circ \upsilon = \iota$. By Proposition 5.6, $\mathrm{O}(U,\iota)$ has r orbits on $\mathcal{E}'$.

Let S be the irreducible component of the root system of U containing $\iota(\mathrm{A}_n)$, and denote by s the number of $\mathrm{W}(S)$-orbits of saturated sublattices of $\mathrm{Q}(S)$ isometric to A_{n+1}. By Lemma 7.8 (applied to $m = n+1$), we have $r \leq s$. By Lemma 7.9 applied to $(R,m) = (S, n+1)$, we either have (a) $s \leq 1$, or (b) $s = 2$, n is even, and either $S \simeq \mathbf{D}_{n+2}$ or $n = 4$ and $S \simeq \mathbf{E}_7$. In case (a) we are done, so we may and do assume $r = s = 2$ and that we are in case (b).

There is a sublattice $\iota(\mathrm{A}_n) \subset Q \subset U$ with $Q \simeq \mathrm{D}_{n+2}$. Indeed, this is obvious for $S \simeq \mathbf{D}_{n+2}$, and it follows from Lemma 7.9 (ii) in the case $S \simeq \mathbf{E}_7$, since the orthogonal of a root in E_7 is isometric to D_6. Let $\upsilon' : \mathrm{A}_n \to \mathrm{D}_{n+1}$ denote the isometric embedding associated to any of the fertile weights ϖ_2, ϖ_{n-2} of A_n: see Example 5.4. By Lemma 7.10 there exists an isometric embedding $\mathrm{D}_{n+1} \to U$ extending ι via υ'. We have $\varpi_2 \cdot \varpi_2 = \varpi_{n-2} \cdot \varpi_{n-2} = \frac{2(n-1)}{n+1}$. By Proposition 5.6 again (applied to $R = \mathbf{A}_n$, η, and $c \equiv \varpi_2$ or ϖ_{n-2}), there is an element $f \in M^\sharp$ with $f \cdot f = 2 - \frac{2(n-1)}{n+1} = \frac{4}{n+1}$, concluding the proof. ☐

Lemma 7.8. *Let U be a lattice and $m \geq 1$. Two isometric embeddings $\mathrm{A}_m \to U$ are in the same $\mathrm{W}(U)^\pm$-orbit if and only if their images are.*

Proof. Fix isometric embeddings $f, g : \mathrm{A}_m \to U$ and $w \in \mathrm{W}(U)$ with $w(\mathrm{Im}\, \mathrm{g}) = \mathrm{Im}\, \mathrm{f}$. We only have to show that there is $w' \in \mathrm{W}(U)^\pm$ such that $w' \circ g = f$. Replacing g with $w \circ g$, we may assume $\mathrm{Im}\, g = \mathrm{Im}\, f$. By Lemma 2.4, we may even assume f and g surjective and $U \simeq \mathrm{A}_m$. The result follows since we have $f = w' \circ g$ for some $w' \in \mathrm{O}(U) = \mathrm{W}(U)^\pm$. ☐

We have used the following lemma about embeddings of root systems, for which we omit the details: see [Ki03, Table 4] for some information and [ChTc] for the full details.

Lemma 7.9. *Let R be an irreducible root system and $m \geq 4$. Then either there is at most one $\mathrm{W}(R)^\pm$-orbit of embeddings $\mathrm{A}_m \to \mathrm{Q}(R)$, or there are two and we are in one of the following cases:*

[26] This is automatic if we have $\mathrm{O}(\mathrm{qres}\, \mathrm{A}_n) = \{\pm 1\}$, or equivalently, if we have $n+1 = p^r$, $2p^r$, or 2^r for some odd prime p and $r \geq 1$. This holds for $n = 4, 5$.

(i) $R \simeq \mathbf{D}_{m+1}$ *with* m *odd, and the two orbits are permuted by* $\mathrm{O}(\mathrm{Q}(R))$*,*

(ii) $m = 5$ *and* $R \simeq \mathbf{E}_7$ *(one orbit with orthogonal* A_2*, the other* $\mathrm{A}_1 \perp \langle 6 \rangle$*),*

(iii) $m = 7$ *and* $R \simeq \mathbf{E}_8$*, and one of the two orbits is not saturated.*

Lemma 7.10. *Let* $n \geq 4$*. Let* $\iota : \mathrm{A}_n \to \mathrm{D}_{n+2}$ *and* $\upsilon : \mathrm{A}_n \to \mathrm{D}_{n+1}$ *be isometric embeddings. Then there exists an extension* $\mathrm{D}_{n+1} \to \mathrm{D}_{n+2}$ *of* ι *via* υ*.*

Proof. This follows from the existence of an isometric embedding $\mathrm{D}_{n+1} \to \mathrm{D}_{n+2}$ (obvious) and the fact that the set of isometric embeddings $\mathrm{A}_n \to \mathrm{D}_{n+2}$ consists of a single $\mathrm{O}(\mathrm{D}_{n+2})$-orbit (Lemma 7.9). □

Remark 7.11. (*The brown arrow*) The arrow $\mathrm{A}_4 \to \mathrm{A}_5$ is the case $n = 4$ of Proposition 7.7. We have thus the unique orbit property for type t vectors of most lattices in $\mathcal{G}_{28,5}$ by Example 7.6, namely all the 2 721 152 ones coming from rank 29 exceptional lattices with no norm 1 vectors (note $4/(n+1) = (5-1)/5$).

As promised above, we provide in Table 7.5 below some information about the maximal/average number of type t vectors of the lattices in the genera at the source of an arrow of Figure 1. We exclude the case of exceptional vectors (detailed in Sect.6), the case of Niemeier lattices (discussed below), and that of roots: an inspection of the Coxeter numbers of root systems shows that the number of roots in any lattice of rank $n \geq 16$ is $\leq 2n(n-1)$, which is always ≤ 1512 for $n \leq 28$. The table shows that the maximum number of type t vectors never exceeds a few thousands, showing that Step A3 is a straightforward task for a computer.

dim	det	t		avg	max	dim	det	t		avg	max
28	5	70	sp	1958.6	11 100	28	5	30	sp	2.5	108
27	6	42	sp	2.8	104	26	1	10	char	1002.6	4424
26	3	6	sp	2.8	6	25	2	10	sp	899	2208

Table 7.5: Average `avg` and maximum `max`, of the *nonzero* numbers of type `t` vectors over the lattices of rank `dim` and determinant `det` in Figure 1.

The situation is quite different for vectors of norm 14 (resp. 6) in Niemeier lattices: there are about respectively 187 billions (resp. 17 millions) in each Niemeier lattice! Of course, the isometry groups of a Niemeier lattice L is huge, so we expect far fewer orbits, but enumerating them with the usual algorithm is not reasonable. One idea to circumvent this problem is to divide the computation in two steps: first determine the orbits of $\mathrm{O}(L)$ in $L/2L$, and then, for a representative $\xi \in L/2L$ of such an orbit, determine the $\mathrm{O}(L;\xi)$-orbits of norm 14 (resp. 6) vectors in $\xi + 2L$. For this purpose, the following straightforward variant of Lemma 3.2 (see also Formula (3.1)) is useful:

Lemma 7.12. *The groupoid of pairs* (L, e)*, with* L *a rank* n *even unimodular lattice and* $e \in L/2L$ *with* $e \cdot e \equiv 2 \bmod 4$*, is equivalent to that of rank* n *even lattices* M *with* $\mathrm{qres}\, M \simeq \mathrm{qres}\, \mathrm{A}_1 \perp -\mathrm{qres}\, \mathrm{A}_1$*, via* $(L, e) \mapsto M := \mathrm{M}_2(L; e)$*. In this correspondence, we have* $L = \{v \in M^\sharp \mid v \cdot v \in \mathbb{Z}\}$ *and* $\frac{1}{2}(e + 2L) = M^\sharp \smallsetminus M$*.*

The isometry classes of M's above of rank 24 are easily determined by first computing orbits of mod 2 vectors in Niemeier lattices (see § 2.17 (e)): we find only 339 classes. In order to conclude, it only remains to determine, for each such M, representatives for the $\mathrm{O}(M)$-orbits of vectors of norm $14/4 = 7/2$ (resp. $6/4 = 3/2$) in $M^\sharp$, and then take their orthogonal in M. The maximum number of such vectors in $M^\sharp$ is now only 32 384 (resp. 88), which makes the orbit computation feasible. This ends the proof of Theorem C. □

Remark 7.13. (*Free masses*) Assume $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ appears in Figure 1. Assume also that the lattice N in $\mathcal{G}'$ is obtained as $L \cap v^\perp$ with L in $\mathcal{G}$ and $v \in L$ of type t. Let s be the size of the $\mathrm{O}(L)$-orbit of v in L, a quantity usually given by the orbit algorithm in A3. Set $e = 1$ if $o = 2$, $e = 2$ otherwise (see Table 7.4). Then the proof of Proposition 7.3 shows $\frac{1}{e}|\mathrm{O}(N)| = |\mathrm{O}(N, w)| = |\mathrm{O}(L, v)| = \frac{1}{s}|\mathrm{O}(L)|$. In particular, we obtain for free the masses of all the classes in $\mathcal{G}'$. A s the Plesken-Souvignier algorithm also allows to find generators of $\mathrm{O}(L, v)$, this also provides an alternative method to determine $\mathrm{O}(N)$ which is sometimes more efficient than the direct one (*e.g.* when L is unimodular).

7.14. The invariant $\mathrm{BV}_{n,p}$ and other methods

Our aim now is to prove Theorem D. The assertion about the black entries of Tables 1.2 and 1.3 follows from a straightforward computation. We shall discuss here some new isometry invariants for the lattices in $\mathcal{G}_{n,p}$. Let us emphasize that the sharpness of these invariants, as for BV itself, is quite empirical and based on computer calculations. We apologize for this, and leave as an important open problem to understand why they work.

The lattices in $\mathcal{G}_{n,p}$ being even, a natural idea is to consider their depth 4 invariant BV_4 of §2.14. Unfortunately, they typically have more than 80 000 norm 4 vectors in our ranges for (n, p), so that the computation of BV_4 for such a lattice is about $40^3 = 64\,000$ times slower than that of BV_3 for a rank 29 unimodular lattices for instance. An alternative solution would be to apply (variants of) BV to short vectors in the dual lattices. We focus here on another (but related) method, whose idea is to first glue to an odd unimodular lattice of rank ≤ 29, for which we already know that BV_3 is sharp by Corollary 4.4, and then apply a marked BV invariant of depth 3 introduced in Definition 2.15.

Consider for this the following Table 7.2, in which we have set $S_5 = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right]$ and $S_7 = \left[\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{smallmatrix}\right]$ (two odd lattices with respective determinant 5 and 7).

$p \setminus n$ mod 8	1	2	3	4	5	6	7	8
3	$\langle 2\rangle \perp \langle 3\rangle$	$\langle 3\rangle$	$\langle 2\rangle \perp \langle 3\rangle$		$\langle 6\rangle$	A_2	$\langle 2\rangle \perp \mathrm{A}_2$	
5	$\langle 10\rangle$		$\langle 2\rangle \perp \langle 5\rangle$	$\langle 5\rangle$	$\langle 2\rangle \perp \langle 5\rangle$		$\langle 2\rangle \perp S_5$	S_5
7	$\langle 2\rangle \perp S_7$	S_7	$\langle 2\rangle \perp S_7$		$\langle 14\rangle$	$\langle 7\rangle$	$\langle 2\rangle \perp \langle 7\rangle$	

Table 7.6: Some lattices whose bilinear residue is *opposite* to that of $\mathcal{G}_{n,p}$

Fact 7.15. *Each (n, p)-entry of Table 7.6 is a lattice A (of rank ≤ 3) satisfying* $\mathrm{res}\, A \simeq -\mathrm{res}\, L$ *for any* $L \in \mathcal{G}_{n,p}$.

Proof. This is a simple case-by-case verification using the discussion of §7.1. It is useful to observe that for n even and $\mathcal{G}_{n,p} \neq \emptyset$, then the lattices in $\mathcal{G}_{n-1,p}$ and $\mathcal{G}_{n+1,p}$ have isomorphic *bilinear* residues $\operatorname{res} L \perp \operatorname{res} \mathrm{A}_1$, with $L \in \mathcal{G}_{n,p}$. □

Fix (n,p) and A as above, and choose L in $\mathcal{G}_{n,p}$. By the gluing construction (Proposition 2.7), the choice of an isometry $\eta : -\operatorname{res} A \overset{\sim}{\to} \operatorname{res} L$ defines a unique unimodular overlattice U of $L \perp A$. The isomorphism class of the pair (L,η) uniquely determines that of (U,ι), with $\iota : A \to U$ the natural inclusion. Since the natural morphism $\mathrm{O}(L) \to \mathrm{O}(\operatorname{res} L)$ is surjective by the trivial equality $\mathrm{O}(\operatorname{res} L) = \{\pm 1\}$, the isomorphism class of (U,ι) actually only depends on the isometry class of L: we denote it by $\widetilde{L}$.

Definition 7.16. *For* $L \in \mathcal{G}_{n,p}$ *we set* $\mathrm{BV}^1_{n,p}(L) = \mathrm{BV}_3(U,\iota)$, *with* $(U,\iota) = \widetilde{L}$.

Note that unless $p = 3$ and $n \equiv 6 \bmod 8$ (the grey cell in Table 7.6), the unimodular lattice U is odd, since its rank is $\not\equiv 0 \bmod 8$. If $\operatorname{rank} U \leq 29$, we know that $\mathrm{BV}_3(U)$ is a sharp (and fast to compute) invariant of U by Corollary 4.4, so it is tempting to hope that $\mathrm{BV}^1_{n,p}(L)$ is a sharp invariant[27] of L as well. Note that there is no red entry $(n,3)$ with $n \equiv 6 \bmod 8$ in Tables 1.2 & 1.3.

Fact 7.17. *The invariant* $\mathrm{BV}^1_{n,p}$ *is sharp on* $\mathcal{G}_{n,p}$ *for all red* (n,p) *in Table 1.2 (so* n *even), and all red* $(n,3)$ *in Table 1.3.*

Proof. It follows from a direct computer calculation. See Example 7.24 for a few CPU time indications. □

For the case $(n,p) = (18,7)$ this invariant is overkill: there are only two isometry classes with the same isomorphism class of root system (namely $\mathbf{A}_{17}$), and they are distinguished by their numbers of vectors of norm 4.

Example 7.18. (*Case* $(n,p) = (28,5)$) For $L \in \mathcal{G}_{28,5}$ and $\widetilde{L} = (U,\iota)$, the lattice U is the exceptional unimodular lattice of rank 29 associated to L, so we did expect $\mathrm{BV}^1_{28,5}$ to be sharp on $\mathcal{G}_{28,5}$ by Theorem E.

Although a computation shows that the invariant $\mathrm{BV}^1_{n,p}$ is also sharp for some other cases of (n,p), such as $(19,5)$, $(21,5)$, $(19,7)$, $(25,7)$ and $(27,5)$, it fails to be so in all cases. For instance, the 1396 lattices in $\mathcal{G}_{23,5}$ only have 1370 distinct BV^1 invariants. Our aim now is to define two variants BV^2 and BV^3 of BV^1, actually faster to compute,[28], the combination of which will eventually allow to provide sharp invariants in all cases except $(23,5)$.

Assume n is odd, $p \leq 7$ and set $m = n+1$ or $m = n-1$ so that $m+p \equiv 1 \bmod 4$. Instead of gluing $L \in \mathcal{G}_{n,p}$ with the lattice A in the (n,p)-entry of Table 7.6, we may rather glue it either with the lattice in the (m,p)-entry of this table, or with the A_1 lattice. In the first (resp. second) case, we denote by $\widetilde{L}'$ (resp. $\widetilde{L}''$) the resulting pair (V,ι). The lattice V has determinant 2 (resp. p). For $p = 5, 7$, V is always odd in the first case, as well as in the second case for $n \equiv p \bmod 4$.

[27] The marked invariant is only marginally slower to compute than the unmarked one, so we refer to [ACh25] and §4 for indications about computation times.

[28] Having a faster invariant can be useful for instance if anyone wants to compute a Hecke operator on $\mathcal{G}_{n,p}$.

Definition 7.19. *For n odd, $p \leq 7$ and $L \in \mathcal{G}_{n,p}$, we set* $\mathrm{BV}^2_{n,p}(L) = \mathrm{BV}_3(\widetilde{L}')$ *and* $\mathrm{BV}^3_{n,p}(L) = \mathrm{BV}_3(\widetilde{L}'')$.

For $i = (i_1, \ldots, i_k) \in \{1, 2, 3\}^k$ and $L \in \mathcal{G}_{n,p}$ we also denote by $\mathrm{BV}^i_{n,p}(L)$ the k-uple $(\mathrm{BV}^{i_1}_{n,p}(L), \ldots, \mathrm{BV}^{i_k}_{n,p}(L))$. A computer calculation shows then:

Fact 7.20. *For each black (resp. red) entry i in the box (n, p) of Table 7.7, the invariant $\mathrm{BV}^i_{n,p}$ is sharp (resp. not sharp) on $\mathcal{G}_{n,p}$.*

$p \setminus n$	19	21	23	25	27
3			$1, 2, 3$	$1, 2, 3$	$1, 2, 3$
5	$1, 2, 3$	$1, 2, 3$	$(1, 2, 3)$	$1, 2, 3, (1, 2), (1, 3), (2, 3)$	$1, 2, 3, (2, 3)$
7	$1, 2, 3$	$2, (1, 3)$	$1, 2, 3, (1, 3)$	$1, 2, 3, (2, 3)$	

Table 7.7: Sharpness of the invariants $\mathrm{BV}^i_{n,p}$ for the red entries (n, p) in Table 1.3

The case $(n, p) = (17, 7)$ is not included in this table because as for $(18, 7)$ it is much easier: there are only two isometry classes in this genus having the same isomorphism class of root system (namely $\mathbf{A}_{15}$), and they are distinguished by their number of vectors of norm 4.

Example 7.21. (*Case* $(n, p) = (25, 5)$) The invariant BV^1 falls short of being sharp in this case: there are 38 749 lattices but 38 746 different BV^1 invariants, hence exactly 3 pairs of ambiguous lattices. For only one of these pairs the two lattices have the same root system, namely $\mathbf{A}_1\,\mathbf{A}_2\,2\mathbf{A}_3\,\mathbf{A}_9\,\mathbf{D}_4$! For each of these 3 pairs, the two lattices are distinguished both by BV^2 and BV^3. Similar exceptional behaviors do repeat for some other values of (n, p), which shows that we may have been quite lucky that the assertion about BV holds in Theorem 4.1. Note also that for $(25, 5)$, the average computation time of BV^1, BV^2 and BV^3 are respectively 330 `ms`, 27 `ms`, 17 `ms`.

At this point, the only remaining genera for which we do not have provided any isometry invariant is $\mathcal{G}_{23,5}$, since $\mathrm{BV}^{(1,2,3)}$ is not sharp in this case by Fact 7.20. We now treat this case in a ad hoc way.

Remark 7.22. (*Case* $(n, p) = (23, 5)$) A computation shows that BV^2 is close to be sharp here: the 1 396 lattices have 1 394 distinct BV^2 invariants. Unfortunately, the two pairs with the same BV^2 have a common root system ($4\mathbf{A}_1\,3\mathbf{A}_2\,2\mathbf{A}_3$ for one pair and $2\mathbf{A}_1\,\mathbf{A}_2\,2\mathbf{A}_3\,\mathbf{A}_4\,\mathbf{A}_5$ for the other), as well as the same BV^1 and BV^3. We may distinguish those last lattices in the following ad hoc way. For L in $\mathcal{G}_{23,5}$, the average number of vectors of norm $n \leq 30$ of the *rescaled dual lattice* $L^\flat := \sqrt{\det L}\, L^\sharp$ is given by Table 7.8. We checked that the invariant $f(L) := \mathrm{BV}_{24}(L^\flat)$ does distinguish the two lattices in each pair, *if we compute* BV *using the absolute variant* described in Remark 2.16.

n	4	11	15	16	19	20	21	24	other ≤ 30
$\#$	0.1	2.4	8.0	24.1	85.4	134.4	21	670.4	0

Table 7.8: Average number $\#$ of norm n elements in $\sqrt{\det L}\; L^\sharp$ for $L \in \mathcal{G}_{23,5}$.

We finally define $\mathrm{BV}_{n,p}$. Assume the (n,p)-box is red in Tables 1.2 or 1.3. If $p = 7$ and $n \in \{17, 18\}$ simply define $\mathrm{BV}_{n,p}(L)$ as the pair formed of the isomorphism class of the root system of L and $\mathrm{r}_4(L)$. Now assume that $(n,p) \notin \{(17,7),(18,7)\}$. If n is even, set $\mathrm{BV}_{n,p} = \mathrm{BV}^1_{n,p}$. If $n \geq 19$ is odd and $(n,p) \neq (23,5)$, define $\mathrm{BV}_{n,p}$ as $\mathrm{BV}^i_{n,p}$ for any black entry i in the (n,p)-box of Table 7.7. If we write $a \prec b$ to mean that a is faster than b, we usually have $\mathrm{BV}^3 \prec \mathrm{BV}^2 \prec \mathrm{BV}^{(2,3)} \prec \mathrm{BV}^1$, hence a best choice for i given by Table 7.7. Finally, for $(n,p) = (23,5)$, set $\mathrm{BV}_{n,p} = (\mathrm{BV}^2, f)$ as in Remark 7.22. From Facts 7.17 & 7.20 and that remark, we deduce:

Corollary 7.23. *Theorem D holds for the definition above of* $\mathrm{BV}_{n,p}$.

The first main application of the invariants $\mathrm{BV}_{n,p}$ is that they allow an independent verification that our lists of lattices in Theorem C are complete, using the mass formula.

Example 7.24. (*Independent check that the list of* 285 825 *representatives of* $\mathcal{G}_{27,3}$ *given in* [ChTa] *is complete*) The computation of all $\mathrm{BV}^1_{27,3}$ invariants takes about 6 `h` 45 `min`, *i.e.* about 85 `ms` per lattice (and they are all distinct). Moreover, applying to the given Gram matrices the variant `qfautors` of the Plesken-Souvignier algorithm mentioned in §2.17 (c), or PARI's `qfauto` for the 9 lattices with no roots,[29] the computation of the order of all isometry groups takes about 8 `h` 45 `min`, *i.e.* about 110 `ms` per lattice (and actually, about 85 `ms` for about 99.8% of the lattices). Statistics for reduced isometry groups of order ≤ 256 are given in Table 7.9 (only 509 lattices have a larger reduced isometry group). The total mass of our lattices coincides with the mass formula of $\mathcal{G}_{27,3}$.[30]

`ord`	2	4	6	8	10	12	16	18	20	24	32	36	40	42	48
#	225 451	44 125	707	8 775	11	1 482	1 922	5	31	896	673	19	50	3	398
`ord`	64	72	80	84	96	104	108	120	128	144	160	192	216	240	256
#	135	58	24	10	261	1	2	11	64	57	8	72	19	32	14

Table 7.9: Number # of lattices in $\mathcal{G}_{27,3}$ with reduced isometry group of order `ord` $\leq$ 256.

The invariants $\mathrm{BV}_{n,p}$ also allow to use alternative methods to compute some genera $\mathcal{G}_{n,p}$. For instance, the case $(23,3)$ can be easily dealt with using Kneser neighbors and $\mathrm{BV}_{23,3}$, and this is how we first determined it.

Example 7.25. (*Alternative determination of* $\mathcal{G}_{27,3}$) We initially determined $\mathcal{G}_{27,3}$ using a "backward" method, represented by the dotted arrow $\mathcal{G}_{26,3} \dashrightarrow \mathcal{G}_{27,3}$ in Figure 1, in the spirit of the method used in §4 to deduce the rank 29 unimodular lattices from rank 27 ones. Indeed, a variant of Proposition 3.1 shows that *the groupoid of pairs* (M, α)*, with* $M \in \mathcal{G}_{27,3}$ *and* α *a root of* M *with* $\mathrm{m}(\alpha) = 1$*, is equivalent to that of pairs* (N, e) *with* $N \in \mathcal{G}_{26,3}$ *and* $e \in N/2N$ *with* $e \cdot e \equiv 2 \bmod 4$. Using `orbmod2`, the known classification of $\mathcal{G}_{26,3}$, and

[29] The order of the isometry groups of the 9 lattices with no roots are 18 720 000, 5 760, 4 608, 1 152, 240, 120, 48, 48, 48. Their direct computation is a bit lengthy (1 `h`). However, it only takes a few seconds if we rather use the associated rank 29 unimodular lattices obtained by gluing with $\langle 2 \rangle \perp \langle 3 \rangle$ (*i.e.* if we rather compute $|\mathrm{O}(\widetilde{L})| = |\mathrm{O}(L)|/2$).

[30] It is 184361388591800313635423567792726086296697/62149408003212888749105351334297600000000.

$\mathrm{BV}_{26,3}$, this allows one to determine all isometry classes in $\mathcal{G}_{27,3}$ with nonempty root system. The lattices $M \in \mathcal{G}_{27,3}$ having a root α with $\mathrm{m}(\alpha) = 2$ are precisely those of the form $\mathrm{A}_1 \perp N$ with $\mathcal{G}_{26,3}$. The 9 remaining lattices without root in $\mathcal{G}_{27,3}$ were found using Kneser neighbors (their total mass is 951709/12480000 by [Ch25, Prop. 6.5] and [Ki03]).

8. Rank 30 unimodular lattices with few roots

Our aim in this last section is to discuss the proof of Theorem B. This is a massive computation, which required more than 100 years[31] of CPU time (single core equivalent). Using the observed sharpness of the BV invariant and the Plesken-Souvignier algorithm, the completeness of these lists can be easily checked independently of the way we found them; given our current implementation, it takes less than 3.5 years (see Remark 8.1). Indeed, for each root system R, we know from the work of King [Ki03] (see also [Ch25, §6.4]) the reduced mass $\mathrm{m}_{30}(R)$ of X_{30}^{R}: see Table 8.1. Again, King's lower bounds for $2\,\mathrm{m}_{30}(R)$ were not too far from the actual size of X_{30}^{R} in the three cases considered above.

R	$\mathrm{m}_{30}(R)$	$\approx 2\mathrm{m}_{30}(R)\cdot 10^{-6}$
$\emptyset$	7180069576834562839/175111372800	82.01
$\mathbf{A}_1$	9242148948311/51840	356.56
$\mathbf{A}_2$	25436628608581/4043520	12.58

Table 8.1: The reduced mass $\mathrm{m}_{30}(R)$, and $2\,\mathrm{m}_{30}(R)$ in millions (rounded to 10^{-2}).

We will not give many details about how we found the lists in Theorem B, as the method is close to the one described in details in [ACh25] for the classification of $\mathrm{X}_{29}^{\emptyset}$. We will content ourselves with giving an overview of the main steps, assuming the reader is familiar with [Ch25, ACh25], and to emphasize some novel difficulties we encountered in the two cases $R = \emptyset$ and $R = \mathbf{A}_1$. The various improvements in lattice algorithms described in § 2.17, as well as the notion of visible isometries explained in [Ch25, §7], were of great help in these new computations.

Remark 8.1. *Assume $L \in \mathrm{X}_{30}$ has no norm 1 vectors. We have*[32] $\mathrm{r}_3(L) = 1520 + 12\,\mathrm{r}_2(L) - 64\,|\mathrm{Exc}\,L|$, *the possible values of* $|\mathrm{Exc}\,L|$ *being given by Remark 6.11. In the case* $\mathrm{R}_2(L) = \emptyset$ *(resp.* $\mathbf{A}_1$, $\mathbf{A}_2$*), it follows that the number of vertices of the graph* $\mathcal{G}_{\leq 3}(L)$ *(see § 2.14) is bounded above by* 760 *(resp.* 773, 799*). Perhaps surprisingly, these quantities are slightly smaller than their* 29*-dimensional analogues discussed at the end of Sect. 4 (the second column of Table 1 in* [NV03] *gives an idea of how these quantities vary with the rank). This makes the sharpness of* $\mathrm{BV}(L)$ *even more remarkable here, and its computation faster: it runs in about* 63 `ms`. *By comparison, it takes about* 30 `ms` *to find a good Gram matrix for such a lattice L, and then about* 250 `ms` *(resp.* 120 `ms`, 82 `ms`*) to compute* $|\mathrm{O}(L)^{\mathrm{red}}|$.

[31] Precise CPU time is difficult to estimate (and may have been significantly higher than stated), as our parallel implementation was not fully optimized.

[32] It follows from similar arguments as in [BV01, §4], and from Remark 6.2.

8.2. Case $R = \emptyset$

We started with an exploration of the d-neighbors of I_{30} having an empty visible root system for d ranging from 61 to 147, following the BNE algorithm described in [ACh25, §5.4]. The number of new lattices we found for each d is indicated in Table 8.2 below. Up to $d = 83$, we enumerated *all* the d-neighbors[33] of I_{30}, but we stopped doing so from $d = 84$ for efficiency reasons, preferring to increase d when the algorithm started to yield fewer new lattices. From $d = 96$ to $d = 147$, we only selected about 10^7 isotropic vectors (and ceased selecting minimal vectors in a line as in [ACh25, Rem. 5.9], which is ineffective here).

d	#	d	#	d	#	d	#	d	#	d	#	d	#	d	#	d	#
61	1	71	130	81	260 291	91	9 124 548	101	289 484	111	174 694	121	58 073	131	18 067	141	1 192
62	0	72	1 177	82	907 179	92	8 053 421	102	323 373	112	124 628	122	51 277	132	10 797	142	911
63	0	73	752	83	638 350	93	4 271 277	103	369 491	113	143 898	123	51 319	133	11 486	143	877
64	0	74	4 986	84	3 378 682	94	2 068 536	104	377 261	114	84 183	124	46 145	134	6 407	144	623
65	4	75	5 678	85	1 846 743	95	1 524 120	105	369 945	115	53 617	125	45 977	135	5 786	145	580
66	4	76	20 249	86	6 085 993	96	137 059	106	343 827	116	69 709	126	35 771	136	3 613	146	526
67	3	77	18 940	87	5 008 910	97	145 590	107	384 739	117	67 457	127	37 791	137	4 035	147	243
68	27	78	103 979	88	12 014 052	98	181 599	108	267 665	118	62 403	128	27 466	138	2 268		
69	30	79	57 901	89	6 972 717	99	190 195	109	301 591	119	64 000	129	26 221	139	2 329		
70	282	80	320 713	90	14 040 632	100	256 585	110	180 120	120	53 835	130	17 898	140	1 181		

Table 8.2: Number # of new lattices found in $X_{30}^{\emptyset}$ as d-neighbors of I_{30}

After this massive computation, about 10^5 lattices remained to be found, but it did not seem reasonable to pursue this strategy further. Note that a specific search for exceptional lattices using the method described in [Ch25, §9.3] led to the discovery of only about 4 000 new lattices, most candidates having already been found. At this point, the remaining mass was

$$1593528554589611/M \text{ with } M = 35022274560 = 29 \cdot 13 \cdot 7 \cdot 5 \cdot 3^4 \cdot 2^{15}.$$

In order to "clean" the denominator, we then searched for neighbors having an isometry of prime order $p \mid M$ and a prescribed characteristic polynomial, using the method of visible isometries described in [Ch25, §7] (see also [ACh25, §6.7] for an example). We stopped after finding about 7 600 new lattices, leaving a remaining mass of $13033918217/M'$ with $M' = 3^2 \cdot 2^{15}$. For instance, we found a lattice with mass $1/232$ for $p = 29$, and two lattices with masses $1/8736$ and $1/134784$ for $p = 13$ (with the characteristic polynomial $\Phi_{13}^2 \Phi_1^6$).

To complete the classification, we then computed the 2-neighbors of a suitably chosen collection $\mathcal{C}$ of already found lattices in X_{30}. As explained in [Ch25, §7.5], the 2-neighbors of a lattice with a large isometry group are good candidates for having a non-trivial (or large) isometry group; conversely, those of a lattice with a trivial isometry group are random enough to be useful in the search for the (many) missing lattices with mass 1/2. We therefore used both kinds of lattices in our choice of $\mathcal{C}$.

More precisely, we first included 40 lattices in $\mathcal{C}$, each given as a d-neighbor of I_{30} for odd d between 65 and 147, and we computed *all* the 2-neighbors of those lattices. We also performed a partial computation of the 2-neighbors of the lattice with mass 1/96 found for $d = 65$, and of 4 lattices with mass 1/32 found for $d = 81$ and 85, in order to start hunting the powers of 2 and 3 in M' (but this turned out to be ineffective). These massive neighbor computations allowed us to find more than 95 000 new lattices, and left a remaining mass of

[33] More precisely, all those defined by an isotropic line with some coordinate coprime to d.

4289033/294912 ≈ 14.5. A posteriori, we know that after this step, only 115 lattices were actually missing, with masses given by Table 8.3 (note that all lattices but one have an isometry group which is a 2-group).

mass	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/512	1/2304	1/32768
#	1	29	38	25	13	4	1	2	1	1

Table 8.3: Number # of lattices with mass mass in the last 115 lattices found

We then added to $\mathcal{C}$ about 200 lattices of the form $\mathrm{I}_1 \perp L$ with $L \in \mathrm{X}_{29}^{\emptyset}$ and mass $\leq 1/16$, as well as a few lattices with mass 1/512, 1/256 and 1/16 obtained using the visible isometry method. We found the remaining lattices by computing, for each of them, "only" about 10^7 2-neighbors. For instance, the two lattices in $\mathrm{X}_{30}^{\emptyset}$ with mass $1/32768 = 1/2^{15}$ and 1/2304 were discovered as 2-neighbors of the same lattice $\mathrm{I}_1 \perp L$, where L is the unique class in $\mathrm{X}_{29}^{\emptyset}$ with mass $1/18432 = 1/(2^{11} \cdot 3^2)$. As an anecdote, the last lattice we found has mass 1/16. The final statistics for the isometry groups in $\mathrm{X}_{30}^{\emptyset}$ are given in Table 8.4.

mass	#	e	mass	#	e	mass	#	e	mass	#	e	mass	#	e	mass	#	e
1/2	81706477	4429936	1/40	33	5	1/128	107	20	1/576	2	2	1/2880	1	0	1/20160	1	1
1/4	583827	85387	1/48	144	63	1/144	13	10	1/600	2	2	1/3072	4	2	1/30720	1	1
1/6	688	195	1/54	1	0	1/160	1	1	1/672	1	0	1/3600	1	1	1/32768	1	0
1/8	25837	5127	1/56	9	2	1/192	35	13	1/768	17	7	1/3840	1	0	1/57600	1	1
1/10	23	0	1/60	7	1	1/232	1	1	1/864	1	1	1/4032	2	2	1/82944	1	1
1/12	791	312	1/64	229	53	1/240	5	2	1/960	2	0	1/4096	2	0	1/134784	1	1
1/16	3429	850	1/72	15	8	1/256	24	4	1/1024	7	2	1/4608	4	2	1/161280	2	1
1/18	6	1	1/80	7	4	1/288	7	5	1/1152	3	2	1/5760	2	1	1/184320	1	0
1/20	34	3	1/84	2	2	1/320	1	0	1/1296	1	0	1/6144	3	0	1/688128	1	0
1/24	408	170	1/96	79	31	1/384	26	9	1/1536	4	2	1/7168	1	0	1/1179648	1	1
1/28	3	0	1/100	1	0	1/448	1	0	1/2048	3	1	1/8736	1	1	1/2419200	1	1
1/32	717	159	1/108	2	1	1/480	5	1	1/2304	2	2	1/9216	1	1	1/41287680	1	0
1/36	6	3	1/120	5	1	1/512	17	4	1/2688	1	0	1/18432	2	1			

Table 8.4: Number # of classes (resp. e of exceptional classes) in $\mathrm{X}_{30}^{\emptyset}$ with mass mass

8.3. Cases $R = \mathbf{A}_1$ and $R = \mathbf{A}_2$

We applied a strategy similar to the one described above, using the visible root system R itself. The $\mathbf{A}_2$ case presented no particular surprises, so we will not say anything about it. The $\mathbf{A}_1$ case was especially challenging. Indeed, despite multiple enumerations of neighbors, it ceased to yield new lattices close to the end, leaving a remaining mass of 3/4. At this stage, finding the missing lattices by the neighbor method amounts to searching for a needle in a haystack. Instead, we used an exceptional degree 3 correspondence on X_{30}, which we call the *triplication method*, and which we now briefly explain.

Let Q be the finite quadratic space $-\mathrm{qres}\,(\mathrm{A}_1 \perp \mathrm{A}_1 \perp \mathrm{A}_1)$, and denote by $\mathcal{H}_n$ the genus of even lattices H of rank n satisfying $\mathrm{qres}\, H \simeq Q$.

Proposition 8.4. *Assume* $n \equiv 6 \bmod 8$. *There is a natural equivalence of groupoids between:*

(i) pairs (L, α) *with* L *a rank* n *unimodular lattice and* α *a root of* L.

(ii) pairs (H, w) *with* H *a lattice in* $\mathcal{H}_{n-1}$ *and* $w \in \mathrm{qres}\, H$ *such that* $\mathrm{q}(w) \equiv 3/4 \bmod \mathbb{Z}$.

In this equivalence, we have $H \simeq L^{\text{even}} \cap \alpha^\perp$ *and* $L^{\text{even}} \simeq (H \perp \mathbb{Z}\alpha) + \mathbb{Z}(w+\alpha/2)$.

Proof. We only sketch the proof. The groupoid in (i) is naturally equivalent to that of pairs (M,α) with M an even lattice in the genus of D_n and α a root of M, via $(L,\alpha) \mapsto (L^{\text{even}},\alpha)$. Any such M may be obtained as the orthogonal of some $\mathrm{D}_2 \simeq \mathrm{A}_1 \perp \mathrm{A}_1$ inside an even unimodular lattice U of rank $n+2$. So $H := M \cap \alpha^\perp$ is the orthogonal in U of some $\mathrm{A}_1 \perp \mathrm{A}_1 \perp \mathrm{A}_1$, showing $H \in \mathcal{H}_{n-1}$. The equivalence between (i) and (ii) is then a consequence of Proposition 2.7 in the even context, with $A = \mathrm{A}_1$ and $H = \mathrm{qres}\, A$. $\square$

We have $Q = \mathbb{Z}/2e_1 \perp \mathbb{Z}/2e_2 \perp \mathbb{Z}/2e_3$ with $\mathrm{q}(e_i) \equiv 3/4 \bmod \mathbb{Z}$ for $1 \le i \le 3$. For each H in $\mathcal{H}_{n-1}$, there are thus exactly 3 elements $w \in \mathrm{qres}\, H$ satisfying $\mathrm{q}(w) \equiv 3/4$, hence 3 corresponding pairs (L,α), explicitly given by the last formula in the proposition. The relation $H \simeq L \cap \alpha^\perp$ shows

$$\mathrm{R}_2(L) \simeq \mathbf{A}_1 \text{ or } \mathbf{A}_2 \iff \mathrm{R}_2(H) = \emptyset.$$

The construction above thus associates to an isometry class $[L] \in \mathrm{X}_n^{\mathbf{A}_1} \sqcup \mathrm{X}_n^{\mathbf{A}_2}$ a 3-element multiset of classes in $\mathrm{X}_n^{\mathbf{A}_1} \sqcup \mathrm{X}_n^{\mathbf{A}_2}$ containing $[L]$. This triple is easily computed. Applied to our list of found lattices in $\mathrm{X}_{30}^{\mathbf{A}_1}$, this method allowed us to produce a list of rank 30 unimodular lattices with root system $\mathbf{A}_1$ or $\mathbf{A}_2$ that is three times larger (but of course, with much redundancy). The computation of the BV invariants of all the new classes happily led us to discover the 3 remaining lattices in $\mathrm{X}_{30}^{\mathbf{A}_1}$, each having the mass $1/4$. This concludes the proof, up to the fact that these last 3 lattices are not yet given as d-neighbors of I_{30}. For this last step we use the following:

Lemma 8.5. *Let* H *be an even lattice* $\mathcal{H}_{n-1}$ *with* $n \equiv 6 \bmod 8$. *Let* W *be the* 3*-element set of* $w \in \mathrm{qres}\ H$ *with* $\mathrm{q}(w) \equiv 3/4 \bmod \mathbb{Z}$, *and for* w *in* W, *let* L_w *be the rank* n *unimodular lattice associated to* (H,w) *under the equivalence of Proposition 8.4. Then* $L_{w'}$ *is a* 2*-neighbor of* L_w *for any* $w' \neq w$ *in* W.

Proof. Set $N = H \perp \mathrm{A}_1$ and write $\mathrm{A}_1 = \mathbb{Z}\alpha$. For $w \in W$ we have $L_w^{\text{even}} = N + \mathbb{Z}(w+\alpha/2)$, and we easily check $L_w = L_w^{\text{even}} + \mathbb{Z}(w'+w'')$ where $\{w,w',w''\} = W$. We have thus $L_w \cap L_{w'} = N + \mathbb{Z}(\alpha/2 + w + w' + w'')$, and $L_w \cap L_{w'}$ has index 2 in L_w and L'_w. $\square$

A neighbor form for the last 3 lattices was finally obtained as follows. For each such lattice L, we computed 2-neighbors of L until we found a lattice L' belonging to our list and represented as a d-neighbor of I_{30} for some odd d (this is very fast). We then computed 2-neighbors of L' with root system $\mathbf{A}_1$, which can easily be done on neighbor forms using [Ch25, Lemma 11.2], until we found one with the same BV invariant as L.

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