LEVEL ONE ALGEBRAIC CUSP FORMS OF CLASSICAL GROUPS OF SMALL RANK

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Abstract. We determine the number of level 1, polarized, algebraic regular, cuspidal automorphic representations of GL_n over $\mathbb{Q}$ of any given infinitesimal character, for essentially all $n \leq 8$. For this, we compute the dimensions of spaces of level 1 automorphic forms for certain semisimple $\mathbb{Z}$-forms of the compact groups SO_7, SO_8, SO_9 (and G_2) and determine Arthur's endoscopic partition of these spaces in all cases. We also give applications to the 121 even lattices of rank 25 and determinant 2 found by Borcherds, to level one self-dual automorphic representations of GL_n with trivial infinitesimal character, and to vector valued Siegel modular forms of genus 3. A part of our results are conditional to certain expected results in the theory of twisted endoscopy.

1. Introduction

1.1. **A counting problem.** Let $n \geq 1$ be an integer. Consider the cuspidal automorphic representations π of GL_n over $\mathbb{Q}$ (see [GGPS66, Ch. 3],[BJ79, §4],[Cog04]) such that :

(a) (polarization) $\pi^\vee \simeq \pi \otimes |\cdot|^w$ for some $w \in \mathbb{Z}$,
(b) (conductor 1) π_p is unramified for each prime p,
(c) (algebraicity) π_∞ is algebraic and regular.

Our main aim in this paper is to give for small values of n, namely for $n \leq 8$, the number of such representations as a function of π_∞. Recall that by the Harish-Chandra isomorphism, the infinitesimal character of π_∞ may be viewed following Langlands as a semisimple conjugacy class in $\mathrm{M}_n(\mathbb{C})$ (see §3.7, §3.11). Condition (c) means[1] that the eigenvalues of this conjugacy class are distinct integers. The opposite of these integers will be called the *weights* of π and we shall denote them by $k_1 > k_2 > \cdots > k_n$. When $n \equiv 0 \bmod 4$, we will eventually allow that $k_{n/2} = k_{n/2+1}$ but to simplify we omit this case in the discussion for the moment. If π satisfies (a), the necessarily unique integer $w \in \mathbb{Z}$ such that $\pi^\vee \simeq \pi \otimes |\cdot|^w$ will be called the *motivic weight* of π, and denoted $w(\pi)$.

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[1] The term *algebraic* here is in the sense of Borel [Bor77, §18.2], and is reminiscent to Weil's notion of Hecke characters of type A_0 : see §3.11. Langlands also uses the term *of type Hodge*, e.g. in [Lan96, §5]. See also [BG], who would employ here the term L-*algebraic*, for a discussion of other notions of algebraicity, as the one used by Clozel in [Clo90].

Problem 1.2. *For any $n \geq 1$, determine the number $\mathrm{N}(k_1, k_2, \cdots, k_n)$ of cuspidal automorphic representations π of GL_n satisfying* (a)*,* (b) *and* (c) *above, and of weights $k_1 > k_2 > \cdots > k_n$.*

An important finiteness result of Harish-Chandra ([HC68, Thm. 1.1]) asserts that this number is indeed finite, even if we omit assumption (a). As far as we know, those numbers have been previously computed only for $n \leq 2$. For $n = 1$, the structure of the idèles of $\mathbb{Q}$ shows that if π satisfies (a), (b) and (c) then $w(\pi) = 2\,k_1$ is even and $\pi = |\cdot|^{-k_1}$. By considering the central character of π, this also shows the relation $n\,w(\pi) = 2\sum_{i=1}^{n} k_i$ for general n. More interestingly, classical arguments show that $\mathrm{N}(k-1, 0)$ coincides with the dimension of the space of cuspidal modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, whose dimension is well-known (see e.g. [Ser70]) and is about[2] $[k/12]$. Observe that up to twisting π by $|\cdot|^{k_n}$, there is no loss of generality in assuming that $k_n = 0$ in the above problem. Moreover, condition (a) implies for $i = 1, \cdots, n$ the relation $k_i + k_{n+1-i} = w(\pi)$.

1.3. **Motivations.** There are several motivations for this problem. A first one is the deep conjectural relations, due on the one hand to Langlands [Lan79], in the lead of Shimura, Taniyama, and Weil, and on the other hand to Fontaine and Mazur [FM95], that those numbers $\mathrm{N}(k_1, k_2, \cdots, k_n)$ share with arithmetic geometry and pure motives[3] over $\mathbb{Q}$. More precisely, consider the three following type of objects :

- (I) Pure motives M over $\mathbb{Q}$, of weight w and rank n, with coefficients in $\overline{\mathbb{Q}}$, which are : simple, of conductor 1, such that $M^\vee \simeq M(w)$, and whose Hodge numbers satisfy $h^{p,q}(M) = 1$ if (p, q) is of the form $(k_i, w - k_i)$ and 0 otherwise.
- $(\mathrm{II})_\ell$ Continuous irreducible representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are unramified outside ℓ, crystalline at ℓ with Hodge-Tate numbers $k_1 > \cdots > k_n$, and such that[4] $\rho^\vee \simeq \rho \otimes \omega_\ell^w$.
- (III) Cuspidal automorphic representations π of GL_n over $\mathbb{Q}$ satisfying (a), (b) and (c) above, of weights $k_1 > k_2 > \cdots > k_n$,

Here ℓ is a fixed prime, and $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_\ell$ are fixed algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_\ell$. To discuss the aforementioned conjectures we need to fix a pair of fields embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_\ell : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$. According to Fontaine and Mazur, Grothendieck's ℓ-adic étale cohomology, viewed with $\overline{\mathbb{Q}}_\ell$ coefficients via ι_ℓ, should induce a bijection between isomorphism classes of motives of type (I) and isomorphism classes of Galois representations of type $(\mathrm{II})_\ell$. Moreover, according to Langlands, the L-function of the ℓ-adic realizations of a motive of type (I), which makes sense via ι_∞ and ι_ℓ, should be the standard L-function of a unique π of type (III), and vice-versa. These conjectural bijections are actually expected to exist in greater generality (any conductor, any weights, not necessarily polarized), but we focus on this case as it is the one we really consider in this paper. In particular, $\mathrm{N}(k_1, \cdots, k_n)$ *is also the conjectural number of isomorphism*

[2]We denote by $[x]$ the floor of the real number x.

[3]The reader is free here to choose his favorite definition of a pure motive [Mot94].

[4]Here ω_ℓ denotes the ℓ-adic cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and our convention is that its Hodge-Tate number is -1.

classes of objects of type (I) *or* $(\mathrm{II})_\ell$ *for any* ℓ. Let us mention that there has been recently important progresses toward those conjectural bijections. First of all, by the works of many authors (including Deligne, Langlands, Kottwitz, Clozel, Harris, Taylor, Labesse, Shin, Ngô and Waldspurger, see [GRFA11],[Shi11] and [CH13]), if π is of type (III) then there is a unique associated semisimple representation $\rho_\pi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ of type $(\mathrm{II})_\ell$ with the same L-function as π (via ι_∞, ι_ℓ), up to the fact that $\rho_{\pi,\iota}$ is only known to be irreducible when $n \leq 5$ (see [CG11]). Second, the advances in modularity results in the lead of Wiles and Taylor, such as the proof of Serre's conjecture by Khare and Wintenberger (see e.g. [Kha06]), or the recent results [BGGT], contain striking results toward the converse statement.

An important source of objects of type (I) or $(\mathrm{II})_\ell$ comes from the cohomology of proper smooth schemes (or stacks) over $\mathbb{Z}$, about which solving problem 1.2 would thus shed interesting lights. This applies in particular to the moduli spaces $\overline{\mathcal{M}_{g,n}}$ of stable curves of genus g with n-marked points and to certain spaces attached to the moduli spaces of principally polarized abelian varieties (see e.g. [BFG11] and [FC90]). As an example, the vanishing of some $\mathrm{N}(k_1, \cdots, k_n)$ translates to a conjectural non-existence theorem about Galois representations or motives. A famous result in this style is the proof by Abrashkin and Fontaine that there are no abelian scheme over $\mathbb{Z}$ (hence no projective smooth curve over $\mathbb{Z}$ of nonzero genus), which had been conjectured by Shafarevich (see [Fon85],[Fon93]). The corresponding vanishing statement about cuspidal automorphic forms had been previously checked by Mestre and Serre (see [Mes86]). See also Khare's paper [Kha07] for other conjectures in this spirit as well as a discussion about the applications to the generalized Serre's conjecture.

A second motivation, which is perhaps more exotic, is the well-known problem of finding an integer $n \geq 1$ such that the cuspidal cohomology $\mathrm{H}^*_{\mathrm{cusp}}(\mathrm{SL}_n(\mathbb{Z}), \mathbb{Q})$ does not vanish. It would be enough to find an integer $n \geq 1$ such that $\mathrm{N}(n-1, \cdots, 2, 1, 0) \neq 0$. Results of Mestre [Mes86], Fermigier [Fer96] and Miller [Mil02] ensure that such an n has to be ≥ 27 (although those works do not assume the self-duality condition). We shall go back to these questions at the end of this introduction.

Last but not least, it follows from Arthur's endoscopic classification [Art11] that the dimensions of various spaces of modular forms for classical reductive groups over $\mathbb{Z}$ have a "simple" expression in terms of these numbers. Part of this paper is actually devoted to explain this relation in very precise and concrete terms. This includes vector valued holomorphic Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$ and level 1 algebraic automorphic forms for the $\mathbb{Z}$-forms of $\mathrm{SO}_{p,q}(\mathbb{R})$ which are semisimple over $\mathbb{Z}$ (such group schemes exist when $p - q \equiv 0, \pm 1 \bmod 8$). It can be used in both ways : either to deduce the dimensions of these spaces of modular forms from the knowledge of the integers $\mathrm{N}(-)$, or also to compute these last numbers from known dimension formulas. We will say much more about this in what follows as this is the main theme of this paper (see Chapter 3).

1.4. **The main result.** We will now state our main theorem. As many results that we prove in this paper, it depends on the fabulous work of Arthur in [Art11]. As explained *loc. cit.*, Arthur's results are still conditional to the stabilization of the twisted trace formula at the moment. All the results below depending on this assumption will be marked by a simple star *. We shall also need to use certain results concerning inner forms of classical groups which have been announced by Arthur (see [Art11, Chap. 9]) but which are not yet available or even precisely stated. We have thus formulated the precise general results that we expect in two explicit Conjectures 3.26 and 3.30. Those conjectures include actually a bit more than what has been announced by Arthur in [Art11], namely also the standard expectation that for Adams-Johnson archimedean Arthur parameters, there is an identification between Arthur's packets in [Art11] and the ones of Adams and Johnson in [AJ87]. The precise special cases that we need are detailed in §3.29. We state in particular Arthur's multiplicity formula in a completely explicit way, in a generality that might be useful to arithmetic geometers. We will go back to the shape of this formula in §1.20.2. All the results below depending on the assumptions of [Art11] as well as on the assumptions 3.26 and 3.30 will be marked by a double star **. Of course, the tremendous recent progresses in this area allow some optimism about the future of all these assumptions ! [5] Besides Arthur's work, let us mention the following results which play a crucial role here : the proof by Chaudouard, Laumon, Ngô and Waldspurger of Langlands' fundamental lemma ([Wald09],[Ngô10],[CLau12]), the works of Shelstad [She08] and Mezo [Mezb] on endoscopy for real groups, and the recent works of Labesse and Waldspurger on the twisted trace formula [LW13].

Theorem ** **1.5.** *Assume $n \leq 8$ and $n \neq 7$. There is an explicit, computable, formula for* $\mathrm{N}(k_1, \cdots, k_n)$.

Although our formulas are explicit, one cannot write them down here as they are much too big : see §1.20.1 for a discussion of the formula. Nevertheless, we implemented them on a computer and have a program which takes $(k_1, k_2, \ldots, k_n)$ as input and returns $\mathrm{N}(k_1, \ldots, k_n)$. When $k_1 - k_n \leq 100$, the computation takes less than ten minutes on our machine[6] : see the website [CR] for some data and for our computer programs. We also have some partial results concerning $\mathrm{N}(k_1, \ldots, k_7)$. This includes an explicit upper bounds for these numbers as well as their values modulo 2, which is enough to actually determine them in quite a few cases (for instance whenever $k_1 - k_7 \leq 26$). On the other hand, as we shall see in Proposition 1.16 below, these numbers are also closely related to the dimensions of the spaces of vector valued Siegel modular forms for $\mathrm{Sp}_6(\mathbb{Z})$. In a remarkable recent work, Bergström, Faber and van der Geer [BFG11] actually found a conjectural explicit formula for those dimensions. Their method is completely different from ours : they study the number of points over finite fields of $\mathcal{M}_{3,n}$ and of certain bundles over the moduli space of principally polarized abelian varieties of dimension 3. Fortunately, in the few hundreds of cases where our work allow to compute this dimension as well, it fits the results found by the formula of these authors ! Even better, if we assume their formula we obtain in turn a conjectural explicit formula for $\mathrm{N}(k_1, \cdots, k_7)$.

[5]Note added in proof : in 2014, Moeglin and Waldspurger have published on the arXiv a series of preprint culminating to a proof of the stabilization of the twisted trace formula, making thus unconditional the results of [Art11] hence the results of this paper which are marked with a simple star.

[6]Four processors Northwood Pentium 4, 2.80 GHz, 5570.56 BogoMIPS.

1.6. **Langlands-Sato-Tate groups.** We not only determine $\mathrm{N}(k_1, \cdots, k_n)$ for $n \leq 8$ (with the caveat above for $n = 7$) but we give as well the conjectural number of π of weights $k_1 > \cdots > k_n$ having any possible *Langlands-Sato-Tate group*. We refer to the appendix B for a brief introduction to this conjectural notion (see also [Ser68, Ch. 1, appendix]). Here are certain of its properties.

First, a representation π as above being given, the Langlands-Sato-Tate group of π (or, for short, its *Sato-Tate group*) is a compact Lie group $\mathcal{L}_\pi \subset \mathrm{GL}_n(\mathbb{C})$, which is well-defined up to $\mathrm{GL}_n(\mathbb{C})$-conjugacy. It is "defined" as the image of the conjectural Langlands group $\mathcal{L}_\mathbb{Z}$ of $\mathbb{Z}$, that we view as a topological following Kottwitz, under the hypothetical morphism $\mathcal{L}_\mathbb{Z} \to \mathrm{GL}_n(\mathbb{C})$ attached to $\pi \otimes |\cdot|^{\frac{w(\pi)}{2}}$ ([Lan79],[Kot88],[Art02]). The natural representation of $\mathcal{L}_\pi$ on $\mathbb{C}^n$ is irreducible and self-dual.

The group $\mathcal{L}_\pi$ is equipped with a collection of conjugacy classes

$$\mathrm{Frob}_p \subset \mathcal{L}_\pi$$

which are indexed by the primes p, and such that for each p the $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of Frob_p is the Satake parameter of π_p multiplied by the scalar $p^{\frac{-w(\pi)}{2}}$. Observe that a necessary condition for this is that the eigenvalues of the Satake parameter of π_p all have absolute value $p^{\frac{w(\pi)}{2}}$. This is the so-called Ramanujan conjecture for π, and it is actually known for each π satisfying (a) and (c) thanks to Deligne's proof of Weil's conjectures and results of Clozel-Harris-Labesse, Shin and Caraiani (see [GRFA11],[Shi11] and [Car12]).

The volume 1 Haar measure of $\mathcal{L}_\pi$ induces a natural measure on its space of conjugacy classes and one of the key expected properties of $\mathcal{L}_\pi$ is that the Frob_p are equidistributed in this space. Let us mention that a pleasant consequence of property (b) of π is that $\mathcal{L}_\pi$ is necessarily connected (as $\mathrm{Spec}(\mathbb{Z})$ is simply connected !). A case-by-case argument, solely based on the fact that $\mathbb{C}^n$ is a self-dual irreducible representation of the connected compact group $\mathcal{L}_\pi$, shows that the list of all the possible Sato-Tate groups is rather small when $n \leq 8$: see Appendix B.

1.7. **The symplectic-orthogonal alternative.** Our main aim now will be to discuss our results in each particular dimension n. Before doing so it will be convenient to introduce more notations. We shall make an important use of automorphic representations π of GL_n satisfying an assumption which is slightly weaker than (c), that we now have to introduce. Assume that π is a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ satisfying property (a) above, so that the integer $w(\pi)$ still makes sense in particular. Consider the property :

(c') The eigenvalues of the infinitesimal character of π_∞, viewed as a semisimple conjugacy class in $\mathrm{M}_n(\mathbb{C})$, are integers. Moreover, each of these eigenvalues has multiplicity one, except perhaps the eigenvalue $-w(\pi)/2$ which is allowed to have multiplicity 2 when $n \equiv 0 \bmod 4$.

A π satisfying (a), (b) and (c') still has weights $k_1 \geq \cdots \geq k_n$ defined as the opposite of the eigenvalues of the infinitesimal character of π_∞, counted with multiplicities. When (c) is not satisfied, then $n \equiv 0 \bmod 4$ and $k_1 > \cdots > k_{n/2} = k_{n/2+1} > \cdots > k_n$. It would not be difficult to extend the conjectural picture suggested by the Langlands and Fontaine-Mazur conjectures to those π's, but we shall not do so here. Let us simply say that when π satisfies (a), (b), (c') but not (c), one still knows how to construct a semisimple continuous Galois representation $\rho_\pi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ unramified outside ℓ and with the same L-function as π : see [GOL12]. It is expected but not known that ρ_π is crystalline at ℓ, and that the Ramanujan conjecture holds for π. However, we know from [TAI12] that ρ_π is Hodge-Tate at ℓ (with Hodge-Tate numbers the k_i) and that for any complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $\mathrm{Trace}\,\rho_\pi(c) = 0$.

We now consider a quite important property of the π satisfying (a), (b) and (c'), namely their *orthogonal-symplectic alternative.*

Definition 1.8. *Let π be a cuspidal automorphic representation of* GL_n *over* $\mathbb{Q}$ *satisfying* (a), (b) *and* (c') *above. We say that π is symplectic if $w(\pi)$ is odd, and orthogonal otherwise.*

Let $k_1 > \cdots > k_n$ denote the weights of π. The relation $n\,w(\pi) = 2\sum_{i=1}^n k_i$ shows that π is necessarily orthogonal if n is odd. Definition 1.8 fits with the conjectural picture described above. For instance, the main theorem of [BC11] asserts that if π is orthogonal (resp. symplectic), and if ρ_π denotes the Galois representation associated to π and $(\iota_\infty, \iota_\ell)$ as discussed above, then there is a nondegenerate symmetric (resp. alternate) $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant pairing $\rho_\pi \otimes \rho_\pi \to \omega_\ell^{-w(\pi)}$ (see also [TAI12] when (c) is not satisfied). As we shall see in §3.8, the definition above also fits with the Arthur-Langlands classification of self-dual cuspidal automorphic representations of GL_n. This means that if π satisfy (a), (b) and (c'), then π is symplectic (resp. orthogonal) if and only if the self-dual representation $\pi \otimes |\cdot|^{w(\pi)/2}$ is so in the sense of Arthur.

We now come to an important notation that we shall use. Assume that π satisfies (a), (b), (c') and is of weights $k_1 > \cdots > k_n$. We will say that π is *centered* if $k_n = 0$. Up to twisting π if necessary, we may focus on centered π's. Assume that π is centered. The symmetry property $k_i + k_{n+1-i} = w(\pi)$ (for $i = 1, \dots, n$) shows that $k_1 = w(\pi)$ is at the same time the biggest weight and the motivic weight of π. Set $r = [n/2]$ and introduce the integers

$$w_i = 2k_i - w(\pi)$$

for $i = 1, \dots, r$. Those numbers will be called the *Hodge weights* of π. Observe that $w_1 = w(\pi)$, $w_i \equiv w(\pi) \bmod 2$ for each i, and that $w_1 > \cdots > w_r \geq 0$. The n weights k_i of π can be recovered from the r Hodge weights w_i : they are the $2r$ integers $\frac{w_1 \pm w_i}{2}$ when $n = 2r$ is even, and the $2r+1$ integers $\frac{w_1 \pm w_i}{2}$ and $w_1/2$ when $n = 2r+1$ is odd. Observe also that $w_r = 0$ if and only if property (c) is not satisfied.

Definition 1.9. *Let $r \geq 1$ be an integer. Let $w_1 > \cdots > w_r$ be nonnegative integers which are all congruent mod 2.*

(i) *If the w_i are odd, we denote by* $\mathrm{S}(w_1, \cdots, w_r)$ *the number of cuspidal automorphic representations π of* GL_{2r} *satisfying* (a), (b), (c)*, which are symplectic, and with Hodge weights $w_1 > \cdots > w_r$.*

(ii) *If the w_i are even, we denote by* $\mathrm{O}(w_1, \cdots, w_r)$ *(resp.* $\mathrm{O}^*(w_1, \cdots, w_r)$*) the number of cuspidal automorphic representations* π *of* GL_{2r} *(resp.* GL_{2r+1}*) satisfying* (a), (b), (c')*, which are orthogonal, and with Hodge weights* $w_1 > \cdots > w_r$*.*

It follows from these definitions that if $k_1 > \cdots > k_n$ are distinct integers, if $k_n = 0$, and if $w_i = 2k_i - k_1$ for $i = 1, \ldots, [n/2]$, then $\mathrm{N}(k_1, \cdots, k_n)$ coincides with : $\mathrm{S}(w_1, \cdots, w_{n/2})$ if n is even and k_1 is odd, $\mathrm{O}(w_1, \cdots, w_{n/2})$ if n is even and k_1 is even, $\mathrm{O}^*(w_1, \cdots, w_{(n-1)/2})$ if n is odd and k_1 is even.

1.10. **Case-by-case description, examples in low motivic weight.** Let us start with the symplectic cases. As already mentioned, a standard translation ensures that for each odd integer $w \geq 1$ the number $\mathrm{S}(w)$ is the dimension of the space $\mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z}))$ of cusp forms of weight $w+1$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. We therefore have the well-known formula

$$\mathrm{S}(w) = \dim \mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z})) = [\frac{w+1}{12}] - \delta_{w \equiv 1 \bmod 12} \cdot \delta_{w>1} \tag{1.1}$$

where δ_P is 1 if property P holds and 0 otherwise. The Sato-Tate group of each π of GL_2 satisfying (a), (b) and (c) is necessarily the compact group $\mathrm{SU}(2)$.

The next symplectic case is to give $\mathrm{S}(w, v)$ for $w > v$ odd positive integers. This case, which is no doubt well known to the experts, may be deduced from Arthur's results [Art11] and a computation by R. Tsushima [Tsu83]. Let $\mathrm{S}_{(w,v)}(\mathrm{Sp}_4(\mathbb{Z}))$ be the space of vector-valued Siegel modular forms of genus 2 for the coefficient systems $\mathrm{Symm}^j \otimes \det^k$ where $j = v - 1$ and $k = \frac{w-v}{2} + 2$ (we follow the conventions in [Geer08, §25]). Using the geometry of the Siegel threefold, Tsushima was able to give an explicit formula for $\dim \mathrm{S}_{(w,v)}(\mathrm{Sp}_4(\mathbb{Z}))$ in terms of (w, v). This formula is already too big to give it here, but see loc. cit. Thm. 4. There is a much simpler closed formula for the Poincaré series of the $\mathrm{S}(w, 1)$ due to Igusa : see [Geer08, §9]. An examination of Arthur's results [Art11] for the Chevalley group $\mathrm{SO}_{3,2} = \mathrm{PGSp}_4$ over $\mathbb{Z}$ shows then that

$$\mathrm{S}(w, v) = \dim \mathrm{S}_{(w,v)}(\mathrm{Sp}_4(\mathbb{Z})) - \delta_{v=1} \cdot \delta_{w \equiv 1 \bmod 4} \cdot \mathrm{S}(w). \tag{1.2}$$

The term which is subtracted is actually the dimension of the Saito-Kurokawa subspace of $\mathrm{S}_{(w,v)}(\mathrm{Sp}_4(\mathbb{Z}))$. That this is the only term to subtract is explained by Arthur's multiplicity formula (see §4.2). We refer to Table 6 for the first nonzero values of $\mathrm{S}(w, v)$. It follows for instance that for $w \leq 23$, then $\mathrm{S}(w, v) = 0$ unless (w, v) is in the following list :

$$(19, 7), (21, 5), (21, 9), (21, 13), (23, 7), (23, 9), (23, 13)$$

In all those cases $\mathrm{S}(w, v) = 1$. Moreover the first w such that $\mathrm{S}(w, 1) \neq 0$ is $w = 37$. The Sato-Tate group of each symplectic π of GL_4 satisfying (a), (b) and (c) is either the compact connected Lie group $\mathrm{Spin}(5) \subset \mathrm{GL}_4(\mathbb{C})$ or $\mathrm{SU}(2) \subset \mathrm{GL}_4(\mathbb{C})$ (symmetric cube of the standard representation). This latter case should only occur when $w = 3v$ (and $\mathrm{S}(v)$ times!), so at least for $w = 33$, and thus should not occur in the examples above. This special case of Langlands functoriality is actually a theorem of Kim and Shahidi [KS02].

Before going further, let us mention that as Bergström and Faber pointed out to us, although Tsushima's explicit formula is expected to hold for all $(w, v) \neq (3, 1)$ it is only proved in [Tsu83] for $w-v \geq 6$. That it holds as well in the remaining cases $w-v=2,4$ (and $v>1$) has actually been recently proved by Taïbi (at least under assumption $*$, see his forthcoming work), and we shall assume it here to simplify the discussion. This actually would not matter for the numerical applications that we will discuss in this introduction. Indeed, the method that we shall describe leads to an independent upper bound on $\mathrm{S}(w,v)$ showing that $\mathrm{S}(w,v)=0$ whenever $w-v\leq 4$ and $w \leq 27$, as predicted by Tsushima's formula[7].

Our first serious contribution is the computation of $\mathrm{S}(w_1,\ldots,w_r)$ for $r=3$ and 4 (and any w_i). Our strategy is to compute first the dimension of the spaces of level 1 automorphic forms for two certain special orthogonal $\mathbb{Z}$-group schemes SO_7 and SO_9 which are reductive over $\mathbb{Z}$. These groups are the special orthogonal groups of the root lattice E_7 and $\mathrm{E}_8 \oplus \mathrm{A}_1$ respectively. They have compact real points $\mathrm{SO}_n(\mathbb{R})$ for $n=7$ and 9 and both have class number 1, so that we are reduced to determine the dimension of the invariants of their integral points, namely the positive Weyl group of E_7 and the Weyl group of E_8, in any given finite dimensional irreducible representation of the corresponding $\mathrm{SO}_n(\mathbb{R})$. We will say more about this computation in §1.20.1 and in Chapter 2. The second important step is to rule out all the endoscopic or non-tempered contributions predicted by Arthur's theory for those groups to get the exact values of $\mathrm{S}(-)$. This is done case-by-case by using the explicit form of Arthur's multiplicity formula that we expect. In the cases of SO_7 (resp. SO_9) there are for instance 9 (resp. 16) multiplicity formulas to determine. They require in particular the computations of $\mathrm{S}(-)$, $\mathrm{O}(-)$ and $\mathrm{O}^*(-)$ for smaller ranks n : we refer to Chapters 5 and 6 for the complete study.

We refer to Tables 7 and 8 for the first non zero values of $\mathrm{S}(w_1,\cdots,w_r)$ for $r=3,4$, and to the url [CR] for much more data. Here is a small sample of our results.

Corollary ** **1.11.** (i) $\mathrm{S}(w_1,w_2,w_3)$ *vanishes for* $w_1<23$.

(ii) *There are exactly* 7 *triples* (w_1,w_2,w_3) *with* $w_1=23$ *such that* $\mathrm{S}(w_1,w_2,w_3)$ *is nonzero :*

$$(23,13,5),\ (23,15,3),\ (23,15,7),\ (23,17,5),\ (23,17,9),\ (23,19,3),\ (23,19,11),$$

and for all of them $\mathrm{S}(w_1,w_2,w_3)=1$.

Corollary ** **1.12.** (i) $\mathrm{S}(w_1,w_2,w_3,w_4)$ *vanishes for* $w_1<25$.

(ii) *There are exactly* 33 *triples* (w_2,w_3,w_4) *such that* $\mathrm{S}(25,w_2,w_3,w_4)\neq 0$ *and for all of them* $\mathrm{S}(25,w_2,w_3,w_4)=1$, *except* $\mathrm{S}(25,21,15,7)=\mathrm{S}(25,23,11,5)=2$ *and* $\mathrm{S}(25,23,15,5)=3$.

The conjectural Sato-Tate group of a symplectic π of GL_6 (resp. GL_8) satisfying (a), (b) and (c) is either $\mathrm{SU}(2)$, $\mathrm{SU}(2)\times\mathrm{SO}(3)$ (resp. $\mathrm{SU}(2)^3/\{\pm 1\}$), or the compact connected, simply connected, Lie group of type C_3 (resp. C_4). This latter case should occur for each of the 7 automorphic representations of Corollary 1.11 (ii) (resp. of the 37 automorphic representations of Corollary 1.12 (ii)).

[7] When $w-v=2$ (resp. $w-v=4$) this follows from case (vi) of §5.5.2 (resp. case (ii) of §7.2.1).

Let us discuss now the orthogonal case. We start with two general useful facts. Although the first one is quite simple, the second one is rather deep and relies on Arthur's proof that the root number of an orthogonal π is always 1 (see Proposition 3.12 and §3.27).

Proposition 1.13. *If r is odd, then* $\mathrm{O}(w_1,\dots,w_r)=0$ *for all* $w_1>w_2>\cdots>w_r$.

Proposition* 1.14. *If* $\frac{1}{2}(\sum_{i=1}^r w_i)\not\equiv[\frac{r+1}{2}] \bmod 2$, *then*

$$\mathrm{O}(w_1,\dots,w_r)=\mathrm{O}^*(w_1,\dots,w_r)=0.$$

Each time we shall write $\mathrm{O}(w_1,\dots,w_r)$ and $\mathrm{O}^*(w_1,\dots,w_r)$ we shall thus assume from now on that $\frac{1}{2}(\sum_i w_i)\equiv[\frac{r+1}{2}] \bmod 2$. Here are those numbers for $r\le 2$.

Theorem 1.15.** (i) $\mathrm{O}^*(w)=\mathrm{S}(\frac{w}{2})$,

(ii) $\mathrm{O}(w,v)=\mathrm{S}(\frac{w+v}{2})\cdot\mathrm{S}(\frac{w-v}{2})$ *if* $v\neq 0$, *and* $\mathrm{O}(w,0)=\frac{\mathrm{S}(w/2)\cdot(\mathrm{S}(w/2)-1)}{2}$,

(iii) $\mathrm{O}^*(w,v)=\mathrm{S}(\frac{w+v}{2},\frac{w-v}{2})$.

Part (i) and (ii) actually only rely on assumption $*$. These identities correspond to some simple cases of Langlands functoriality related to the exceptional isogenies $\mathrm{SL}_2(\mathbb{C})\to\mathrm{SO}_3(\mathbb{C})$ (*symmetric square*), $\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{SO}_4(\mathbb{C})$ (*tensor product*) and $\mathrm{Sp}_4(\mathbb{C})\to\mathrm{SO}_5(\mathbb{C})$ (*reduced exterior square*). As we shall show, they are all consequences of Arthur's work : see Chapter 4. Another tool in our proof is a general, elementary, lifting result for isogenies between Chevalley groups over $\mathbb{Z}$. From the point of view of Langlands conjectures, it asserts that the Langlands group of $\mathbb{Z}$, a compact connected topological group, is simply connected (see Appendix B).

Our main remaining contribution in the orthogonal case is thus the assertion in Thm. 1.5 about $\mathrm{O}(w_1,w_2,w_3,w_4)$. We argue as before by considering this time the special orthogonal group SO_8 over $\mathbb{Z}$ of the root lattice E_8. It also has class number 1 as E_8 is the unique even unimodular lattice in rank 8. For some reasons related to twisted endoscopy between SO_8 and Sp_6, the precise numbers that we compute are the $\mathrm{O}(w_1,w_2,w_3,w_4)$ when $w_4\neq 0$, as well as the numbers

$$2\cdot\mathrm{O}(w_1,w_2,w_3,0)+\mathrm{O}^*(w_1,w_2,w_3).$$

When this latter number is ≤ 1, it is thus necessarily equal to $\mathrm{O}^*(w_1,w_2,w_3)$, which leads first to the following partial results for the orthogonal representations π of in dimension $n=7$. See Table 10 and the url [CR] for more results.

Corollary 1.16.** (i) $\mathrm{O}^*(w_1,w_2,w_3)$ *vanishes for* $w_1<24$.

(ii) *There are exactly* 8 *triples* (w_1,w_2,w_3) *with* $w_1\le 26$ *such that* $\mathrm{O}^*(w_1,w_2,w_3)\neq 0$, *namely*

$$(24,16,8),(26,16,10),(26,20,6),(26,20,10),(26,20,14),$$

$$(26,24,10),(26,24,14),(26,24,18),$$

in which cases $\mathrm{O}^*(w_1,w_2,w_3)=1$.

Observe that our approach does not allow to tackle this case directly as there is no semisimple $\mathbb{Z}$-group of type C_3 with compact real points (and actually of type C_l for any $l \geq 3$). On the other hand, our results allow to compute in a number of cases the dimension of the space $S_{w_1,w_2,w_3}(\mathrm{Sp}_6(\mathbb{Z}))$ of vector valued Siegel modular forms of whose infinitesimal character, a semisimple element in $\mathfrak{so}_7(\mathbb{C})$, has distinct eigenvalues $\pm\frac{w_1}{2}, \pm\frac{w_2}{2}, \pm\frac{w_3}{2}, 0$, where $w_1 > w_2 > w_3$ are even positive integers. Indeed, we deduce from Arthur's multiplicity formula that :

Proposition 1.17.** $\dim S_{w_1,w_2,w_3}(\mathrm{Sp}_6(\mathbb{Z})) = \mathrm{O}^*(w_1,w_2,w_3) + \mathrm{O}(w_1,w_3) \cdot \mathrm{O}^*(w_2)$
$$+\delta_{w_2\equiv 0 \bmod 4} \cdot (\delta_{w_2=w_3+2} \cdot \mathrm{S}(w_2-1) \cdot \mathrm{O}^*(w_1) + \delta_{w_1=w_2+2} \cdot \mathrm{S}(w_2+1) \cdot \mathrm{O}^*(w_3)).$$

In particular, in turns out that Corollary 1.16 and Theorem 1.15 allow to determine the dimension of $S_{w_1,w_2,w_3}(\mathrm{Sp}_6(\mathbb{Z}))$ when $w_1 \leq 26$ (which makes 140 cases). We refer to Chapter 9 for more about this and to the website [CR] for some results. We actually explain in this chapter how to compute *for any genus g* the dimension of the space of Siegel cusp forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$ of any given regular infinitesimal character in terms of various numbers $\mathrm{S}(-)$, $\mathrm{O}(-)$ and $\mathrm{O}^*(-)$.

The problem of the determination of $\dim S_{w_1,w_2,w_3}(\mathrm{Sp}_6(\mathbb{Z}))$ has been solved by Tsuyumine in [Tsuy86] when $w_1 - w_3 = 4$ (scalar valued Siegel modular forms of weight $k = \frac{1}{2}(w_1+2)$). As already said when we discussed Theorem 1.5, it has also been studied recently in general by Van der Geer, Bini, Bergström and Faber, see e.g. [BFG11] for the latest account of their beautiful results. In this last paper, the authors give in particular a (partly conjectural) table for certain values of $\dim S_{w_1,w_2,w_3}(\mathrm{Sp}_6(\mathbb{Z}))$: see Table 1 loc. cit. We checked that this table fits our results. In turn, their results allow not only to determine conjecturally each $\mathrm{O}^*(w_1,w_2,w_3)$, but $\mathrm{O}(w_1,w_2,w_3,0)$ as well by our work. Let us mention that those authors not only compute dimensions but also certain Hecke eigenvalues.

The Sato-Tate group of an orthogonal π of GL_7 satisfying (a), (b) and (c) is either $\mathrm{SO}(3)$, $\mathrm{SO}(7)$ or the compact Lie group of type G_2. In order to enumerate the conjectural number of π having this latter group as Sato-Tate group there is a funny game we can play with the reductive group G_2 over $\mathbb{Z}$ such that $\mathrm{G}_2(\mathbb{R})$ is compact, namely the automorphism group scheme of the Coxeter octonions. We compute in Chapter 8 the dimension of the spaces of level 1 automorphic forms for this $\mathbb{Z}$-group G_2.[8] Assuming Langlands and Arthur's conjectures for the embedding of dual groups $\mathrm{G}_2(\mathbb{C}) \to \mathrm{SO}_7(\mathbb{C})$ we are able to compute the conjectural number

$$\mathrm{G}_2(v,u)$$

of orthogonal π of GL_7 with Hodge weights $v+u > v > u$ and whose Sato-Tate group is either $\mathrm{SO}(3)$ or G_2 as an explicit function of v and u. Of course it is easy to rule out from this number the contribution of the $\mathrm{SO}(3)$ case, which occurs $\mathrm{S}(u/2)$ times when

[8]Added in 2014 : after this work was completed, G. Savin informed us that S. Sullivan and himself have studied a similar problem in their paper *A trace formula for G_2*, which is available at the address `http://content.lib.utah.edu/cdm/ref/collection/etd3/id/2421`.

$v = 2u$ (sixth symmetric power of the standard representation). We obtain in particular a conjectural minoration of $\mathrm{O}^*(v+u, v, u)$ in general which matches beautifully the results of corollary (ii) above : the first three π's should have G_2 as Sato-Tate groups, and the five others SO(7). We refer to Table 11 for a sample of results and to the url [CR] for much more. This also confirms certain similar predictions in [BFG11]. There is actually another way, still conjectural but perhaps accessible nowadays, to think about $\mathrm{G}_2(v, u)$, using twisted endoscopy for a triality automorphism for PGSO_8. This concerns triples of weights (w, v, u) with $w = v+u$. With this theory in mind, it should follow that whenever $\mathrm{O}(v + u, v, u) = 1$ the unique orthogonal π of GL_7 with Hodge weights $v + u > v > u$ should have G_2 or SO(3) as Sato-Tate group. This criterion applies to the first three π's given by the corollary (ii) (the Sato-Tate group SO(3) being obviously excluded in these cases) and thus comforts the previous predictions.

Modular forms of level one for the Chevalley group of type G_2, and whose Archimedean component is a quaternionic discrete series, have been studied by Gan, Gross and Savin in [GGS02]. They define a notion of Fourier coefficients for those modular forms and give interesting examples of Eisenstein series and of two exceptional theta series coming from the modular forms of level 1 and trivial coefficient of the anisotropic form of F_4 over $\mathbb{Q}$. Table 11 shows that the first cusp form for this G_2 whose conjectural transfer to GL_7 is cuspidal should occur for the weight $k = 8$, which is the first integer k such that $\mathrm{G}_2(2k, 2k-2) \neq 0$. Modular forms for the anisotropic $\mathbb{Q}$-form of G_2 have also been studied by Gross, Lansky, Pollack and Savin in [GS98], [GP05], [LP02] and [Pol98], partly in order to find $\mathbb{Q}$-motives with Galois group of type G_2, a problem initially raised by Serre [Ser94]. The automorphic forms they consider there are not of level 1, but of some prime level p and Steinberg at this prime p.

Let us now give a small sample of results concerning $\mathrm{O}(w_1, w_2, w_3, w_4)$ for $w_4 > 0$, see Table 9 and [CR] for more values.

Corollary 1.18.** (i) *$\mathrm{O}(w_1, w_2, w_3, w_4)$ vanishes for $w_1 < 24$.*

(ii) *The (w_1, w_2, w_3, w_4) with $0 < w_4 < w_1 \leq 26$ such that $\mathrm{O}(w_1, w_2, w_3, w_4) \neq 0$ are*

$$(24, 18, 10, 4), (24, 20, 14, 2), (26, 18, 10, 2), (26, 18, 14, 6), (26, 20, 10, 4), (26, 20, 14, 8),$$

$$(26, 22, 10, 6), (26, 22, 14, 2), (26, 24, 14, 4), (26, 24, 16, 2), (26, 24, 18, 8), (26, 24, 20, 6),$$

and for all of them $\mathrm{O}(w_1, w_2, w_3, w_4) = 1$.

The Sato-Tate group of an orthogonal π of GL_8 satisfying (a), (b) and (c) can be a priori either SU(2), SO(8), $(\mathrm{SU}(2) \times \mathrm{Spin}(5))/\{\pm 1\}$, SU(3) (adjoint representation) or Spin(7). The case SU(3) should actually never occur (see Appendix B). Moreover, it is not difficult to check that the Spin(7) case may only occur for Hodge weights (w_1, w_2, w_3, w_4) such that $w_4 = |w_1 - w_2 - w_3|$, in which case it occurs exactly $\mathrm{O}^*(v_1, v_2, v_3) - \mathrm{G}_2(v_2, v_3) \cdot \delta_{v_1 = v_2 + v_3}$ times, where $(v_1, v_2, v_3) = (w_2 + w_3, w_1 - w_3, w_1 - w_2)$. But for each of the six Hodge weights (w_1, w_2, w_3, w_4) of Corollary 1.18 (ii) such that $w_4 = |w_1 - w_2 - w_3|$, we have $v_1 \neq v_2 + v_3$ (i.e. $w_1 \neq w_2 + w_3$) and the number $\mathrm{O}^*(v_1, v_2, v_3) + 2 \cdot \mathrm{O}(v_1, v_2, v_3, 0)$ that we computed is 1, so that $\mathrm{O}^*(v_1, v_2, v_3) = 1$. It follows that in these six cases the Sato-Tate groups should be Spin(7), and thus in the six remaining ones it must be SO(8) (it is easy to rule out SU(2) and $(\mathrm{SU}(2) \times \mathrm{Spin}(5))/\{\pm 1\}$).

1.19. **Generalizations.** At the moment, we cannot compute $\mathrm{N}(k_1, \cdots, k_n)$ for $n > 8$ because we don't know neither the dimensions of the spaces of vector valued Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$ in genus $g \geq 4$, nor the number of level 1 automorphic representations π of $\mathrm{SO}_{p,q}$ such that π_∞ is a discrete series when $p+q > 10$. We actually have in our database the dimensions of the spaces of level 1 automorphic forms of the special orthogonal $\mathbb{Z}$-group SO_{15} of the root lattice $\mathrm{E}_7 \oplus \mathrm{E}_8$ (note that the class number is 2 in this case). They lead to certain upper bound results concerning the number of symplectic π in dimension $n \leq 14$ that we won't give here. However, they contain too many unknowns to give as precise results as the ones we have described so far for $n \leq 8$, because of the inductive structure of the dimension formulas.

1.20. **Methods and proofs.**

1.20.1. We now discuss a bit more the methods and proofs. As already explained, a first important technical ingredient to obtain all the numbers above is to be able to compute, say given a finite subgroup Γ of a compact connected Lie group G, and given a finite dimensional irreducible representation V of G, the dimension

$$\dim V^\Gamma$$

of the subspace of vectors in V which are fixed by Γ. This general problem is studied in Chapter 2 (which is entirely unconditional). The main result there is an explicit general formula for $\dim V^\Gamma$ as a function of the extremal weight of V, which is made explicit in the cases alluded above. Our approach is to write

$$\dim V^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_V(\gamma)$$

where $\chi_V : G \to \mathbb{C}$ is the character of V. The formula we use for χ_V is a degeneration of the Weyl character formula which applies to possibly non regular elements and which was established in [CC09]. Fix a maximal torus T in G, with character group X, and a set $\Phi^+ \subset X$ of positive roots for the root system (G, T). Let V_λ be the irreducible representation with highest weight λ. Then

$$\dim V_\lambda^\Gamma = \sum_{j \in J} a_j \, e^{\frac{2i\pi}{N} \langle b_j, w_j(\lambda+\rho)-\rho \rangle} \, P_j(\lambda)$$

where N is the lcm of the orders of the elements of Γ, $a_j \in \mathbb{Q}(e^{2i\pi/N})$, b_j is a certain cocharacter of T, w_j is a certain element in the Weyl group W of (G, T), and P_j is a certain rational polynomial on $X \otimes \mathbb{Q}$ which is a product of at most $|\Phi^+|$ linear forms. For each $\gamma \in T$ let $W_\gamma \subset W$ be the Weyl group of the connected centralizer of γ in G with respect to T. Then

$$|J| = \sum_{\gamma} |W/W_\gamma|$$

where γ runs over a set of representatives of the G-conjugacy classes of elements of Γ.

In practice, this formula for $\dim V_\lambda^\Gamma$ is quite insane. Consider for instance $G = \mathrm{SO}_7(\mathbb{R})$ and $\Gamma = \mathrm{W}^+(E_7)$ the positive Weyl group of the root system of type E_7 : this is the case we need to compute $\mathrm{S}(w_1, w_2, w_3)$. Then $|J| = 725$, $N = 2520$ and $|\Phi^+| = 9$: it certainly impossible to explicitly write down this formula in the present paper. This is however nothing (in this case!) for a computer and we refer to § 2.4 for some details about the computer program we wrote using PARI/GP [GP]. This program is available at the url [CR]. Let us mention that we use in an important way some tables of Carter [Cart72] giving the characteristic polynomials of all the conjugacy classes of a given Weyl group in its reflection representation.

1.20.2. The second important ingredient we need is Arthur's multiplicity formula in a various number of cases. Concretely this amounts to determining a quite large collection of signs. This is discussed in details in Chapter 3, in which we specify Arthur's general results to the case of classical semisimple $\mathbb{Z}$-groups G. This leads first to a number of interesting properties of the automorphic representations π satisfying (a), (b) and (c') of this introduction. Of course, a special attention is given to the groups G with $G(\mathbb{R})$ compact, hence to the integral theory of quadratic forms. We restrict our study to the representations in the discrete spectrum of G which are unramified at each finite place. At the Archimedean place we are led to review some properties of the packets of representations defined by Adams-Johnson in [AJ87]. We explain in particular in the appendix A the parameterization of the elements of these packets by the characters of the dual component group in the spirit of Adams paper [Ada11] in the discrete series case. For our purposes, we need to apply Arthur's results to a number of classical groups of small rank, namely

$$\mathrm{SL}_2,\ \mathrm{Sp}_4,\ \mathrm{Sp}_6,\ \mathrm{SO}_{2,2},\ \mathrm{SO}_{3,2},\ \mathrm{SO}_7,\ \mathrm{SO}_8 \text{ and } \mathrm{SO}_9.$$

When $G(\mathbb{R})$ is compact, Arthur's multiplicity formula takes a beautifully simple form, in which the half-sum of the positive roots on the dual side plays an important role. Let G be any semisimple $\mathbb{Z}$-group such that $G(\mathbb{R})$ is compact. We do not assume that G is classical here and state the general conjectural formula. Let $\mathcal{L}_\mathbb{Z}$ denote the Langlands group of $\mathbb{Z}$ and let

$$\psi : \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$$

be a global Arthur parameter such that ψ_∞ is an Adams-Johnson parameter : see [Art89] as well as the appendices A and B. Denote by π_ψ the irreducible admissible representation of $G(\mathbb{A})$ which is $G(\widehat{\mathbb{Z}})$-spherical and with the Satake parameters and infinitesimal character determined by ψ according to Arthur's recipe[9]. Denote also by $e(\psi)$ the (finite) number of $\widehat{G}$-conjugacy classes of global Arthur parameters ψ' as above such that $\pi_{\psi'} \simeq \pi_\psi$ (for most ψ we have $e(\psi) = 1$). The multiplicity $m(\pi_\psi)$ of π_ψ in $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$

[9]This means that ψ corresponds to π_ψ in the sense of the appendix B, assumption (L5). This uniquely determines π_ψ as $G(\mathbb{R})$ is compact and connected.

should be given in general by

$$m(\pi_\psi) = \begin{cases} e(\psi) & \text{if } \rho^\vee_{|\mathrm{C}_\psi} = \varepsilon_\psi, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

The group C_ψ is by definition the centralizer of $\operatorname{Im}\psi$ in $\widehat{G}$, it is a finite group. As explained in the appendix A, it is always an elementary abelian 2-group. The character ε_ψ is defined by Arthur in [Art89]. The character $\rho^\vee$ is defined as follows. Let $\varphi_{\psi_\infty} : \mathrm{W}_\mathbb{R} \longrightarrow \widehat{G}$ the Langlands parameter associated by Arthur to ψ_∞. First, the centralizer in $\widehat{G}$ of $\varphi_{\psi_\infty}(\mathrm{W}_\mathbb{C})$ is a maximal torus $\widehat{T}$ of $\widehat{G}$, so that $\varphi_{\psi_\infty}(z) = z^\lambda \overline{z}^{\lambda'}$ for some $\lambda \in \frac{1}{2}\mathrm{X}_*(\widehat{T})$ and all $z \in \mathrm{W}_\mathbb{C}$, and λ is dominant with respect to a unique Borel subgroup $\widehat{B}$ of $\widehat{G}$ containing $\widehat{T}$. Let $\rho^\vee$ denote the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$. As G is semisimple over $\mathbb{Z}$ and $G(\mathbb{R})$ is compact, this is actually a character of $\widehat{T}$. By construction, we have $\mathrm{C}_\psi \subset \widehat{T}$, and thus formula (1.3) makes sense. The second important statement is that any automorphic representation of G which is $G(\widehat{\mathbb{Z}})$-spherical has the form π_ψ for some ψ as above.

1.21. **Application to Borcherds even lattices of rank 25 and determinant 2.** We end this introduction by discussing two other applications. The first one is very much in the spirit of the work of the first author and Lannes [CL14]. It concerns the genus of euclidean lattices $L \subset \mathbb{R}^{25}$ of covolume $\sqrt{2}$ which are *even*, in the sense that $x \cdot x \in 2\mathbb{Z}$ for each $x \in L$. A famous computation by Borcherds in [Bor84] asserts that there are up to isometry exactly 121 such lattices. It follows that there are exactly 121 level 1 automorphic representations of the special orthogonal group SO_{25} over $\mathbb{Z}$ of the root lattice $\mathrm{E}_8^3 \oplus \mathrm{A}_1$, for the trivial coefficient. The dual group of SO_{25} is $\mathrm{Sp}_{24}(\mathbb{C})$.

Observe now our tables : we have found exactly 23 cuspidal automorphic representations π of GL_n (for any n) satisfying conditions (a), (b) and (c) above, centered and with motivic weight ≤ 23, namely :

(a) the trivial representation of GL_1,
(b) 7 representations of GL_2,
(c) 7 symplectic representations of GL_4,
(d) 7 symplectic representations of GL_6,
(e) The orthogonal representation of GL_3 symmetric square of the representation of GL_2 of motivic weight 11 associated to a generator of $\mathrm{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

We refer to § 3.18 for the notion of global Arthur parameters for a $\mathbb{Z}$-group such as SO_{25}. This is the non-conjectural substitute used by Arthur for the conjugacy classes of morphisms $\psi : \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{24}(\mathbb{C})$ with finite centralizers that we just discussed in §1.20.2 (see Appendix B). We have now this first crazy coincidence, which is easy to check with a computer.

Proposition 1.22. *There are exactly* 121 *global Arthur parameters for* SO_{25} *which have trivial infinitesimal character that one can form using only those* 23 *cuspidal automorphic representations.*

See the table of Appendix D for a list of these parameters, using notations of §3.18. One also uses the following notation : if $\mathrm{S}(w_1,\dots,w_r)=1$ we denote by $\Delta_{w_1,\cdots,w_r}$ the twist by $|\cdot|^{w_1/2}$ of the unique centered $\pi\in\Pi(\mathrm{GL}_{2r})$ satisfying (i) to (iii) and with Hodge weights $w_1,\cdots,w_r$. When $\mathrm{S}(w_1,\dots,w_r)=k>1$ we denote by $\Delta^k_{w_1,\dots,w_r}$ any of the k representations of GL_{2r} with this latter properties.

The second miracle is that for each of the 121 parameters ψ that we found in Proposition 1.22, the unique level 1 automorphic representation π_ψ of $\mathrm{SO}_{25}(\mathbb{A})$ in the packet $\Pi(\psi)$ (see Def. 3.21) has indeed a nonzero multiplicity, that is multiplicity 1. In other words, we have the following theorem.

Theorem** 1.23. *The* 121 *level* 1 *automorphic representations of* SO_{25} *with trivial coefficient are the ones given in Appendix D.*

The 24 level 1 automorphic representations of O_{24} with trivial coefficient ("associated" to the 24 Niemeier lattices) and the 32 level 1 automorphic representations of SO_{23} with trivial coefficient (associated to the 32 even lattices of rank 23 of covolume $\sqrt{2}$) had been determined in [CL14]. As in *loc. cit.*, observe that given the shape of Arthur's multiplicity formula, the naive probability that Theorem 1.23 be true was close to 0 (about 2^{-450} here in we take in account the size of C_ψ for each ψ), so something quite mysterious seems to occur for these small dimensions and trivial infinitesimal character. The miracle in all these cases is that whenever we can write down some ψ, then $\rho^\vee_{|\mathrm{C}_\psi}$ is always equal to ε_ψ.

It is convenient for us to include here the proof of this theorem, although it uses freely the notations of Chapter 3.

Proof — To check that each parameter has multiplicity one, we apply for instance the following simple claim already observed in [CL14]. Let $\psi=(k,(n_i),(d_i),(\pi_i))$ be a global Arthur parameter for $\mathrm{SO}_{8m\pm1}$ with trivial infinitesimal character. Assume there exists an integer $1\le i\le k$ such that $\pi_i=1$ and π_j is symplectic if $j\neq i$. Then the unique $\pi\in\Pi(\psi)$ has a nonzero multiplicity (hence multiplicity 1) if and only if for each $j\neq i$ one has either $\varepsilon(\pi_j)=1$ or $d_j<d_i$. Indeed, when the infinitesimal character of ψ is trivial the formula for $\rho^\vee(s_j)$ given in §3.30.1 shows that $\rho^\vee(s_j)=\varepsilon(\pi_j)$ for each $j\neq i$. The claim follows as $\varepsilon_\psi(s_j)=\varepsilon(\pi_j)^{\mathrm{Min}(d_i,d_j)}$ by definition (see §3.27), as d_i is even but d_j is odd.

Among the 21 symplectic π's above of motivic weight ≤ 23, one observes that exactly 4 of them have epsilon factor -1, namely

$$\Delta_{17},\Delta_{21},\Delta_{23,9}\text{ and }\Delta_{23,13}.$$

A case-by-case check at the list concludes the proof thanks to this claim except for the parameter

$$\mathrm{Sym}^2\Delta_{11}[2]\oplus\Delta_{11}[9]$$

which is the unique parameter which is not of the form above. But it is clear that for such a ψ one has $\varepsilon_\psi = 1$ and one observes that $\rho^\vee_{|C_\psi} = 1$ as well in this case, which concludes the proof (see §3.30.1). ☐

1.24. **A level** 1**, non-cuspidal, tempered automorphic representation of** GL_{28} **over** $\mathbb{Q}$ **with weights** $0, 1, 2, \cdots, 27$**.** None of the 121 automorphic representations of SO_{25} discussed above is tempered. This is clear since none of the 21 symplectic π's above admit the Hodge weight 1. Two representations in the list are not too far from being tempered however, namely the ones whose Arthur parameters are

$$\Delta_{23,13,5} \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [4],$$

$$\Delta_{23,15,3} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [2].$$

It is thus tempting to consider the following problem : for which integers n can we find

- a partition $n = \sum_{i=1}^r n_i$ in integers $n_i \geq 1$,
- for each $i = 1, \cdots, r$, a cuspidal automorphic representation π_i of GL_{n_i} satisfying assumptions (a), (b) and (c), and of motivic weight $n-1$,

such that the parabolically induced representation

$$\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r$$

of $\mathrm{GL}_n(\mathbb{A})$ has the property that the eigenvalues of the infinitesimal character of π_∞ are all the integers between 0 and $n-1$?

By assumption, the π_i are non necessarily centered but share the same motivic weight $n-1$, so that π is essentially tempered. It follows that the L-function

$$\mathrm{L}(\pi, s) = \prod_i \mathrm{L}(\pi_i, s)$$

of such a π shares much of the analytic properties of the L-function of a cuspidal π' of GL_n satisfying (a), (b) and (c) and with weights $n-1, n-2, \ldots, 1, 0$: they have the same Archimedean factors and both satisfy Ramanujan's conjecture. In particular, it seems that the methods of [Fer96], hence his results in §9 *loc. cit.*, apply to these more general L-functions. They say that such an L-function (hence such a π) does not exist if $1 < n < 23$, and even if $n = 24$ if one assumes the Riemann hypothesis. This is fortunately compatible with our previous result !

On the other hand, our tables allow to show that the above problem has a positive answer for $n = 28$, which leads to a very interesting L-function in this dimension.

Theorem 1.25.** *There is a non cuspidal automorphic representation of* GL_{28} *over* $\mathbb{Q}$ *which satisfies* (a), (b) *and* (c), *and whose weights are all the integers between* 0 *and* 27, *namely the twist by* $|\cdot|^{\frac{1-n}{2}}$ *of*

$$\Delta_{27,23,9,1} \oplus \Delta_{25,13,3} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}.$$

It simply follows from the observation that $\mathrm{S}(27,23,9,1) = \mathrm{S}(25,13,3) = \mathrm{S}(21,5) = \mathrm{S}(19,7) = 1$. Actually, it is remarkable that our whole tables only allow to find a single representation with these properties, and none in rank $1 < n < 28$. It seems quite reasonable to conjecture that this is indeed the only one in rank 28 and that there are none in rank $1 < n < 28$. From the example above, one easily deduces examples for any even $n \geq 28$. On the other hand, the first odd $n > 1$ for which our tables allow to find a suitable π is $n = 31$ (in which case there are several).

Acknowledgements: We thank Jean Lannes for a number of stimulating discussions about lattices. This paper actually owes a lot to the work [CL14] in which the authors compute in particular the Kneser p-neighborhood graph of the Niemeier lattices for each prime p. Certain observations here concerning the shape of Arthur's multiplicity formula, e.g. formula (1.3), were partially realized there, as well as the feeling that it would be possible to obtain Theorem 1.23. The first author is also grateful to Colette Moeglin and Olivier Taïbi for useful discussions about Arthur's results. We also thank Xavier Caruso for his help in creating tables, as well as Daniel Bump, Wee Teck Gan, Benedict Gross, Guy Henniart and Jean-Pierre Serre for their remarks. In all our computations, we heavily used the PARI/GP computer system. Last but not least, it is a pleasure to thank J. Arthur for his beautiful conjectures and monumental work [Art11], without which this paper would not have existed.

Contents

2. Polynomial invariants of finite subgroups of compact connected Lie groups

2.1. **The setting.**

Let G be a compact connected Lie group and consider

$$\Gamma \subset G$$

a finite subgroup. Let V be a finite dimensional complex continuous representation of G. The general problem addressed in this chapter is to compute the dimension

$$\dim V^{\Gamma}$$

of the subspace $V^{\Gamma} = \{v \in V, \gamma(v) = v \ \ \forall \gamma \in \Gamma\}$ of Γ-invariants in V. Equivalently,

$$\dim V^{\Gamma} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_V(\gamma) \tag{2.1}$$

where $\chi_V : G \to \mathbb{C}$ is the character of V. One may of course reduce to the case where V is irreducible and we shall most of the time do so. In order to apply formula (2.1) it is enough to know :

(a) The value of the character χ_V on each conjugacy class in G,

(b) For each $\gamma \in \Gamma$, a representative of the conjugacy class $c(\gamma)$ of γ in G.

Of course, $c(\gamma)$ only depends on the conjugacy class of γ in Γ, but the induced map $c : \mathrm{Conj}(\Gamma) \to \mathrm{Conj}(G)$ needs not to be injective in general. Here $\mathrm{Conj}(H)$ denotes the set of conjugacy classes of the group H.

We will be especially interested in cases where $\Gamma \subset G$ are fixed, but with V varying over all the possible irreducible representations of G. With this in mind, observe that problem (b) has to be solved once, but problem (a) for infinitely many V whenever $G \neq \{1\}$.

Consider for instance the group $\Gamma \subset \mathrm{SO}_3(\mathbb{R})$ of positive isometries of a given regular tetrahedron in the euclidean $\mathbb{R}^3$ with center 0. Each numbering of the vertices of the tetrahedron defines an isomorphism

$$\Gamma \simeq \mathfrak{A}_4$$

and we fix one. For each odd integer $n \geq 1$ denote by V_n the n-dimensional irreducible representations of $\mathrm{SO}_3(\mathbb{R})$. This representation V_n is well-known to be unique up to isomorphism, and if $g_\theta \in \mathrm{SO}_3(\mathbb{R})$ is a non-trivial rotation with angle θ then

$$\chi_{V_n}(g_\theta) = \frac{\sin(n\frac{\theta}{2})}{\sin(\frac{\theta}{2})}.$$

The group Γ has 4 conjugacy classes, with representatives $1, (12)(34), (123), (132)$ and respective orders $1, 3, 4, 4$. These representatives act on $\mathbb{R}^3$ as rotations with respective angles $0, \pi, 2\pi/3, 2\pi/3$. Observe that (123) and (132) are conjugate in $\mathrm{SO}_3(\mathbb{R})$ but not in Γ. Formula (2.1) thus writes

$$\dim V_n^\Gamma = \frac{1}{12}(n + 3\,\frac{\sin(n\pi/2)}{\sin(\pi/2)} + 8\,\frac{\sin(n\pi/3)}{\sin(\pi/3)}) = \begin{cases} \lceil \frac{n}{12} \rceil & \text{if} \quad n \equiv 1, 7, 9 \bmod 12, \\ \lfloor \frac{n}{12} \rfloor & \text{if} \quad n \equiv 3, 5, 11 \bmod 12. \end{cases}$$

This formula is quite simple but already possesses some features of the general case.

2.2. **The degenerate Weyl character formula.** A fundamental ingredient for the above approach is a formula for the character $\chi_V(g)$ where V is any irreducible representations of G and $g \in G$ is any element as well. When g is either central or regular, such a formula is given by Weyl's dimension formula and Weyl's character formula respectively. These formulas have been extended by Kostant to the more general case where the centralizer of g is a Levi subgroup of G, and by the first author and Clozel in general in [CC09, Prop. 1.9]. Let us now recall this last result.

We fix once and for all a maximal torus $T \subset G$ and denote by

$$X = X^*(T) = \mathrm{Hom}(T, \mathbb{S}^1)$$

the character group of T. We denote by $\Phi = \Phi(G,T) \subset X \otimes \mathbb{R}$ the root system of (G,T) and $W = W(G,T)$ its Weyl group. We choose $\Phi^+ \subset \Phi$ a system of positive roots, say with base Δ, and we fix as well a W-invariant scalar product $(,)$ on $X \otimes \mathbb{R}$. Recall that a dominant weight is an element $\lambda \in X$ such that $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta$. The Cartan-Weyl theory defines a canonical bijection

$$\lambda \mapsto V_\lambda$$

between the dominant weights and the irreducible representations of G. The representation V_λ is uniquely characterized by the following property. If V is a representation of G, denote by $P(V) \subset X$ the subset of $\mu \in X$ appearing in $V_{|T}$. If we consider the partial ordering on X defined by $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda$ is a finite sum of elements of Δ, then λ is the maximal element of $P(V_\lambda)$. One says that λ is the highest weight of V_λ.

Let us fix some dominant weight $\lambda \in X$. Recall that the inclusion $T \subset G$ induces a bijection

$$W \backslash T \xrightarrow{\sim} \mathrm{Conj}(G),$$

it is thus enough to determine $\chi_{V_\lambda}(t)$ for any $t \in T$. Fix some $t \in T$ and denote by

$$M = C_G(t)^0$$

the neutral component of the centralizer of t in G. Of course, $t \in T \subset M$ and T is maximal torus of M. Set $\Phi_M^+ = \Phi(M,T) \cap \Phi^+$ and consider the set

$$W^M = \{w \in W, w^{-1}\Phi_M^+ \subset \Phi^+\}.$$

Let ρ and $\rho_M \in X \otimes \mathbb{R}$ denote respectively the half-sum of the elements of Φ^+ and of Φ_M^+. If $w \in W^M$, we set $\lambda_w = w(\lambda + \rho) - \rho_M \in X \otimes \mathbb{R}$. Observe that

$$2\frac{(\alpha, \lambda_w)}{(\alpha, \alpha)} \in \mathbb{N}, \quad \forall\, \alpha \in \Phi_M^+.$$

It follows that λ_w is a dominant weight for some finite covering of M, that we may choose to be the smallest finite covering $\widetilde{M} \to M$ for which $\rho - \rho_M$ becomes a character. This is possible as $2\frac{(\alpha,\rho-\rho_M)}{(\alpha,\alpha)} \in \mathbb{Z}, \ \forall \alpha \in \Phi_M^+$. It follows from the Weyl dimension formula that the dimension of the irreducible representation of $\widetilde{M}$ with highest weight λ_w is $P_M(\lambda_w)$ where we set

$$P_M(v) = \prod_{\alpha\in\Phi_M^+} \frac{(\alpha, v+\rho_M)}{(\alpha,\rho_M)} \quad \forall v \in X \otimes \mathbb{R}.$$

We need two last notations before stating the main result. We denote by $\varepsilon : W \to \{\pm 1\}$ the signature, and for $x \in X$ it will be convenient to write t^x for $x(t)$. It is well-known that $w(\mu+\rho) - \rho \in X$ for all $w \in W$ and $\mu \in X$.

Proposition 2.3. *(Degenerate Weyl character formula) Let $\lambda \in X$ be a dominant weight, $t \in T$ and $M = C_G(t)^0$. Then*

$$\chi_{V_\lambda}(t) = \frac{\sum_{w\in W^M} \varepsilon(w)\cdot t^{w(\lambda+\rho)-\rho} \cdot P_M(w(\lambda+\rho)-\rho_M)}{\prod_{\alpha\in\Phi^+\setminus\Phi_M^+}(1-t^{-\alpha})}.$$

Proof — This is the last formula in the proof of [CC09, Prop. 1.9]. Note that it is unfortunately incorrectly stated in the beginning of that proof that up to replacing G by a finite covering one may assume that ρ and ρ_M are characters. It is however not necessary for the proof to make any reduction on the group G. Indeed, we rather have to introduce the inverse image $\widetilde{T}$ of T in the covering $\widetilde{M}$ defined above and argue as *loc. cit.* but in the Grothendieck group of characters of $\widetilde{T}$. The argument given there shows that for any element $z \in \widetilde{T}$ whose image in T is t, we have

$$\chi_{V_\lambda}(t) = z^{\rho_M-\rho}\frac{\sum_{w\in W^M}\varepsilon(w) z^{\lambda_w} P_M(\lambda_w)}{\prod_{\alpha\in\Phi^+\setminus\Phi_M^+}(1-t^{-\alpha})}.$$

We conclude as $\lambda_w + \rho_M - \rho = w(\lambda+\rho) - \rho \in X$, so $z^{\rho_M-\rho} z^{\lambda_w} = t^{w(\lambda+\rho)-\rho}$. □

Let us mention another related application of Weyl's character formula due to Kostant, called *Kostant multiplicity formula* : if $H \subset G$ is a compact connected subgroup, this is a formula for the multiplicity of a given irreducible representation of H in the restriction of V_λ to H. See Lepowsky Ph. D. dissertation [Lep70, Ch. II §1] for a precise statement and a proof when Lie(H) contains a regular element of Lie(G) (we thank Daniel Bump for pointing out this reference to us).

2.4. **A computer program.** We now return to the main problem discussed in §2.1. We fix a compact connected Lie group G and a finite subgroup $\Gamma \subset G$. In order to enumerate the irreducible representations of G we fix as in the previous paragraph a maximal torus $T \subset G$ and a subset Φ^+ of positive roots for (G, T). For each dominant weight λ one thus has a unique irreducible representation V_λ with highest weight λ, hence a number

$$\dim(V_\lambda^\Gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \chi_{V_\lambda}(t(\gamma)),$$

where for each $\gamma \in \Gamma$ we define $t(\gamma)$ to be any element in T which is conjugate to γ in G. The last ingredient to be given for the computation is thus a list of these elements $t(\gamma) \in T$, which is a slightly more precise form of problem (b) of §2.1. Recall that the elements of T may be described as follows. Denote by $X^\vee = \mathrm{Hom}(\mathbb{S}^1, T)$ the cocharacter group of T and $\langle\, ,\, \rangle : X \otimes X^\vee \to \mathbb{Z}$ the canonical perfect pairing. If $\mu \in X^\vee \otimes \mathbb{C}$, denote by $e^{2i\pi\mu}$ the unique element $t \in T$ such that

$$\forall \lambda \in X, \ \ \lambda(t) = e^{2i\pi\langle\lambda,\mu\rangle}.$$

The map $\mu \mapsto e^{2i\pi\mu}$ defines an isomorphism $(X^\vee \otimes \mathbb{C})/X^\vee \overset{\sim}{\to} T$.

We thus wrote a computer program with the following property. It takes as input :

(a) The based root datum of (G, T, Φ^+), i.e. the collection $(X, \Phi, \Delta, X^\vee, \Phi^\vee, \langle , \rangle, \iota)$, where $\Phi^\vee \subset X^\vee$ is the set of coroots of (G, T) and $\iota : \Phi \to \Phi^\vee$ is the bijection $\alpha \mapsto \alpha^\vee$.

(b) A finite set of pairs $(\mu_j, C_j)_{j\in J}$, where $\mu_j \in X^\vee \otimes \mathbb{Q}$ and $C_j \in \mathbb{N}$, with the property that there exists a partition $\Gamma = \coprod_{j\in J} \Gamma_j$ such that $|\Gamma_j| = C_j$ and each element of Γ_j is conjugate in G to the element $e^{2i\pi\mu_j} \in T$.

(c) A dominant weight $\lambda \in X$.

It returns $\dim(V_\lambda^\Gamma) = |\Gamma|^{-1} \sum_{j\in J} C_j\, \chi_{V_\lambda}(e^{2i\pi\mu_j})$.

Recall that for $\alpha \in \Phi^+$ and $v \in X \otimes \mathbb{R}$ one has the relation $2\frac{(v,\alpha)}{(\alpha,\alpha)} = \langle v, \alpha^\vee\rangle$, thus (a), (b) and (c) contain indeed everything needed to evaluate the degenerate Weyl character formula. Although in theory the Weyl group W of (G, T) may be deduced from (a) we also take it as an input in practice. The program computes in particular for each $t_j = e^{2i\pi\mu_j}$ the root system of $M_j = C_G(t_j)^0$ and the set W^{M_i}. Of course it is often convenient to take $X = X^\vee = \mathbb{Z}^n$ with the canonical pairing. A routine in PARI/GP may be found at the url [CR].

2.5. **Some numerical applications.** We shall present in this paper four numerical applications of our computations. They concern the respective compact groups

$$G = \mathrm{SO}_7(\mathbb{R}), \ \ \mathrm{SO}_8(\mathbb{R}), \ \ \mathrm{SO}_9(\mathbb{R}), \ \ \text{and} \ \ \mathrm{G}_2$$

and each time a very specific finite subgroup Γ. We postpone to § 8.2 the discussion of the case G_2 and concentrate here on the first three cases. The general context is as follows.

Let V be a finite dimensional vector space over $\mathbb{R}$ and let $R \subset V$ be a reduced root system in the sense of Bourbaki [Bou81, Chap. VI §1]. Let $\mathrm{W}(R)$ denote the Weyl group of R and fix a $\mathrm{W}(R)$-invariant scalar product on V, so that

$$\mathrm{W}(R) \subset \mathrm{O}(V).$$

Assume that R is irreducible. Then V is irreducible as a representation of $\mathrm{W}(R)$ ([Bou81, Chap. VI §2]). Let $\varepsilon : \mathrm{W}(R) \to \{\pm 1\}$ the signature of $\mathrm{W}(R)$, i.e. $\varepsilon(w) = \det(w)$ for each $w \in \mathrm{W}(R)$, and set

$$\mathrm{W}(R)^+ = \mathrm{W}(R) \cap \mathrm{SO}(V).$$

We are in the general situation of this chapter with $G = \mathrm{SO}(V)$ and $\Gamma = \mathrm{W}(R)^+$. Beware that the root system Φ of $(\mathrm{SO}(V), T)$ is not the root system R above ! We choose the standard based root datum for $(\mathrm{SO}(V), T)$ as follows. If $l = \lfloor \frac{\dim(V)}{2} \rfloor$ we set $X = X^\vee = \mathbb{Z}^l$, equipped with the canonical pairing : if (e_i) denotes the canonical basis of $\mathbb{Z}^n$, then $\langle e_i, e_j \rangle = \delta_{i=j}$. There are two cases depending whether $\dim(V)$ is odd or even :

(i) $\dim(V) = 2l + 1$. Then $\Phi^+ = \{e_i, e_i \pm e_j, 1 \leq i < j \leq n\}$, $e_i^\vee = 2e_i$ for all i, and $(e_i \pm e_j)^\vee = e_i \pm e_j$ for all $i < j$.

(ii) $\dim(V) = 2l$. Then $\Phi^+ = \{e_i \pm e_j, 1 \leq i < j \leq n\}$ and $(e_i \pm e_j)^\vee = e_i \pm e_j$ for all $i < j$.

The dominant weights are thus the $\lambda = (n_1, \cdots, n_l) = \sum_{i=1}^{l} n_i e_i \in X$ such that $n_1 \geq n_2 \geq \cdots \geq n_l \geq 0$ if $\dim(V) = 2l + 1$, and such that $n_1 \geq n_2 \geq \cdots \geq n_{l-1} \geq |n_l|$ if $\dim(V) = 2l$.

Consider now the input (b) for the program. Recall that at least if $\dim(V)$ is odd, the conjugacy class of any element $g \in \mathrm{SO}(V)$ is uniquely determined by the characteristic polynomial of g acting on V. It turns out that for any reduced root system R, the characteristic polynomial of each conjugacy class of elements of $\mathrm{W}(R)$ has been determined by Carter in [Cart72]. We make an important use of these results, especially when R is of type E_7 and E_8 for the applications here, in which case it is given in Tables 10 and 11 *loc. cit.*

2.5.1. *Case I : R is of type E_7.* Then $-1 \in \mathrm{W}(R)$ and $\mathrm{W}(R) = \mathrm{W}(R)^+ \times \{\pm 1\}$, so the conjugacy classes in $\mathrm{W}(R)^+$ coincide with the conjugacy classes in $\mathrm{W}(R)$ belonging to $\mathrm{W}(R)^+$, i.e. with determinant 1. From Table 10 *loc.cit.* one sees that $\mathrm{W}(R)^+$ has exactly 27 conjugacy classes (c_j) and for each of them it gives its order C_j and its characteristic polynomial, from which we deduce μ_j : this is the datum we need for (b). For each dominant weight $\lambda = (n_1, n_2, n_3) \in \mathbb{Z}^3$ our computer program then returns $\dim(V_\lambda)^{\mathrm{W}(R)^+}$: see Table 2 for a sample of results and to the url [CR] for much more.

2.5.2. *Case II : R is of type E_8.* This case presents two little differences compared to the previous one. First the characteristic polynomial of an element $g \in \mathrm{SO}(V)$ does only determine its $\mathrm{O}(V)$-conjugacy class as $\dim(V) = 8$ is even. It determines its $\mathrm{SO}(V)$ conjugacy class if and only if ± 1 is an eigenvalue of g. Let $C \subset \mathrm{W}(R)^+$ be a $\mathrm{W}(R)$-conjugacy class and let P be its characteristic polynomial. If ± 1 is a root of P, there is thus a unique conjugacy class in $\mathrm{SO}(V)$ with this characteristic polynomial. Otherwise, C meets exactly two conjugacy classes in $\mathrm{SO}(V)$, it follows that $C = C_1 \coprod C_2$ where the C_i are $\mathrm{W}(R)^+$-conjugacy classes permuted by any element in $\mathrm{W}(R)\backslash\mathrm{W}(R)^+$, and in particular $|C_1| = |C_2|$. It follows that the table of Carter gives input (b) as well in this case.

We refer to Table 3 for a sample of values of nonzero $\dim(V_\lambda^{\mathrm{W}(R)^+})$ for $\lambda = (n_1, n_2, n_3, n_4)$ dominant with[10] $n_4 \geq 0$. As $-1 \in \mathrm{W}(R)^+$, one must have $n_1 + n_2 + n_3 + n_4 \equiv 0 \bmod 2$.

2.5.3. *Case III : the Weyl group of E_8 as a subgroup of* $\mathrm{SO}_9(\mathbb{R})$. This case is slightly different and we start with some general facts, keeping the setting of the beginning of § 2.5. Consider now the representation of $\mathrm{W}(R)$ on $V \oplus \mathbb{R}$ defined by $V' = V \oplus \varepsilon$. The map $w \mapsto (w, \varepsilon(w))$ defines an injective group homomorphism

$$\mathrm{W}(R) \hookrightarrow \mathrm{SO}(V'),$$

and we are thus again in the general situation of this chapter with this time $G = \mathrm{SO}(V')$ and $\Gamma = \mathrm{W}(R)$.

Consider now the special case of a R of type E_8, so that $\dim(V') = 9$. Table 11 of Carter gives the characteristic polynomials for the action of V of each $\mathrm{W}(R)$-conjugacy class in $\mathrm{W}(R)$, from which we immediately deduce the characteristic polynomial for the action of $V' = V \oplus \varepsilon$, hence the associated conjugacy class in $\mathrm{SO}(V')$ as $\dim(V') = 9$ is odd. This is the datum (b) we need for computing $\dim(V_\lambda)^{\mathrm{W}(R)^+}$: see Table 4 for a sample of values.

2.6. **Reliability.** Of course, there is some possibility that we have made mistakes during the implementation of the program of § 2.4 or of the characteristic polynomials from Carter's tables. This seems however unlikely due to the very large number of verifications we have made.

The first trivial check is that the sum of characteristic polynomials of all the elements of Γ in cases I and II is

$$|\mathrm{W}(R)^+|(X^{\dim(V)} + (-1)^{\dim(V)})$$

as it should be.

The second check is that our computer program for $\dim(V_\lambda^\Gamma)$ always returns a positive integer ... and it does in the several hundreds of cases we have tried. As observed in the introduction, a priori each term in the sum of the degenerate Weyl character formula is not an integer but an element of the cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is a N-th root of unity ($N = 2520$ in both cases, and we indeed computed in this number field with PARI GP, see the url [CR]). This actually makes a really good check for both the degenerate Weyl character formula and Carter's tables.

We will present two more evidences in the paper. One just below using a specific family of irreducible representations of $\mathrm{W}(R)^+$ for which one can compute directly the dimension of the $\mathrm{W}(R)^+$-invariants. The other one will be done much later in Chapters 5, 6, 7, where

[10] One easily sees that $\dim V_\lambda^{\mathrm{W}(R)^+} = \dim V_{\lambda'}^{\mathrm{W}(R)^+}$ if $\lambda = (n_1, n_2, n_3, n_4)$ and $\lambda' = (n_1, n_2, n_3, -n_4)$. Better, the triality $(n_1, n_2, n_3, n_4) \mapsto (\frac{n_1+n_2+n_3+n_4}{2}, \frac{n_1+n_2-n_3-n_4}{2}, \frac{n_1-n_2+n_3-n_4}{2}, \frac{-n_1+n_2+n_3-n_4}{2})$ preserves as well the table. This has a natural explanation when we identify $\mathrm{W}(R)$ as a certain orthogonal group over $\mathbb{Z}$ as in § 7.1, see [Gro96] and the forthcoming [CL14].

we shall check that our computations beautifully confirm the quite intricate Arthur's multiplicity formula in a large number of cases as well.

2.7. **A check : the harmonic polynomial invariants of a Weyl group.** We keep the notations of § 2.5. For each integer $n \geq 0$, let $\mathrm{Pol}_n(V)$ denote the space homogeneous polynomials on V of degree n and consider the two formal power series in $\mathbb{Z}[[t]]$:

$$P_R(t) = \sum_{n\geq 0} \dim(\mathrm{Pol}_n(V)^{\mathrm{W}(R)})\, t^n,$$

$$A_R(t) = \sum_{n\geq 0} \dim((\mathrm{Pol}_n(V)\otimes\varepsilon)^{\mathrm{W}(R)})\, t^n.$$

By [Bou81, Chap. V §6], if $l = \dim(V)$ and $m_1, \cdots, m_l$ are the exponents of $\mathrm{W}(R)$, then

$$P_R(t) = \prod_{i=1}^{l}(1-t^{m_i+1})^{-1} \quad \text{and} \quad A_R(t) = t^{|R|/2}P_R(t).$$

Let $\mathbf{\Delta}$ be "the" $\mathrm{O}(V)$-invariant Laplace operator on V. It induces an $\mathrm{O}(V)$-equivariant surjective morphism $\mathrm{Pol}_{n+2}(V)\to\mathrm{Pol}_n(V)$, whose kernel

$$\mathrm{H}_n(V)\subset\mathrm{Pol}_n(V)$$

is the space of harmonic polynomials of degree n on V. This is an irreducible representation of $\mathrm{SO}(V)$ if $\dim(V)\neq 2$, namely the irreducible representation with highest weight $ne_1 = (n, 0, \cdots, 0)$ (see e.g. [GW98, §5.2.3]). One deduces the following corollary.

Corollary 2.8. (i) $\sum_{n\geq 0}\dim(\mathrm{H}_n(V)^{\mathrm{W}(R)^+})t^n = (1-t^2)(1+t^{|R|/2})P_R(t)$.

(ii) $\sum_{n\geq 0}\dim(\mathrm{H}_n(V')^{\mathrm{W}(R)})t^n = (1+t^{1+|R|/2})P_R(t)$.

Proof — The generating series of $\dim(\mathrm{Pol}_n(V)^{\mathrm{W}^+(R)})$ is $P_R(t)+A_R(t)$, thus the first assertion follows from the $\mathrm{W}(R)$-equivariant exact sequence $0\to\mathrm{H}_{n+2}(V)\to\mathrm{Pol}_{n+2}(V)\overset{\mathbf{\Delta}}{\to}\mathrm{Pol}_n(V)\to 0$. Observe that (i) holds whenever R is irreducible or not. Assertion (ii) follows then from (i) applied to the root system $R\cup A_1$ in $V' = V\oplus\mathbb{R}$. □

We are not aware of an infinite family (V_i) of irreducible representations of $\mathrm{SO}(V)$ other than the $\mathrm{H}_i(V)$ with a simple close formula for $\dim V_i^{\mathrm{W}(R)^+}$. We end with some examples. Consider for instance the special case where R is of type E_7. The exponents of $\mathrm{W}(R)$ are 1, 5, 7, 9, 11, 13, 17, and $|R| = 18\cdot 7 = 126$. The power series of the corollary (i) thus becomes

$$\frac{1+t^{63}}{(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})(1-t^{14})(1-t^{18})}$$

$$= 1+t^6+t^8+t^{10}+2\,t^{12}+2\,t^{14}+2\,t^{16}+4\,t^{18}+4\,t^{20}+4\,t^{22}+7\,t^{24}+7\,t^{26}+8\,t^{28}+o(t^{28})$$

In the case R is of type E_8, the exponents of $\mathrm{W}(R)$ are $1, 7, 11, 13, 17, 19, 23, 29$ and $|R| = 8 \cdot 30 = 240$, so the power series of the corollary (i) is

$$\frac{1 + t^{120}}{(1 - t^8)(1 - t^{12})(1 - t^{14})(1 - t^{18})(1 - t^{20})(1 - t^{24})(1 - t^{30})}$$

$$= 1+t^8+t^{12}+t^{14}+t^{16}+t^{18}+2\,t^{20}+t^{22}+3\,t^{24}+2\,t^{26}+3\,t^{28}+3\,t^{30}+5\,t^{32}+3\,t^{34}+6\,t^{36}+o(t^{36})$$

The power series in case (ii) for R of type E_8 thus starts with

$$1+t^2+t^4+t^6+2\,t^8+2\,t^{10}+3\,t^{12}+4\,t^{14}+5\,t^{16}+6\,t^{18}+8\,t^{20}+9\,t^{22}+12\,t^{24}+14\,t^{26}+17\,t^{28}+o(t^{28})$$

In the three cases, those numbers turn out to perfectly fit our computations of the previous paragraph with the degenerate Weyl character formula: see the url [CR].

3. Automorphic representations of classical groups : review of Arthur's results

In this section, we review Arthur's recent results [Art11] on the endoscopic classification of discrete automorphic representations of classical groups. Our main aim is to apply it to the level 1 automorphic representations of certain very specific classical groups schemes defined over the ring of integers $\mathbb{Z}$, namely the ones which are reductive over $\mathbb{Z}$, for which the theory is substantially simpler.

3.1. **Classical semisimple groups over $\mathbb{Z}$.** By a $\mathbb{Z}$-group we shall mean an affine group scheme over $\mathbb{Z}$ which is of finite type. Besides the $\mathbb{Z}$-group SL_n, the symplectic $\mathbb{Z}$-group Sp_{2g} and their respective isogeny classes, we shall mainly focus on a collection of special orthogonal $\mathbb{Z}$-groups that we shall briefly recall now. We refer to [Ser70, Ch. IV & V], [Gro96], [Con11, Appendix C] and [CL14] for a more complete discussion.

Let L be a *quadratic abelian group* of rank n, which means that L is a free abelian group of rank n equipped with a quadratic form, that we will denote by $\mathrm{q} : L \to \mathbb{Z}$. We denote by $\mathrm{O}_L \subset \mathrm{Aut}_L$ the *orthogonal group scheme over $\mathbb{Z}$ associated to L*. Recall that by definition, if A is any commutative ring then $\mathrm{O}_L(A)$ is the subgroup of the general linear group $\mathrm{Aut}(L \otimes A)$ consisting of the elements g satisfying $\mathrm{q}_A \circ g = \mathrm{q}_A$, where $\mathrm{q}_A : L \otimes_{\mathbb{Z}} A \to A$ is the extension of scalars of q. The *isometry group* of L is the group $\mathrm{O}(L) := \mathrm{O}_L(\mathbb{Z}) \subset \mathrm{Aut}(L)$.

A quadratic abelian group L has a *determinant* $\det(L) \in \mathbb{Z}$, which is by definition the determinant of the symmetric bilinear form $x \cdot y = \mathrm{q}(x+y) - \mathrm{q}(x) - \mathrm{q}(y)$ on L. We say that L is *nondegenerate* if $\det(L) = \pm 1$ or $\det(L) = \pm 2$. This terminology is non standard, but will be convenient for us. Note that $x \cdot x = 2\mathrm{q}(x) \in 2\mathbb{Z}$, thus $(x, y) \mapsto x \cdot y$ is alternate on $L/2L$. This forces n to be even (resp. odd) if $\det(L) = \pm 1$ (resp. ± 2). Define also the *signature* of L as the signature (p, q) of $\mathrm{q}_{\mathbb{R}}$.

Assume now that L is nondegenerate. If n is even, then O_L is smooth over $\mathbb{Z}$. It has exactly two connected components and we shall denote by $\mathrm{SO}_L \subset \mathrm{O}_L$ the neutral one. This $\mathbb{Z}$-group may be also described as the kernel of the Dickson-Dieudonné morphism $\mathrm{O}_L \to \mathbb{Z}/2\mathbb{Z}$, which refines the usual homomorphism $\det : \mathrm{O}_L \to \mu_2$ defined for any quadratic abelian group L. When 2 is not a zero divisor in the commutative ring A, it turns out that $\mathrm{SO}_L(A) = \{g \in \mathrm{O}_L(A), \det(g) = 1\}$, but this does not hold in general. If n is odd, we simply define $\mathrm{SO}_L \subset \mathrm{O}_L$ as the kernel of det, and we have $\mathrm{O}_L \simeq \mu_2 \times \mathrm{SO}_L$. In all cases, SO_L is then reductive over $\mathbb{Z}$, and actually semisimple if $n \neq 2$ (see [Con11, Appendix C],[Bor91, V.23.6]). We also set $\mathrm{SO}(L) := \mathrm{SO}_L(\mathbb{Z})$.

Let L be a nondegenerate quadratic abelian group of rank n. The following two important properties hold (see [Ser70, Ch. V] when n is even) :

(i) If (p, q) denotes the signature of L, then $p - q \equiv -1, 0, 1 \bmod 8$.

(ii) For each prime ℓ, there is a $\mathbb{Z}_\ell$-basis of $L \otimes \mathbb{Z}_\ell$ in which $\mathrm{q}_{\mathbb{Z}_\ell}$ has the form

$$\begin{cases} x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n & \text{if } n \equiv 0 \bmod 2, \\ x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} + (-1)^{[n/2]}\frac{1}{2}\det(L)\ x_n^2 & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

In standard terminology, part (ii) implies that the nondegenerate quadratic abelian groups of given signature and determinant form a single genus.

We now briefly discuss certain aspects of the classification of non degenerate quadratic abelian groups L, starting with the definite case, which is not only the most important one for us but the most difficult case as well. A standard reference for this is the book by Conway and Sloane [CS99]. Replacing q by $-$q, there is no loss of generality in restricting to the positive ones (of signature $(n, 0)$), in which case such an L may be viewed as a lattice in the euclidean space $L \otimes \mathbb{R}$. Consider thus the standard euclidean space $\mathbb{R}^n$, with scalar product $(x_i) \cdot (y_i) = \sum_{i=1}^n x_i y_i$, and denote by

$$\mathcal{L}_n$$

the set of lattices $L \subset \mathbb{R}^n$ such that the map $x \mapsto \frac{x \cdot x}{2}$ defines a structure of nondegenerate quadratic abelian group on L. It is equivalent to ask that the lattice L is even, i.e. $x \cdot x \in 2\mathbb{Z}$ for each $x \in L$, and that L has covolume 1 (resp. $\sqrt{2}$) if n is even (resp. odd). The euclidean isometry group $\mathrm{O}(\mathbb{R}^n)$ naturally acts on $\mathcal{L}_n$ and we shall denote by

$$\mathcal{X}_n = \mathrm{O}(\mathbb{R}^n) \backslash \mathcal{L}_n$$

the quotient set. The map sending a positive definite quadratic abelian group L to the isometry class of the euclidean lattice L inside $L \otimes \mathbb{R}$ defines then a bijection between the set of isomorphism classes of non degenerate quadratic abelian groups of rank n and $\mathcal{X}_n$. The set $\mathcal{X}_n$ is a finite by reduction theory. Here is what seems to be currently known about its cardinality $h_n = |\mathcal{X}_n|$, thanks to works of Mordell, Witt, Kneser, Niemeier, and Borcherds :

$$h_1 = h_7 = h_8 = h_9 = 1, \;\; h_{15} = h_{16} = 2, \;\; h_{17} = 4, \;\; h_{23} = 32, \;\; h_{24} = 24, \;\; h_{25} = 121$$

In all those cases explicit representatives of $\mathcal{X}_n$ are known, and we recall some of them just below : see for instance [CS99] and [Bor84]. When $n \geq 31$ then the Minkowski-Siegel-Smith mass formula shows that $\mathcal{X}_n$ is huge, and h_n has not been determined in any case. One sometimes need to consider the set

$$\widetilde{\mathcal{X}}_n = \mathrm{SO}(\mathbb{R}^n) \backslash \mathcal{L}_n$$

of direct isometry classes of even lattices $L \in \mathcal{L}_n$. One has a natural surjective map $\widetilde{\mathcal{X}}_n \to \mathcal{X}_n$. The inverse image of the class of a lattice L has one element if $\mathrm{O}(L) \neq \mathrm{SO}(L)$, and two elements otherwise. In particular, $\widetilde{\mathcal{X}}_n \overset{\sim}{\to} \mathcal{X}_n$ if n is odd.

Some important even euclidean lattices are related to root systems as follows. Let $R \subset \mathbb{R}^n$ be a root system of rank n such that each $x \in R$ satisfies $x \cdot x = 2$. In particular, the irreducible components of R are of type A, D or E. The set R generates a lattice of $\mathbb{R}^n$ denoted by $\mathrm{Q}(R)$ in [Bou81, Ch. VI §1], that we view as a quadratic abelian group via the quadratic form $x \mapsto \frac{x \cdot x}{2}$. It is called the *root lattice* associated to R. It contains exactly the same information as R, because of the well known property $R = \{x \in Q(R), x \cdot x = 2\}$. The Cartan matrix of the root system R is symmetric and is a Gram matrix for the bilinear form of $\mathrm{Q}(R)$; its determinant is the index of connexion of R. It follows that the root lattices A_1, E_7, E_8 and $\mathrm{E}_8 \oplus \mathrm{A}_1$, associated respectively to root systems of type A_1, E_7, E_8 and $E_8 \coprod A_1$, are nondegenerate of ranks $n = 1, 7, 8$

and 9. Up to isometry, they are the unique such lattices in these dimensions, and the only ones we shall really need in this paper. The isometry groups of these lattices, and more generally of root lattices, are well known. Indeed, the isometry group of $\mathrm{Q}(R)$ is by definition the group denoted $\mathrm{A}(R)$ in [Bou81, Ch. VI §1]. It contains the Weyl group $\mathrm{W}(R)$ as a normal subgroup. Moreover, if $B \subset R$ is a basis of R, and if $\Gamma \subset \mathrm{A}(R)$ denotes de subgroup preserving B, then Γ is isomorphic to the automorphism group of the Dynkin diagram of R and the group $\mathrm{A}(R)$ is a semi-direct product of Γ by $\mathrm{W}(R)$ by [Bou81, Ch. VI no 1.5, Prop. 16]. It follows that in the four cases above, we have $\mathrm{O}(\mathrm{Q}(R)) = \mathrm{W}(R)$.

In general dimension $n = 8k + s$ with $s = -1$ (resp. $s = 0$, resp. $s = 1$). We obtain an example of positive definite nondegenerate quadratic abelian group Λ_n by considering the orthogonal direct sum $\mathrm{E}_7 \oplus \mathrm{E}_8^{k-1}$ (resp. E_8^k, resp. $\mathrm{E}_8^k \oplus \mathrm{A}_1$). Let us call it the *standard* positive definite quadratic abelian group of rank n. We will simply write

$$\mathrm{O}_n \text{ and } \mathrm{SO}_n$$

for O_{Λ_n} and SO_{Λ_n}. More generally, if $p \geq q$ are nonnegative integers, and if $p - q \equiv -1, 0, 1 \bmod 8$, the orthogonal direct sum of q hyperbolic planes[11] over $\mathbb{Z}$ and of Λ_{p-q} is a quadratic abelian group of signature (p, q), that we shall call *standard* as well for this signature. When $q > 0$ it turns out to be the only nondegenerate quadratic abelian group of signature (p, q) up to isometry (see [Ser70, Ch. V],[Gro96]), and we shall simply denote by $\mathrm{SO}_{p,q}$ its special orthogonal group scheme. When $|p - q| \leq 1$, this is a Chevalley group. In low dimension, we have the following exceptional isomorphisms over $\mathbb{Z}$:

$$\mathrm{SO}_{1,1} \simeq \mathbb{G}_m, \quad \mathrm{SO}_{2,1} \simeq \mathrm{PGL}_2, \quad \mathrm{SO}_{3,2} \simeq \mathrm{PGSp}_4,$$

as well as a central isogeny $\mathrm{SO}_{2,2} \to \mathrm{PGL}_2 \times \mathrm{PGL}_2$.

Remark 3.2. If L is any of the standard quadratic abelian group defined above, then it always contains elements α such that $\alpha \cdot \alpha = 2$. If s_α denotes the orthogonal symmetry with respect to such an α then $\mathrm{s}_\alpha \in \mathrm{O}(L) \backslash \mathrm{SO}(L)$, and the conjugation by s_α defines a $\mathbb{Z}$-automorphism of SO_L.

3.3. **Discrete automorphic representations.** Let G by a semisimple $\mathbb{Z}$-group. Denote by $\Pi(G)$ the set of isomorphism classes of complex representations π of $G(\mathbb{A})$ such that $\pi \simeq \pi_\infty \otimes \pi_f$, where :

(i) π_f is a smooth irreducible complex representation of $G(\mathbb{A}_f)$, and π_f is *unramified*, *i.e.* such that $\pi_f^{G(\widehat{\mathbb{Z}})} \neq 0$,

(ii) π_∞ is an irreducible unitary representation of $G(\mathbb{R})$.

Of course $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ denotes the adèle ring of $\mathbb{Q}$ and $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adèles. Denote by $\mathcal{H}(G)$ the complex Hecke-algebra of the pair $(G(\mathbb{A}_f), G(\widehat{\mathbb{Z}}))$. By well-known results of Satake and Tits ([Sat63],[Tit79]), the ring $\mathcal{H}(G)$ is commutative, so

[11] The hyperbolic plane over $\mathbb{Z}$ is the abelian group $\mathbb{Z}^2$ equipped with the quadratic form $\mathrm{q}(x, y) = xy$.

that $\dim \pi_f^{G(\widehat{\mathbb{Z}})} = 1$ for each unramified smooth irreducible complex representation π_f of $G(\mathbb{A}_f)$.

Recall that the homogeneous space $G(\mathbb{Q})\backslash G(\mathbb{A})$ has a nonzero $G(\mathbb{A})$-invariant Radon measure (Weil) of finite volume (Borel, Harish-Chandra, see [Bor63, §5]). Consider the Hilbert space

$$\mathcal{L}(G) = \mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}))$$

of square-integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ for this measure which are $G(\widehat{\mathbb{Z}})$-invariant on the right ([BJ79, §4],[GGPS66, Ch. 3]). This space $\mathcal{L}(G)$ is equipped with a unitary representation of $G(\mathbb{R})$ by right translations and with an action of the Hecke algebra $\mathcal{H}(G)$ commuting with $G(\mathbb{R})$. The subspace $\mathcal{L}_{\mathrm{disc}}(G) \subset \mathcal{L}(G)$ is defined as the closure of the sum of the irreducible closed subspaces for the $G(\mathbb{R})$-action, it is stable by $\mathcal{H}(G)$. A fundamental result of Harish-Chandra [HC68, Ch. 1 Thm. 1] asserts that each irreducible representation of $G(\mathbb{R})$ occurs with finite multiplicity in $\mathcal{L}(G)$. It follows that

$$\mathcal{L}_{\mathrm{disc}}(G) = \overline{\bigoplus_{\pi \in \Pi(G)}} m(\pi)\ \pi_\infty \otimes \pi_f^{G(\widehat{\mathbb{Z}})}, \qquad (3.1)$$

where the integer $m(\pi) \geq 0$ is the multiplicity of π as a sub-representation of $\mathcal{L}(G)$. We denote by

$$\Pi_{\mathrm{disc}}(G) \subset \Pi(G)$$

the subset of π such that $m(\pi) \neq 0$ and call them the *discrete automorphic representations of the $\mathbb{Z}$-group G*. A classical result of Gelfand and Piatetski-Shapiro [GGPS66] asserts that the subspace of cuspforms of G, which is stable by $G(\mathbb{R})$ and $\mathcal{H}(G)$, is included in $\mathcal{L}_{\mathrm{disc}}(G)$ and we denote by

$$\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G)$$

the subset of π consisting of cusp forms.

3.4. **The case of Chevalley and definite semisimple $\mathbb{Z}$-groups.** All those automorphic representations have various models, depending on the specific group G and the kind of π_∞ we are interested in. We shall content ourselves with the following classical descriptions. Consider the class set of G

$$\mathrm{Cl}(G) = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}}).$$

This is a finite set by [Bor63, §5] and we set $h(G) = |\mathrm{Cl}(G)|$. A well-known elementary fact (see *loc. cit.* §2) is that

$$h(\mathrm{SL}_n) = h(\mathrm{PGL}_n) = h(\mathrm{Sp}_n) = h(\mathrm{PGSp}_n) = 1,$$

and part of what we said in § 3.1 amounts to saying as well that $h(\mathrm{SO}_{p,q}) = 1$ if $pq \neq 0$. More generally, the strong approximation theorem of Kneser ensures that $h(G) = 1$ if G is simply connected and $G(\mathbb{R})$ has no compact factor (see [PR94]). Recall that a *Chevalley group* is a split semisimple $\mathbb{Z}$-group. We refer to [SGA3] and [Con11] for the general theory of Chevalley groups.

Proposition 3.5. *Let G be a Chevalley group. Then $h(G) = 1$ and the inclusion $G(\mathbb{R}) \to G(\mathbb{A})$ induces a homeomorphism*

$$G(\mathbb{Z})\backslash G(\mathbb{R}) \to G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}).$$

Moreover, $G(\mathbb{R})/G(\mathbb{Z})$ is connected and $\mathrm{Z}(G)(\mathbb{R}) = \mathrm{Z}(G)(\mathbb{Z})$.

Proof — We refer to [SGA3, Exp. XXII §4.2, §4.3] and [Con11, Chap. 6] for central isogenies between semisimple group schemes. Let $s : G_{\mathrm{sc}} \to G$ be a central isogeny with G_{sc} simply connected. The $\mathbb{Z}$-group G_{sc} is a Chevalley group as well. Let T be a maximal $\mathbb{Z}$-split torus in G and let $T_{\mathrm{sc}} \subset G_{\mathrm{sc}}$ be the split maximal torus defined as the inverse image of T by s. Recall that for any field k, we have the following simple facts from Galois cohomology :

(i) $s(G_{\mathrm{sc}}(k))$ is a normal subgroup of $G(k)$,

(ii) $T(k)s(G_{\mathrm{sc}}(k)) = G(k)$.

In particular, $G(\mathbb{A}_f) = T(\mathbb{A}_f)s(G_{\mathrm{sc}}(\mathbb{A}_f))G(\widehat{\mathbb{Z}})$. But $T(\mathbb{A}_f) = T(\mathbb{Q})T(\widehat{\mathbb{Z}})$ as T is $\mathbb{Z}$-split. It follows that

$$G(\mathbb{A}_f) = T(\mathbb{Q})T(\widehat{\mathbb{Z}})s(G_{\mathrm{sc}}(\mathbb{A}_f))G(\widehat{\mathbb{Z}}) = T(\mathbb{Q})s(G_{\mathrm{sc}}(\mathbb{A}_f))G(\widehat{\mathbb{Z}}), \tag{3.2}$$

where the last equality comes from (i) above. But $h(G_{\mathrm{sc}}) = 1$ by the strong approximation theorem, thus $h(G) = 1$ as well by the above identity.

Observe now that the map of the first statement is trivially injective, and even surjective as $h(G) = 1$. It is moreover continuous and open, as $G(\mathbb{R}) \times G(\widehat{\mathbb{Z}})$ is open in $G(\mathbb{A})$, hence a homeomorphism.

Let us check that $G(\mathbb{R})/G(\mathbb{Z})$ is connected. By (ii) again, observe that

$$G(\mathbb{R}) = s(G_{\mathrm{sc}}(\mathbb{R}))T(\mathbb{R}).$$

But $G_{\mathrm{sc}}(\mathbb{R})$ is connected as G_{sc} is connected and simply connected, by a classical result of Steinberg. We conclude as $T(\mathbb{Z}) \subset G(\mathbb{Z})$ meets every connected component of $T(\mathbb{R})$, since T is $\mathbb{Z}$-split. The last assertion follows from the following simple fact applied to $A = \mathrm{Z}(G)$: if A is a finite multiplicative $\mathbb{Z}$-group scheme, then the natural map $A(\mathbb{Z}) \to A(\mathbb{R})$ is bijective (reduce to the case $A = \mu_n$). □

When $G = \mathrm{PGSp}_{2g}$ or Sp_{2g}, the cuspidal automorphic representations π of G such that π_∞ is a holomorphic discrete series representation are closely related to vector valued Siegel cuspforms : see e.g. [AS01].

A semisimple $\mathbb{Z}$-group G will be said *definite* if $G(\mathbb{R})$ is compact. This is somewhat the opposite case of Chevalley groups, but a case of great interest in this paper. For instance the semisimple $\mathbb{Z}$-group SO_n defined in §3.1 is definite and there is a natural bijection

$$\mathrm{Cl}(\mathrm{SO}_n) \overset{\sim}{\to} \widetilde{X}_n, \tag{3.3}$$

because the rank n definite quadratic forms over $\mathbb{Z}$ form a single genus, as recalled in §3.1 (see [Bor63, §2]). If G is definite, then

$$\mathcal{L}(G) = \mathcal{L}_{\rm disc}(G)$$

by the Peter-Weyl theorem, and the discrete automorphic representations of G have very simple models. Automorphic forms for definite semisimple $\mathbb{Z}$-groups are a special case of "algebraic modular forms" in the sense of Gross [Gro99].

Proposition 3.6. *Let G be a semisimple definite $\mathbb{Z}$-group and let (ρ, V) be an irreducible continuous representation of $G(\mathbb{R})$. The vector space $\mathrm{Hom}_{G(\mathbb{R})}(V, \mathcal{L}(G))$ is canonically isomorphic to the space of covariant functions*

$$M_\rho(G) = \{f : G(\mathbb{A}_f)/G(\widehat{\mathbb{Z}}) \to V^*,\;\; f(\gamma g) = {}^t\rho(\gamma)^{-1} f(g)\;\; \forall \gamma \in G(\mathbb{Q}), g \in G(\mathbb{A}_f)\}.$$

In particular, $\dim(M_\rho(G)) = \sum_{\pi \in \Pi_{\rm disc}(G), \pi_\infty \simeq V} m(\pi)$.

The canonical bijection of the statement is $\varphi \mapsto (g \mapsto (v \mapsto \varphi(v)(1 \times g)))$, where $\varphi \in \mathrm{Hom}_{G(\mathbb{R})}(V, \mathcal{L}(G))$, $v \in V$ and $g \in G(\mathbb{A}_f)$. If $g_1, \cdots, g_{h(G)} \in G(\mathbb{A}_f)$ are representatives for the classes in $\mathrm{Cl}(G)$, the evaluation map $f \mapsto (f(g_i))$ defines thus a bijection

$$M_\rho(G) \overset{\sim}{\to} \prod_{i=1}^{h(G)} (V^*)^{\Gamma_i}$$

where Γ_i is the finite group $G(\mathbb{R}) \cap g_i^{-1} G(\mathbb{Q}) g_i$. In particular, to compute $M_\rho(G)$ we are reduced to compute invariants of the finite group $\Gamma_i \subset G(\mathbb{R})$ in the representation V, what we have already studied in Chapter 2. Indeed, the compact group $G(\mathbb{R})$ is always connected by a classical result of Chevalley [Bor91, V.24.6 (c) (ii)].

Of course if $g_i = 1$, then $\Gamma_i = G(\mathbb{Z})$. In the example of the group $G = \mathrm{SO}_n$, if $L_i \in \mathcal{L}_n$ is the lattice corresponding to g_i via the bijection (3.3), then $\Gamma_i = \mathrm{SO}(L_i)$. Later, we will study in details the cases $G = \mathrm{SO}_n$ where $n = 7, 8$ and 9, and the definite semisimple $\mathbb{Z}$-group G_2.

3.7. **Langlands parameterization of $\Pi_{\rm disc}(G)$.** In this paragraph, we discuss a parameterization of the elements of $\Pi(G)$ due to Satake and Harish-Chandra, according to Langlands point of view [Lan67, §2]. An important role is played by the Langlands dual group of G, for which a standard reference is Borel's paper [Bor77].

Let G be any semisimple $\mathbb{Z}$-group. As observed by Gross in [Gro96], the natural action of the absolute Galois group of $\mathbb{Q}$ on the based root datum of $G_{\overline{\mathbb{Q}}}$ is trivial, as each non trivial number field has a ramified prime. It follows that the $\mathbb{Q}$-group $G_{\mathbb{Q}}$ is an inner form of a split Chevalley group over $\mathbb{Q}$. In particular, the Langlands dual group of G may simply be defined as a complex semisimple algebraic group $\widehat{G}$ equipped with an isomorphism between the dual based root datum of $\widehat{G}$ and the based root datum of $G_{\overline{\mathbb{Q}}}$ (see [Bor77]). The group $\widehat{G}$ itself is well defined up to inner automorphism. When G is either PGL_n, Sp_{2n} or of the form SO_L for L a nondegenerate quadratic abelian group of rank n, it is well-known that $\widehat{G}$ is respectively isomorphic to

$$\mathrm{SL}_n(\mathbb{C}),\ \mathrm{SO}_{2n+1}(\mathbb{C}),\ \mathrm{Sp}_{n-1}(\mathbb{C})\ (n \text{ odd}) \ \text{ or } \ \mathrm{SO}_n(\mathbb{C})\ (n \text{ even}).$$

If $G = \mathrm{SO}_L$, then both $G_{\mathbb{Q}}$ and the $G(\mathbb{A}_f)$-conjugacy class of $G(\widehat{\mathbb{Z}}) \subset G(\mathbb{A}_f)$ only depend on the signature $L \otimes \mathbb{R}$ by the property of the genus of L discussed in §3.1. We shall thus loose nothing in assuming once and for all that L is the standard quadratic abelian group as defined *loc. cit.*

If H is the group of $\mathbb{C}$-points of a complex semisimple algebraic group over $\mathbb{C}$, we shall denote by

$$\mathcal{X}(H)$$

the set of collections (c_v) indexed by the places v of $\mathbb{Q}$, where each c_p (resp. c_∞) is a semisimple conjugacy class in H (resp. $\mathrm{Lie}_{\mathbb{C}}(H)$). If $\pi \in \Pi(G)$, then π_f is isomorphic to the restricted tensor product over all primes p of irreducible smooth representations π_p of $G(\mathbb{Q}_p)$ which are well defined up to isomorphism and unramified (i.e. $\pi_p^{G(\mathbb{Z}_p)} \neq 0$). By Langlands' interpretation of the work of Harish-Chandra and Satake, we have a natural parameterization map

$$c : \Pi(G) \longrightarrow \mathcal{X}(\widehat{G}), \quad \pi \mapsto (c_p(\pi)),$$

where :

(i) for each prime p the semisimple conjugacy class $c_p(\pi)$ is Satake parameter of π_p.

(ii) $c_\infty(\pi)$ is the conjugacy class defined by the infinitesimal character of π_∞ and the Harish-Chandra isomorphism.

See [Lan67, §2], [Bor77] and [Gro98] for a discussion of Satake's parameterization in those terms. We recall parameterization (ii) for the convenience of the reader (see e.g. Delorme's survey [Del97] for precise references). Let $\mathfrak{g}$ be the complex Lie algebra of $G(\mathbb{C})$. If V is a unitary representation of $G(\mathbb{R})$, the subspace $V^\infty \subset V$ of indefinitely differentiable vectors for the action of $G(\mathbb{R})$ carries an action of the enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$, and is dense in V. If V is irreducible, a version of Schur's lemma implies that the center $\mathrm{Z}(\mathrm{U}(\mathfrak{g}))$ of $\mathrm{U}(\mathfrak{g})$ acts by scalars on V^∞, which thus defines a $\mathbb{C}$-algebra homomorphism $\mathrm{Z}(\mathrm{U}(\mathfrak{g})) \to \mathbb{C}$ called the *infinitesimal character* of V. The last point to understand the meaning of (ii) above is that there is a canonical bijection between $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathrm{Z}(\mathrm{U}(\mathfrak{g})), \mathbb{C})$ and the set of semisimple conjugacy classes in $\widehat{\mathfrak{g}}$, that we now recall.

Let $\mathfrak{t}$ be a Cartan algebra of $\mathfrak{g}$ and let W denote the Weyl group of $(\mathfrak{g}, \mathfrak{t})$. The Harish-Chandra isomorphism is a canonical isomorphism $\mathrm{Z}(\mathrm{U}(\mathfrak{g})) \overset{\sim}{\to} (\mathrm{Sym}\, \mathfrak{t})^{\mathrm{W}}$. It follows that an infinitesimal character may be viewed as a W-orbit of elements in the dual vector space $\mathrm{Hom}(\mathfrak{t}, \mathbb{C})$. But the L-group datum defining $\widehat{G}$ naturally identifies $\mathrm{Hom}(\mathfrak{t}, \mathbb{C})$ with a Cartan algebra $\widehat{\mathfrak{t}}$ of $\widehat{\mathfrak{g}}$, and W with the Weyl group of $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{t}})$. It follows that the set of W-orbits of elements in $\mathrm{Hom}(\mathfrak{t}, \mathbb{C})$ is in canonically bijection with the set of semisimple conjugacy classes in $\widehat{\mathfrak{g}}$.

A result of Harish-Chandra asserts that up to isomorphism, there are at most finitely many irreducible unitary representations of $G(\mathbb{R})$ of any given infinitesimal character [KNA86, Corollary 10.37]. When $G(\mathbb{R})$ is compact, in which case it is necessarily connected by a theorem of Chevalley already cited in §3.4, the situation is much simpler. Indeed, let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra and let $\rho \in \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$ be the half-sum of the positive roots of $(\mathfrak{g}, \mathfrak{t}, \mathfrak{b})$. The infinitesimal character of the irreducible representation V_λ of $G(\mathbb{R})$ of highest weight $\lambda \in \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$ relative to $\mathfrak{b}$ is the conjugacy class of the element $\lambda + \rho$ ([DIX74, §7.4.6]), viewed as an element of $\widehat{\mathfrak{t}} = \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$. In particular, it uniquely determines V_λ.

3.8. **Arthur's symplectic-orthogonal alternative.** By a *classical semisimple group over* $\mathbb{Z}$ we shall mean either Sp_{2g} for $g \geq 1$, or SO_L for L a standard quadratic form over $\mathbb{Z}$ of rank $\neq 2$. In particular, SO_L is either SO_n or $\mathrm{SO}_{p,q}$ defined in §3.1. The *classical Chevalley groups* are the $\mathbb{Z}$-groups Sp_{2g}, $\mathrm{SO}_{p,q}$ with $p-q \in \{0,1\}$, and the trivial $\mathbb{Z}$-group SO_1. The definite classical semisimple groups over $\mathbb{Z}$ are the SO_n.

Fix G a classical semisimple group over $\mathbb{Z}$. Arthur's classification describes $\Pi_{\mathrm{disc}}(G)$ in terms of the $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_m)$ for various m's, and our aim from now is to recall this classification. We shall denote by

$$\mathrm{St} : \widehat{G} \hookrightarrow \mathrm{SL}_n(\mathbb{C})$$

the standard representation of its dual group, which defines in particular the integer $n = n(G)$. For instance $n(\mathrm{Sp}_{2g}) = 2g+1$ and $n(\mathrm{SO}_m) = 2[m/2]$. This group homomorphism defines in particular a natural map $\mathcal{X}(\widehat{G}) \to \mathcal{X}(\widehat{\mathrm{PGL}_n})$ that we shall still denote by St.

Theorem* 3.9. *(Arthur) For any $n \geq 1$ and any given self-dual $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ there is a unique classical Chevalley group G^π with $n(G^\pi) = n$ such that there exists $\pi' \in \Pi_{\mathrm{disc}}(G^\pi)$ satisfying $\mathrm{St}(c(\pi')) = c(\pi)$.*

This is [ART11, Thm. 1.4.1]. As $n(G^\pi) = n$, the only possibilities for G^π are thus $G^\pi = \mathrm{SO}_1$ if $n = 1$, $G^\pi = \mathrm{Sp}_{n-1}$ if $n > 1$ is odd, $G^\pi = \mathrm{SO}_{\frac{n}{2},\frac{n}{2}-1}$ or $\mathrm{SO}_{\frac{n}{2},\frac{n}{2}}$ if n is even. This last case only exists for $n > 2$, which forces $G^\pi = \mathrm{SO}_{2,1} \simeq \mathrm{PGL}_2$ if $n = 2$.

As self-dual $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ will be said *orthogonal* (resp. *symplectic*) if $\widehat{G^\pi}$ is isomorphic to a complex special orthogonal group (resp. symplectic group). For short, we shall define the *sign* of π

$$s(\pi) \in \{\pm 1\}$$

to be 1 if π is orthogonal, -1 otherwise. If n is odd then π is necessarily orthogonal, i.e. $s(\pi) = 1$.

Definition 3.10. *Let $n \geq 1$ be an integer. We denote by :*

- $\Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n) \subset \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ *the subset of self-dual π, i.e. such that $\pi^\vee \simeq \pi$,*
- $\Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathrm{PGL}_n) \subset \Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n)$ *the subset of symplectic π,*
- $\Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_n) \subset \Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n)$ *the subset of orthogonal π.*

We have $\Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n) = \Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathrm{PGL}_n) \coprod \Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_n)$.

3.11. **The symplectic-orthogonal alternative for polarized algebraic regular cuspidal automorphic representations of GL_n over $\mathbb{Q}$.** Let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ satisfying (a), (b) and (c') of §1.7. The self-dual representation $\pi' := \pi \otimes |\cdot|^{\frac{w(\pi)}{2}}$ has a trivial central character, hence defines a self-dual element of $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$. Our aim in this paragraph is to show that π is orthogonal (resp. symplectic) in the sense of §1.7 if and only if π' is so in the sense of Arthur (§3.8). A key role will be played by condition (c') on π. This forces us to discuss first Langlands parameterization for $\mathrm{GL}_n(\mathbb{R})$ in more details than we have done so far. We refer to [Lan73], [Bor77], and especially [Kna94], for more details.

Recall that the Weil group of $\mathbb{C}$ is the topological group $\mathrm{W}_\mathbb{C} := \mathbb{C}^*$. The Weil group of $\mathbb{R}$, denoted $\mathrm{W}_\mathbb{R}$, is a non-split extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathrm{W}_\mathbb{C}$, for the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^\times$. The set $\mathrm{W}_\mathbb{R}\backslash\mathrm{W}_\mathbb{C}$ contains a unique $\mathrm{W}_\mathbb{C}$-conjugacy class of elements $j \in \mathrm{W}_\mathbb{R}\backslash\mathrm{W}_\mathbb{C}$ such that $j^2 = -1$ (as elements of $\mathbb{C}^*$), and we fix once and for all such an element. According to Langlands parameterization, if π is a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ then the unitary representation π_∞ of $\mathrm{GL}_n(\mathbb{R})$ is uniquely determined up to isomorphism by its *Langlands parameter*. By definition, this is an isomorphism class of continuous semisimple representations

$$\mathrm{L}(\pi_\infty) : \mathrm{W}_\mathbb{R} \to \mathrm{GL}_n(\mathbb{C}).$$

It refines the infinitesimal character of π_∞, viewed as a semisimple conjugacy class in $\mathrm{M}_n(\mathbb{C}) = \mathrm{Lie}_\mathbb{C}\widehat{\mathrm{GL}_n}$, which may actually be read on the restriction of $\mathrm{L}(\pi_\infty)$ to $\mathrm{W}_\mathbb{C}$. Concretely, this restriction is a direct sum of continuous homomorphisms $\chi_i : \mathbb{C}^* \to \mathbb{C}^*$ for $i = 1, \cdots, n$, which are unique up to reordering. For each i, there are unique $\lambda_i, \mu_i \in \mathbb{C}$ such that $\lambda_i - \mu_i \in \mathbb{Z}$, satisfying

$$\chi_i(z) = z^{\lambda_i}\overline{z}^{\mu_i}$$

for all $z \in \mathbb{C}^*$: the n eigenvalues of the infinitesimal character of π_∞ are the elements $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. We have used Langlands convenient notation : if $\lambda, \mu \in \mathbb{C}$ satisfy $\lambda - \mu \in \mathbb{Z}$, and if $z \in \mathbb{C}^*$, then $z^\lambda\overline{z}^\mu$ denotes the element of $\mathbb{C}^*$ defined by $(z\overline{z})^{\frac{\lambda+\mu}{2}}(\frac{z}{|z|})^{\lambda-\mu}$.

Langlands parameters are easy to classify, as the irreducible continuous representations of $\mathrm{W}_\mathbb{R}$ are either one dimensional or induced from a one dimensional representation of $\mathrm{W}_\mathbb{C}$. The first ones are described thanks to the natural isomorphism $\mathrm{W}_\mathbb{R}^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{R}^*$ sending $z \in \mathrm{W}_\mathbb{C}$ to $z\overline{z}$: they have thus the form $|\cdot|^s$ or $\varepsilon_{\mathbb{C}/\mathbb{R}}|\cdot|^s$, where $s \in \mathbb{C}$ and $\varepsilon_{\mathbb{C}/\mathbb{R}}(x) = x/|x|$ is the sign character. The irreducible continuous 2-dimensional representations of $\mathrm{W}_\mathbb{R}$ are the $\mathrm{I}_w \otimes |\cdot|^s$ for $w > 0$ and $s \in \mathbb{C}$, where we have set for any integer $w \geq 0$

$$\mathrm{I}_w = \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}\, z^{-w/2}\overline{z}^{w/2}.$$

Proposition 3.12. *Let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ satisfying (a), (b) and (c') of the introduction, of Hodge weights w_i. Then*

$$\mathrm{L}(\pi_\infty) \otimes |\cdot|^{w(\pi)/2} \simeq \begin{cases} \bigoplus_{i=1}^{\frac{n}{2}} \mathrm{I}_{w_i} & \text{if } n \equiv 0 \bmod 2, \\ \varepsilon_{\mathbb{C}/\mathbb{R}}^{\frac{n-1}{2}} \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} \mathrm{I}_{w_i} & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

Moreover, if w and n are even then $n \equiv 0 \bmod 4$.

The main ingredient in the proof of this proposition is the following special case of Clozel's purity lemma [Clo90, Lemma 4.9].

Lemma 3.13. *(Clozel's purity lemma) Let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$. Assume that the eigenvalues of the infinitesimal character of π_∞ are in $\frac{1}{2}\mathbb{Z}$. Then there is an element $w \in \mathbb{Z}$ such that $\mathrm{L}(\pi_\infty) \otimes |\cdot|^{w/2}$ is a direct sum of representations of the form 1, $\varepsilon_{\mathbb{C}/\mathbb{R}}$, or $\mathrm{I}_{w'}$ for $w' \in \mathbb{Z}$.*

Proof — (of Proposition 3.12) We apply Clozel's purity lemma to π. Condition (a) on π ensures that $\mathrm{L}(\pi_\infty)^* \simeq \mathrm{L}(\pi_\infty) \otimes |\cdot|^{w(\pi)}$. As both the $\mathrm{I}_{w'}$, 1 and $\varepsilon_{\mathbb{C}/\mathbb{R}}$ are self-dual, it follows that the element w given by the purity lemma coincides with $w(\pi)$. Condition (c') on π, and the relation $-k_i + w(\pi)/2 = -w_i$ for $i = 1, \cdots, [n/2]$, concludes the proof when $\frac{w(\pi)}{2}$ is not a weight of π (e.g. when $w(\pi)$ is odd). By assumption (c'), if $\frac{w(\pi)}{2}$ is a weight of π then it has multiplicity 1 if n is odd and 2 if $n \equiv 0 \bmod 4$. But by condition (b) on π and the structure of the idèles of $\mathbb{Q}$, the global central character of π is $|\cdot|^{-\frac{n\,w(\pi)}{2}}$, so that $\det(\mathrm{L}(\pi_\infty) \otimes |\cdot|^{w(\pi)/2}) = 1$. Observe that for any $w' \in \mathbb{Z}$ we have

$$\det(\mathrm{I}_{w'}) = \varepsilon_{\mathbb{C}/\mathbb{R}}^{w'+1}.$$

The proposition follows when n is odd, as well as when $n \equiv w(\pi) \equiv 0 \bmod 2$ since

$$\mathrm{I}_0 \simeq 1 \oplus \varepsilon_{\mathbb{C}/\mathbb{R}}.$$

□

We are now able to state a strengthening of Arthur's Theorem 3.9, which is more precise at the infinite place. Indeed, let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ be self-dual and let

$$\mathrm{L}(\pi_\infty) : \mathrm{W}_\mathbb{R} \longrightarrow \mathrm{SL}(n, \mathbb{C})$$

be the Langlands parameter of π_∞. Arthur shows that $\mathrm{L}(\pi_\infty)$ maybe conjugated into $\mathrm{St}(\widehat{G^\pi}) \subset \mathrm{SL}(n, \mathbb{C})$ ([Art11, Thm. 1.4.2]). Note that the $\widehat{G}^\pi$-conjugacy class of the resulting Langlands parameter

$$\widetilde{\mathrm{L}}(\pi_\infty) : \mathrm{W}_\mathbb{R} \longrightarrow \widehat{G^\pi} \tag{3.4}$$

is not quite canonical, but so is its $\mathrm{Out}(\widehat{G^\pi})$-orbit.

Corollary 3.14. *Let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ satisfying* (a), (b) *and* (c') *of the introduction. Then π is orthogonal (resp. symplectic) in the sense of §1.7 if and only if $\pi \otimes |\cdot|^{w(\pi)/2}$ is so in the sense of §3.8.*

Proof — Consider the self-dual representation $\pi' = \pi \otimes |\cdot|^{w(\pi)/2}$ in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$. By Proposition 3.12, $\mathrm{L}(\pi'_\infty) : \mathrm{W}_\mathbb{R} \to \mathrm{SL}_n(\mathbb{C})$ is a direct sum of distinct irreducible self-dual representations of $\mathrm{W}_\mathbb{R}$. It follows that if $\mathrm{L}(\pi'_\infty)$ preserves a nondegenerate pairing on $\mathbb{C}^n$ then each irreducible subspace is nondegenerate as well. Moreover, the 2-dimensional representation I_w has determinant $\varepsilon_{\mathbb{C}/\mathbb{R}}^{w+1}$ and may be conjugate into $\mathrm{O}_2(\mathbb{C})$ if and only if w is even. The result follows as $\mathrm{L}(\pi_\infty)$ may be conjugate into $\mathrm{St}(\widehat{G^\pi})$ by the aforementioned result of Arthur. □

It will be convenient in the sequel to adopt a slightly different point of view, although eventually equivalent, on the representations π studied in the introduction. Consider the cuspidal automorphic representations π of GL_n over $\mathbb{Q}$ such that :

(i) (self-dual) $\pi^\vee \simeq \pi$,

(ii) (conductor 1) π_p is unramified for each prime p,

(iii) (regular half-algebraicity) the representation $\mathrm{L}(\pi_\infty)$ is multiplicity free and the eigenvalues of the infinitesimal character of π_∞ are in $\frac{1}{2}\mathbb{Z}$.

Such a π necessarily has a trivial central character, hence may be viewed as well as an element of $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$.

Proposition 3.15. *The map* $\pi \mapsto \pi \otimes |\cdot|^{w(\pi)/2}$ *defines a bijection between the set of centered cuspidal automorphic representations of* GL_n *over* $\mathbb{Q}$ *satisfying conditions* (a), (b) *and* (c') *(§1.1,§1.7) and the set of cuspidal automorphic representations of* GL_n *over* $\mathbb{Q}$ *satisfying* (i), (ii) *and* (iii) *above.*

Proof — Proposition 3.12 shows that if π satisfies (a), (b) and (c') then $\pi \otimes |\cdot|^{w(\pi)/2}$ satisfies (i), (ii) and (iii). It also shows that the map of the statement is injective. Assume conversely that π satisfies (i), (ii) and (iii). Clozel's purity lemma 3.13 and condition (iii) imply that $\mathrm{L}(\pi_\infty)$ is a direct sum of non-isomorphic representations of the form $1, \varepsilon_{\mathbb{C}/\mathbb{R}}$ or $\mathrm{I}_{w'}$ for $w' > 0$. As explained in the proof of corollary 3.14, it follows from the existence of $\widetilde{\mathrm{L}}(\pi_\infty)$ that each of these summands has the same symplectic/orthogonal alternative than π. Recall that $\mathrm{I}_{w'}$ preserves a nondegenerate symplectic pairing if and only if w' is odd.

Assume first that π is symplectic. Then the representations 1, $\varepsilon_{\mathbb{C}/\mathbb{R}}$ and $\mathrm{I}_{w'}$ for $w' \equiv 0 \bmod 2$ do not occur in $\mathrm{L}(\pi_\infty)$. In other words, n is even and

$$\mathrm{L}(\pi_\infty) \simeq \bigoplus_{i=1}^{\frac{n}{2}} \mathrm{I}_{w_i} \tag{3.5}$$

for some unique odd positive integers $w_1 > \cdots > w_{n/2}$.

Assume now that π is orthogonal. If n is odd, we have

$$\mathrm{L}(\pi_\infty) \simeq \chi \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} \mathrm{I}_{w_i}, \tag{3.6}$$

where $\chi \in \{1, \varepsilon_{\mathbb{C}/\mathbb{R}}\}$ and for some unique even positive integers $w_1 > \cdots > w_{n/2}$. As π has trivial central character, we have $\det(\mathrm{L}(\pi_\infty)) = 1$, thus $\chi = \varepsilon_{\mathbb{C}/\mathbb{R}}^{\frac{n-1}{2}}$ is uniquely determined. If n is even, and if 0 is not an eigenvalue of the infinitesimal character of π_∞, then

$$\mathrm{L}(\pi_\infty) \simeq \bigoplus_{i=1}^{\frac{n}{2}} \mathrm{I}_{w_i}, \tag{3.7}$$

for some unique even positive integers $w_1 > \cdots > w_{n/2}$. If 0 is an eigenvalue of the infinitesimal character of π_∞, it has necessarily multiplicity 2 and the two characters 1 and $\varepsilon_{\mathbb{C}/\mathbb{R}}$ occur in $\mathrm{L}(\pi_\infty)$, so that the above isomorphism still holds for some unique even nonnegative integers $w_1 > \cdots > w_{n/2}$, and with $w_{n/2} = 0$. Note that $n \equiv 0 \bmod 4$ if n is even, as $\det(\mathrm{L}(\pi_\infty)) = 1$.

In all these cases, we have defined a sequence of nonnegative integers $w_1 > \cdots > w_{[n/2]}$ having the same parity. The cuspidal automorphic representation $\pi \otimes |\cdot|^{-w_1/2}$ of GL_n satisfies (a), (b) and (c'), for the motivic weight w_1 and the Hodge weights w_i (and is centered). ☐

Definition 3.16. *We denote by* $\Pi^\perp_{\mathrm{alg}}(\mathrm{PGL}_n) \subset \Pi^\perp_{\mathrm{cusp}}(\mathrm{PGL}_n)$ *the subset of* π *satisfying* (i), (ii) *and* (iii) *above. For* $* = \mathrm{o}$ *or* s *we also set* $\Pi^*_{\mathrm{alg}}(\mathrm{PGL}_n) = \Pi^*_{\mathrm{cusp}}(\mathrm{PGL}_n) \cap \Pi^\perp_{\mathrm{alg}}(\mathrm{PGL}_n)$ *(see Definition 3.10).*

Definition 3.17. *If* $\pi \in \Pi^\perp_{\mathrm{alg}}(\mathrm{PGL}_n)$, *its* Hodge weights

$$w_1 > \cdots > w_{[n/2]}$$

are the Hodge weights of the cuspidal automorphic representation π_0 *of* GL_n *over* $\mathbb{Q}$ *such that* $\pi \simeq \pi_0 \otimes |\cdot|^{w(\pi_0)/2}$ *given by Proposition 3.15.*

They are odd if π *is symplectic and even otherwise. They determine* π_∞ *by the formula* (3.5) if n is even (*resp. by the formula* (3.6) *if* n *is odd*).

3.18. **Arthur's classification : global parameters.** Let G be a classical semisimple group over $\mathbb{Z}$ and let $n = n(G)$. Define $s(G) \in \{\pm 1\}$ by $s(G) = 1$ if $\widehat{G}$ is a special orthogonal group, -1 otherwise. Denote by $\Psi_{\mathrm{glob}}(G)$ the set of quadruples

$$(k, (n_i), (d_i), (\pi_i))$$

where $1 \leq k \leq n$ is an integer, where for each $1 \leq i \leq k$ then $n_i \geq 1$ is an integer and d_i is a divisor of n_i, and where $\pi_i \in \Pi^\perp_{\mathrm{cusp}}(\mathrm{PGL}_{n_i/d_i})$, such that :

(i) $\sum_{i=1}^k n_i = n$,
(ii) for each i, $s(\pi_i)(-1)^{d_i+1} = s(G)$,
(iii) if $i \neq j$ and $(n_i, d_i) = (n_j, d_j)$ then $\pi_i \neq \pi_j$.

The set $\Psi_{\mathrm{glob}}(G)$ only depends on $n(G)$ and $s(G)$. Two elements $(k, (n_i), (d_i), (\pi_i))$ and $(k', (n'_i), (d'_i), (\pi'_i))$ in $\Psi_{\mathrm{glob}}(G)$ are said *equivalent* if $k = k'$ and if there exists $\sigma \in \mathfrak{S}_k$ such that $n'_i = n_{\sigma(i)}$, $d'_i = d_{\sigma(i)}$ and $\pi'_i = \pi_{\sigma(i)}$ for each i. An element of $\Psi_{\mathrm{glob}}(G)$ will be called a *global Arthur parameter* for G. The class $\underline{\psi}$ of $\psi = (k, (n_i), (d_i), (\pi_i))$ will also be denoted symbolically by

$$\underline{\psi} = \pi_1[d_1] \oplus \pi_2[d_2] \oplus \cdots \oplus \pi_k[d_k].$$

In the writing above we shall replace the symbol $\pi_i[d_i]$ by $[d_i]$ if $n_i = d_i$ (as then π_i is the trivial representation), and by π_i if $d_i = 1$ and $n_i \neq d_i$.

Let $\psi \in \Psi_{\rm glob}(G)$. Recall that for each integer $d \geq 1$, the $\mathbb{C}$-group SL_2 has a unique irreducible $\mathbb{C}$-representation ν_d of dimension d, namely $\mathrm{Sym}^{d-1}(\mathbb{C}^2)$. Condition (i) on ψ allows to define a morphism

$$\rho_\psi : \prod_{i=1}^{k} \mathrm{SL}_{n_i/d_i} \times \mathrm{SL}_2 \longrightarrow \mathrm{SL}_n$$

(canonical up to conjugation by $\mathrm{SL}_n(\mathbb{C})$) obtained as the direct sum of the representations $\mathbb{C}^{n_i/d_i} \otimes \nu_{d_i}$. One obtains this way a canonical map

$$\rho_\psi : \prod_{i=1}^{k} \mathcal{X}(\mathrm{SL}(n_i/d_i)) \times \mathcal{X}(\mathrm{SL}_2) \longrightarrow \mathcal{X}(\mathrm{SL}_n).$$

A specific element of $\mathcal{X}(\mathrm{SL}_2)$ plays an important role in Arthur's theory : it is the element $e = (e_v)$ defined by

$$e_p = \mathrm{diag}(p^{1/2}, p^{-1/2})$$

(positive square roots) for each prime p, and by

$$e_\infty = \mathrm{diag}(1/2, -1/2).$$

As is well known, $e = c(1)$ where $1 \in \Pi_{\rm disc}(\mathrm{PGL}_2)$ is the trivial representation.

Theorem* 3.19. *(Arthur's classification) Let G be any classical semisimple group over $\mathbb{Z}$ and let $\pi \in \Pi_{\rm disc}(G)$. There is a $\psi(\pi) = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\rm glob}(G)$ unique up to equivalence such that*

$$\mathrm{St}(c(\pi)) = \rho_\psi(\prod_{i=1}^{k} c(\pi_i) \times e).$$

When G is a Chevalley group this follows from [Art11, Thm. 1.5.2], otherwise it expected to be part of Arthur's treatment of inner forms of quasi-split classical groups over $\mathbb{Q}$ (see the last chapter *loc.cit.* ; note that the special case needed here, namely for pure inner forms, is presumably much simpler because none of the difficulties mentioned by Arthur seems to occurs.). The uniqueness of $\psi(\pi)$ up to equivalence is actually due to Jacquet-Shalika [JS81]. The part of the theorem concerning the infinitesimal character is a property of Shelstad's transfer : see [She08, Lemma 15.1], [MEza, Lemma 25].

Definition 3.20. *The global Arthur parameter $\psi(\pi)$ will be called the global Arthur parameter of π.*

For instance if $1_G \in \Pi_{\rm disc}(G)$ denotes the trivial representation of G, then it is well-known that the Arthur parameter of 1_G is $[n(G)]$, unless $\widehat{G} \simeq \mathrm{SO}_{2m}(\mathbb{C})$ in which case it is $[1] \oplus [n(G) - 1]$.

Let $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\rm glob}(G)$. The associated triple $(k, (n_i, d_i))$, taken up to permutations of the (n_i, d_i), will be called the *endoscopic type* of ψ. One usually says that ψ is *stable* if $k = 1$ and *endoscopic* otherwise. The generalized Ramanujan conjecture asserts that each π_i is tempered. We shall thus say that ψ is *tempered* if $d_i = 1$ for all i. If $\psi = \psi(\pi)$, the Ramanujan conjecture asserts then that π is tempered if and only if $\psi(\pi)$ is. In some important cases, e.g. the special case where $G(\mathbb{R})$ is compact, this conjecture is actually known in most cases : see Corollary 3.24. We will say that π is *stable*, *endoscopic* or *formally tempered* if $\psi(\pi)$ is respectively stable, endoscopic or tempered. We will also talk about the endoscopic type of a π for the endoscopic type of $\psi(\pi)$.

Our last task is to explain Arthur's converse to the theorem above, namely to decide whether a given $\psi \in \Psi_{\rm glob}(G)$ is in the image of the map $\pi \mapsto \psi(\pi)$. This is the content of the so-called *Arthur's multiplicity formula*. Our aim until the end of this chapter will be to state certain special cases of this formula.

3.21. **The packet $\Pi(\psi)$ of a $\psi \in \Psi_{\rm glob}(G)$.** Fix $\psi = (k,(n_i),(d_i),(\pi_i)) \in \Psi_{\rm glob}(G)$. If p is a prime number, define

$$\Pi_p(\psi)$$

as the set of isomorphism classes of $G(\mathbb{Z}_p)$-spherical (i.e. unramified) irreducible smooth representations of $G(\mathbb{Q}_p)$ whose Satake parameter s_p, a semisimple conjugacy class in $\widehat{G}$, satisfies

$$\mathrm{St}(s_p) = \rho_\psi(\prod_{i=1}^{k} c_p(\pi_i) \times e_p).$$

This relation uniquely determines the $\mathrm{Out}(\widehat{G})$-orbit of s_p. It follows that $\Pi_p(\psi)$ is a singleton, unless $\widehat{G} \simeq \mathrm{SO}(2m,\mathbb{C})$ and $\mathrm{St}(s_p)$ does not possess the eigenvalue ± 1 (which implies that each n_i is even), in which case it has exactly 2 elements.

We shall now associate to ψ a $\mathrm{Out}(\widehat{G})$-orbit of equivalence classes of *Archimedean Arthur parameters* for $G_{\mathbb{R}}$, which will eventually lead in some cases to a definition of a set $\Pi_\infty(\psi)$ of irreducible unitary representations of $G(\mathbb{R})$. Denote by $\Psi(G_{\mathbb{R}})$ the set of such parameters, i.e. of continuous homomorphisms

$$\psi_{\mathbb{R}} : \mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}$$

which are $\mathbb{C}$-algebraic on the $\mathrm{SL}_2(\mathbb{C})$-factor and such that the image of any element of $\mathrm{W}_{\mathbb{R}}$ is semisimple. Two such parameters are said *equivalent* if they are conjugate under $\widehat{G}$. An important invariant of an equivalence class of parameters $\psi_{\mathbb{R}}$ is its *infinitesimal character*

$$z_{\psi_{\mathbb{R}}}$$

which is a semisimple conjugacy class in $\widehat{\mathfrak{g}}_{\mathbb{C}}$ given according to a recipe of Arthur : see e.g. §A.2 for the general definition. It is also the infinitesimal character of the Langlands parameter $\mathrm{W}_{\mathbb{R}} \to \widehat{G}$ associated by Arthur to $\psi_{\mathbb{R}}$.

We now go back to the global Arthur parameter ψ. By assumption (ii) on ψ, each space $\mathbb{C}^{n_i/d_i} \otimes \nu_{d_i}$ carries a natural representation of $\widehat{G^{\pi_i}} \times \mathrm{SL}_2(\mathbb{C})$ which preserves a nondegenerate bilinear form unique up to scalars, which is symmetric if $s(\widehat{G}) = 1$ and antisymmetric otherwise. One thus obtains a $\mathbb{C}$-morphism

$$r_\psi : \prod_{i=1}^{k} \widehat{G^{\pi_i}} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}.$$

The collection of $\widetilde{\mathrm{L}}((\pi_i)_\infty) : \mathrm{W}_\mathbb{R} \to \widehat{G^{\pi_i}}$ (see (3.4)) defines by composition with r_ψ a morphism

$$\psi_\infty : \mathrm{W}_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$

which is by definition *the Archimedean Arthur parameter associated to* ψ. The $\mathrm{Out}(\widehat{G})$-orbit of the equivalence class of ψ_∞ only depends on $\underline{\psi}$. In particular, only the $\mathrm{Out}(\widehat{G})$-orbit of its infinitesimal character is well-defined, with this caveat in mind we shall still denote it by z_{ψ_∞}. By definition we have

$$\mathrm{St}(z_{\psi_\infty}) = \rho_\psi(\prod_{i=1}^{k} c_\infty(\pi_i) \times e_\infty), \tag{3.8}$$

which also determines z_{ψ_∞} uniquely.

Consider the following two properties of an Arthur parameter $\psi_\mathbb{R} \in \Psi(G_\mathbb{R})$:

(a) $z_{\psi_\mathbb{R}}$ is the infinitesimal character of a finite dimensional, irreducible, $\mathbb{C}$-representation of $G(\mathbb{C})$,

(b) $\mathrm{St} \circ \psi_\mathbb{R}$ is a multiplicity free representation of $W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C})$.

If ψ satisfies (a) (resp (b)) then so does $\tau \circ \psi$ where $\tau \in \mathrm{Aut}(\widehat{G})$. In particular, it makes sense to say that ψ_∞ satisfies (a) if $\psi \in \Psi_{\mathrm{glob}}(G)$.

Definition 3.22. *Let $\Psi_{\mathrm{alg}}(G) \subset \Psi_{\mathrm{glob}}(G)$ be the subset of ψ such that ψ_∞ satisfies (a).*

The following lemma is mainly a consequence of Clozel's purity lemma 3.13.

Lemma 3.23. *Let $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{alg}}(G)$. If $s(G) = 1$, assume that $n(G) \not\equiv 2 \bmod 4$. Then :*

(i) *ψ_∞ satisfies (b).*

(ii) *For each $i = 1, \dots, k$, we have $\pi_i \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_{n_i/d_i})$,*

(iii) *Each n_i is even, except one of them if $n(G)$ is odd, and except perhaps exactly two of them if $s(G) = 1$ and $n(G) \equiv 0 \bmod 4$. Moreover, if $s(G) = 1$ and if n_i is even, then $n_i \equiv 0 \bmod 4$.*

Proof — Let X be the semisimple conjugacy class $\mathrm{St}(z_{\psi_\infty}) \subset \mathrm{M}_n(\mathbb{C})$. Of course, we have $X = -X$. Property (a) on ψ_∞ is equivalent to the following properties :

(a1) the eigenvalues of X are integers if $s(G) = 1$, and half odd integers otherwise,

(a2) and these eigenvalues are distinct, except if $s(G) = 1$, $n(G) \equiv 0 \bmod 2$, and if 0 is an eigenvalue of X. In this exceptional case, that we shall call (E), the eigenvalue 0 has multiplicity 2 and the other eigenvalues have multiplicity 1.

In particular, assertion (i) follows from (a2) if (E) does not hold. For each i, the eigenvalues of $c_\infty(\pi_i)\otimes \mathrm{Sym}^{d_i-1}(e_\infty)$ are among those of X. Consider the set I of integers $i\in\{1,\dots,k\}$ such that $c_\infty(\pi_i)$ does not have distinct eigenvalues. It follows that $|I|\le 1$, and if $i\in I$ then $d_i=1$ and 0 is the only multiple eigenvalue of $\mathrm{c}_\infty(\pi_i)$ (and has multiplicity 2). Fix $1\le i\le k$. The eigenvalues of $c_\infty(\pi_i)$ are in $\frac{1}{2}\mathbb{Z}$ by (a1), thus it follows from Clozel's purity lemma 3.13 that $\mathrm{L}((\pi_i)_\infty)$ is a direct sum of representations of $\mathrm{W}_\mathbb{R}$ of the form I_w, 1 or $\varepsilon_{\mathbb{C}/\mathbb{R}}$. We have $\pi_i\in\Pi_{\mathrm{alg}}^\perp$ unless $i\in I$ and the two characters occurring in $\mathrm{L}((\pi_i)_\infty)$ are both 1 or both $\varepsilon_{\mathbb{C}/\mathbb{R}}$. This proves assertion (ii) when (E) does not hold, in which case $I=\emptyset$. This also shows that if $I=\{i\}$ then $n_i\equiv 0 \bmod 2$.

Observe that the assertion (iii) of the lemma is obvious if $s(G)=-1$. Indeed, for each i then $s(\pi_i)=(-1)^{d_i}$, so if d_i is odd then π_i is symplectic and thus n_i/d_i is even. It follows that we may assume from now on that $s(G)=1$.

Let $J\supset I$ be the set of integers $i\in\{1,\dots,k\}$ such that 0 is an eigenvalue of $c_\infty(\pi_i)\otimes \mathrm{Sym}^{d_i-1}(e_\infty)$. Then $1\le|J|\le 2$, and $|J|=1$ if $n(G)$ is odd. Let $i\notin J$, we claim that $n_i\equiv 0\bmod 4$. Indeed, this is clear if d_i is even as then π_i is symplectic. If d_i is odd, then $\pi_i\in\Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_{n_i/d_i})$ as $i\notin I$, and $c_\infty(\pi_i)$ does not contain the eigenvalue 0. It follows that n_i/d_i is even, in which case $n_i/d_i\equiv 0\bmod 4$ by Proposition 1.13. In particular, we have the congruence

$$\sum_{j\in J} n_j \equiv n(G) \bmod 4. \tag{3.9}$$

This proves assertion (iii) of the lemma.

Assume now that $I\neq\emptyset$ and let $i\in I$. Then $J=I=\{i\}$, and also $n(G)\equiv 0\bmod 4$ by assumption, so $n_i\equiv 0\bmod 4$. Of course we have $\det(\mathrm{L}((\pi_i)_\infty))=1$. But $\mathrm{L}((\pi_i)_\infty)$ is a direct sum of $n_i/2-1$ non isomorphic representations of the form I_w with $w>0$ and w even, and of two characters χ_1 and χ_2 among 1 and $\varepsilon_{\mathbb{C}/\mathbb{R}}$. The congruence $n_i\equiv 0\bmod 4$ implies that $\chi_1\chi_2=\varepsilon_{\mathbb{C}/\mathbb{R}}$. In other words, $\chi_1\neq\chi_2$ and thus $\pi_i\in\Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_{n_i/d_i})$. This ends the proof of assertion (ii) of the lemma.

It only remains to prove (i) in case (E). Observe that ψ_∞ does not satisfy (b) if and only if the representation $\mathrm{St}\circ\psi_\infty$ of $\mathrm{W}_\mathbb{R}\times\mathrm{SL}_2(\mathbb{C})$ contains either twice the character 1 or twice the character $\varepsilon_{\mathbb{C}/\mathbb{R}}$ (with trivial action of $\mathrm{SL}_2(\mathbb{C})$). This can only happen if either $J=I$ or $|J|=2$, $I=\emptyset$ and $d_i=1$ for all $i\in J$. In the case $I=J$, we conclude by the previous paragraph. If $|J|=2$, n_i is odd for each $i\in J$, and the congruence (3.9) shows that exactly one of the two n_i, $i\in J$, is congruent to 1 modulo 4 (resp. to 3 modulo 4). But $\pi_j\in\Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_{n_j})$ for $j\in J$, thus $\mathrm{L}((\pi_j)_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}$ if $n_j\equiv 1\bmod 4$ and 1 otherwise (see (3.6)). □

This is the first important motivation for the consideration of the properties (a) and (b). The second is that if $G(\mathbb{R})$ is compact, and if $\pi\in\Pi_{\mathrm{disc}}(G)$, then $\psi(\pi)_\infty$ obviously satisfies (a), as well as (b) because $n(G)\equiv -1,0,1\bmod 8$.

Corollary* 3.24. *Assume that $n(G)\not\equiv 2\bmod 4$ if $s(G)=1$. If $\pi\in\Pi_{\mathrm{disc}}(G)$ is such that π_∞ has the infinitesimal character of a finite dimensional irreducible $\mathbb{C}$-representation of $G(\mathbb{C})$, and if $\psi(\pi)=(k,(n_i),(d_i),(\pi_i))$, then $\pi_i\in\Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_{n_i/d_i})$ for each i.*

In particular, each π_i satisfies the Ramanujan conjecture, unless perhaps if $s(G) = s(\pi_i) = d_i = 1$, $n(G) \equiv n_i \equiv 0 \bmod 2$, and $\mathrm{St}(c_\infty(\pi_i))$ contains the eigenvalue 0.

Proof — It only remains to justify the statement about Ramanujan conjecture, but this follows from Lemma 3.23 (ii) and the results of Clozel-Harris-Labesse, Shin and Caraiani recalled in §1.6. □

We shall exclude from now on the particular case $s(G) = 1$ and $n(G) \equiv 2 \bmod 4$, i.e. we assume that

$$\widehat{G} \not\simeq \mathrm{SO}(4m+2, \mathbb{C}).$$

We already said that $G(\mathbb{R})$ is an inner form of a split group. As $\widehat{G} \simeq \mathrm{SO}(4m+2, \mathbb{C})$, it is also an inner form of a compact group (this is of course obvious if $G(\mathbb{R})$ is already compact). A parameter $\psi_\mathbb{R} \in \Psi(G_\mathbb{R})$ satisfying conditions (a) and (b) above is called *an Adams-Johnson parameter* for $G_\mathbb{R}$. The set of these parameters is denoted by

$$\Psi_{\mathrm{AJ}}(G_\mathbb{R}) \subset \Psi(G_\mathbb{R}).$$

We refer to the Appendix A for a general discussion about them, and more precisely to Definition A.4 and the discussion that follows. For $\psi_\mathbb{R} \in \Psi_{\mathrm{AJ}}(G_\mathbb{R})$, Adams and Johnson have defined in [AJ87] a finite set $\Pi(\psi_\mathbb{R})$ of (cohomological) irreducible unitary representations of $G(\mathbb{R})$. In the notations of this appendix, the group $G(\mathbb{R})$ is isomorphic to a group of the form G_t for some $t \in \mathfrak{X}_1(T)$. Recall that up to inner isomorphisms, G_t only depends on the W-orbit of $t\mathrm{Z}(G)$. We fix such an isomorphism between $G(\mathbb{R})$ and $G_{[t]}$ and set $\Pi(\psi_\mathbb{R}) = \Pi(\psi, G_{[t]})$. As $\mathrm{Aut}(G(\mathbb{R})) \neq \mathrm{Int}(G(\mathbb{R}))$ in general, this choice of an isomorphism might be problematic in principle. However, a simple case-by-case inspection shows that for any classical semisimple $\mathbb{Z}$-group G the natural map $\mathrm{Out}(G) \to \mathrm{Out}(G(\mathbb{R}))$ is surjective, so that this choice virtually plays no role in the following considerations. We shall say more about this when we come to the multiplicity formula.

Let $\psi \in \Psi_{\mathrm{alg}}(G)$. If $\mathrm{Out}(\widehat{G}) = 1$, or more generally if the $\mathrm{Out}(\widehat{G})$-orbit of the equivalence class of ψ_∞ has one element, we set

$$\Pi_\infty(\psi) = \Pi(\psi_\infty).$$

In the remaining case, we define $\Pi_\infty(\psi)$ as the disjoint union of the two sets $\Pi(\psi_\mathbb{R})$ where $\psi_\mathbb{R}$ is an equivalence class of parameters in the $\mathrm{Out}(\widehat{G})$-orbit of ψ_∞. Recall from §A.7 that the isomorphism $G(\mathbb{R}) \to G_t$ fixed above furnishes a canonical parameterization map

$$\tau : \Pi_\infty(\psi) \longrightarrow \mathrm{Hom}(\mathrm{C}_{\psi_\infty}, \mathbb{C}^\times).$$

The presence of C_{ψ_∞} in the target, rather than S_{ψ_∞}, follows from the fact that $G(\mathbb{R})$ is a pure inner form of a split group and from Lemma A.14. When the $\mathrm{Out}(\widehat{G})$-orbit of the equivalence class of ψ_∞ has two elements, say ψ_1, ψ_2, there is a canonical way of identifying C_{ψ_1} and C_{ψ_2}, thus it is harmless to denote them by the same name C_{ψ_∞}.

Definition 3.25. *If $\psi \in \Psi_{\mathrm{alg}}(G)$ set $\Pi(\psi) = \{\pi \in \Pi(G),\ \pi_v \in \Pi_v(\psi)\ \forall v\}$.*

The first conjecture we are in position to formulate is a comparison between the Arthur packet attached to a $\psi_{\mathbb{R}} \in \Psi_{\mathrm{AJ}}(G_{\mathbb{R}})$, as defined in his book [Art11, §2.2] by twisted endoscopy when $G_{\mathbb{R}}$ is split, and the packet $\Pi(\psi_{\mathbb{R}})$ of Adams and Johnson recalled above (in a slightly weak sense in the case $\widehat{G} = \mathrm{SO}_{2r}(\mathbb{C})$). It seems widely believed that they indeed coincide, although no proof seems to have been given yet. A first consequence would be the following conjecture. Observe that this conjecture is obvious when $G(\mathbb{R})$ is compact.

Conjecture 3.26. *If $\pi \in \Pi_{\mathrm{disc}}(G)$ and if π_∞ has the infinitesimal character of a finite dimensional irreducible $\mathbb{C}$-representation of $G(\mathbb{C})$ then $\pi \in \Pi(\psi(\pi))$.*

So far we have defined for each $\psi \in \Psi_{\mathrm{alg}}(G)$ a set $\Pi(\psi)$ as well as a parameterization map τ of $\Pi_\infty(\psi)$. This set is e.g. a singleton when $G = \mathrm{SO}_n$ with n odd, and it is finite, in bijection with $\Pi_\infty(\psi)$, if $\mathrm{Out}(\widehat{G}) = 1$. Arthur's multiplicity formula is a formula for $m(\pi)$ for each $\pi \in \Pi(\psi)$, at least when $\mathrm{Out}(\widehat{G}) = 1$. This formula contains a last ingredient that we now study.

3.27. **The character ε_ψ of C_ψ.** Consider some $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{alg}}(G)$ and denote by C_ψ the centralizer of $\mathrm{Im}(r_\psi)$ in $\widehat{G}$. This is an elementary abelian 2-group that we may describe as follows.

Observe that $\mathrm{St} \circ r_\psi$ is a direct sum of k non-isomorphic irreducible representations of $\prod_{i=1}^k \widehat{G^{\pi_i}} \times \mathrm{SL}_2(\mathbb{C})$, say $\oplus_{i=1}^k V_i$, where V_i factors through a representation of $\widehat{G^{\pi_i}} \times \mathrm{SL}_2(\mathbb{C})$ whose dimension is n_i. Observe that by Lemma 3.23 (iii) each n_i is even, except perhaps exactly one or two of them when $\widehat{G}$ is an orthogonal group. If $1 \leq i \leq k$ is such that n_i is even, there is a unique element

$$s_i \in \widehat{G}$$

such that $\mathrm{St}(s_i)$ acts as $-\mathrm{Id}$ on V_i and as Id on each V_j with $j \neq i$. Of course, we have $s_i^2 = 1$ and $s_i \in \mathrm{C}_\psi$, and the following lemma is clear.

Lemma 3.28. *C_ψ is generated by $\mathrm{Z}(\widehat{G})$ and by the elements s_i, where $i = 1, \ldots, k$ is such that n_i is even.*

A first important ingredient in Arthur's multiplicity formula is Arthur's character

$$\varepsilon_\psi : \mathrm{C}_\psi \longrightarrow \{\pm 1\}.$$

It has been defined by Arthur in full generality in [Art89]. We shall apply formula (1.5.6) of [Art11]. By definition, ε_ψ is trivial on $\mathrm{Z}(\widehat{G}) \subset \mathrm{C}_\psi$. In the special case here, we thus only have to give the $\varepsilon_\psi(s_i)$. As the representation $\nu_a \otimes \nu_b$ of $\mathrm{SL}_2(\mathbb{C})$ has exactly $\mathrm{Min}(a, b)$ irreducible factors, the formula loc. cit. is thus easily seen to be

$$\varepsilon_\psi(s_i) = \prod_{j \neq i} \varepsilon(\pi_i \times \pi_j)^{\mathrm{Min}(d_i, d_j)} \tag{3.10}$$

where $\varepsilon(\pi_i \times \pi_j) = \pm 1$ is the sign such that

$$\mathrm{L}(1 - s, \pi_i \times \pi_j) = \varepsilon(\pi_i \times \pi_j)\mathrm{L}(s, \pi_i \times \pi_j).$$

Here $\mathrm{L}(s, \pi_i \times \pi_j)$ is the completed L-function of $\pi_i \times \pi_j$, and the functional equation above is due to Jacquet, Shalika and Piatetski-Shapiro : see [Cog04, Ch. 9] for a survey. An important result of Arthur asserts that $\varepsilon(\pi_i \times \pi_j) = 1$ if $s(\pi_i)s(\pi_j) = 1$ [Art11, Thm. 1.5.3], so that in the product (3.10) we may restrict to the j such that $s(\pi_j) \neq s(\pi_i)$.

The cuspidal automorphic representation π_i is unramified at each finite place, and also quite specific at the infinite place : it belongs to $\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_{n_i/d_i})$ by Lemma 3.23 (ii)). It follows that one has an explicit formula for $\varepsilon(\pi_i \times \pi_j)$ in terms of the Hodge weights of π_i and π_j. The precise recipe is as follows. There is a unique collection of complex numbers

$$\varepsilon(r) \in \{1, i, -1, -i\}$$

defined for all the isomorphism classes of continuous representations $r : \mathrm{W}_{\mathbb{R}} \to \mathrm{GL}_m(\mathbb{C})$ which are trivial on $\mathbb{R}_{>0} \subset \mathrm{W}_{\mathbb{C}}$, such that :

(i) $\varepsilon(r \oplus r') = \varepsilon(r)\varepsilon(r')$ for all r, r',

(ii) $\varepsilon(\mathrm{I}_w) = i^{w+1}$ for any integer $w \geq 0$,

(iii) $\varepsilon(1) = 1$.

As $\mathrm{I}_0 \simeq 1 \oplus \varepsilon_{\mathbb{C}/\mathbb{R}}$, it follows that $\varepsilon(\varepsilon_{\mathbb{C}/\mathbb{R}}) = i$. For instance, if $w, w' \geq 0$ are integers, then

$$\varepsilon(\mathrm{I}_w \otimes \mathrm{I}_{w'}) = (-1)^{1+\mathrm{Max}(w,w')},$$

as $\mathrm{I}_w \otimes \mathrm{I}_{w'} \simeq \mathrm{I}_{w+w'} \oplus \mathrm{I}_{|w-w'|}$.

If $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n)$ and $\pi' \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_{n'})$ then both $\mathrm{L}(\pi_\infty)$ and $\mathrm{L}(\pi'_\infty)$ are trivial on $\mathbb{R}_{>0}$ (see §3.11), and one has

$$\varepsilon(\pi \times \pi') = \varepsilon(\mathrm{L}(\pi_\infty) \otimes \mathrm{L}(\pi'_\infty)). \tag{3.11}$$

See [Tat79, §4] (the epsilon factor is computed here with respect to $x \mapsto e^{2i\pi x}$), [Art11, §1.3], and Cogdell's lectures [Cog04, Ch. 9]. This allows to compute the character ε_ψ in all cases. See [CL14] for some explicit formulas.

We are now able to prove Proposition 1.14 of the introduction.

Proof — (of Proposition 1.14) Let $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_n)$ and consider its global epsilon factor $\varepsilon(\pi) := \varepsilon(\pi \times 1)$. Arthur's result [Art11, Thm. 1.5.3] ensures that $\varepsilon(\pi) = 1$ as π is orthogonal. On the other hand, if $w_1 > \cdots > w_{[n/2]}$ are the Hodge weights of π then the formulas (3.6) and (3.7) show that

$$\varepsilon(\pi) = \begin{cases} (-1)^{\frac{\sum_{j=1}^{[n/2]}(w_j+1)}{2}} & \text{if } n \not\equiv 3 \bmod 4, \\ (-1)^{\frac{1+\sum_{j=1}^{[n/2]}(w_j+1)}{2}} & \text{otherwise.} \end{cases}$$

□

3.29. **Arthur's multiplicity formula.** Let G be a classical semisimple group over $\mathbb{Z}$ such that $\widehat{G} \neq \mathrm{SO}(4m+2,\mathbb{C})$ and let $\psi = (k,(n_i),(d_i),(\pi_i)) \in \Psi_{\mathrm{alg}}(G)$. Following Arthur, set

$$m_\psi = \begin{cases} 2 & \text{if } s(\widehat{G}) = 1 \text{ and } n_i \equiv 0 \bmod 2 \text{ for all } 1 \leq i \leq k, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the following equivalence relation $\sim$ on $\Pi(G)$. The relation $\sim$ is trivial (i.e. equality) unless $\widehat{G}$ is an even orthogonal group, in which case one may assume that $G = \mathrm{SO}_L$ is a standard even orthogonal group. Consider the outer automorphism s of the $\mathbb{Z}$-group G induced by the conjugation by any $s_\alpha \in \mathrm{O}(L)$ as in Remark 3.2. If $\pi, \pi' \in \Pi(G)$ we define $\pi \sim \pi'$ if $\pi_v \in \{\pi'_v, \pi'_v \circ s\}$ for each v.

For $\pi \in \Pi(G)$, recall that $m(\pi)$ denotes the multiplicity of π in $\mathrm{L}^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Recall that we have defined a group C_ψ in §3.27, as well as a group C_{ψ_∞} in §3.21. By definition there is a canonical inclusion

$$\mathrm{C}_\psi \subset \mathrm{C}_{\psi_\infty}.$$

Conjecture 3.30. *(Arthur's multiplicity formula) Let $\psi = (k,(n_i),(d_i),(\pi_i)) \in \Psi_{\mathrm{alg}}(G)$ and let $\pi \in \Pi(\psi)$. Then*

$$\sum_{\pi' \in \Pi(\psi), \pi' \sim \pi} m(\pi') = \begin{cases} 0 & \text{if } \tau(\pi_\infty)_{|\mathrm{C}_\psi} \neq \varepsilon_\psi, \\ m_\psi & \text{otherwise.} \end{cases}$$

Observe that $\{\pi' \in \Pi(\psi), \pi \sim \pi'\}$ is the singleton $\{\pi\}$ unless $\widehat{G}$ is an even orthogonal group.

At the moment this multiplicity formula is still conjectural in the form stated here. However, when G is a Chevalley group, it is a Theorem* by [Art11, Thm. 1.5.2] if we replace the parameterized set $(\Pi_\infty(\psi), \tau)$ above by the one abstractly defined by Arthur [Art11, Thm. 1.5.1]. Actually, an extra subtlety arises in Arthur's work because the Archimedean packets he constructs *loc. cit.* are a priori multisets rather than sets. The resulting possible extra multiplicities have been neglected here to simplify the exposition, as they are actually expected not to occur according to Arthur (and even more so for the Adams-Johnson packets). Note also that Arthur's formula even holds for all the global parameters $\psi \in \Psi_{\mathrm{glob}}(G)$. The case of a general G has also been announced by Arthur : see Chap. 9 loc. cit.

In this paper, we shall use this conjecture only in the following list of special cases. In each case we will explicit completely the multiplicity formula in terms of the Hodge weights of the π_i appearing in ψ. We have already done so for the term ε_ψ in the previous paragraph. In each case we also discuss the dependence of the multiplicity formula on the choice of the identification of $G(\mathbb{R})$ that we have fixed in §3.21 to define $\tau(\pi_\infty)$.

3.30.1. *The definite odd orthogonal group* $G = \mathrm{SO}_{2r+1}$.

In this case $r \equiv 0, 3 \bmod 4$ and $\widehat{G} = \mathrm{Sp}_{2r}(\mathbb{C})$. Consider the standard based root datum for $(\widehat{G}, \widehat{B}, \widehat{T})$ with $\mathrm{X}^*(\widehat{T}) = \mathbb{Z}^r$ with canonical basis (e_i) and

$$\Phi^+(\widehat{G}, \widehat{T}) = \{2e_i, 1 \leq i \leq r\} \cup \{e_i \pm e_j, 1 \leq i < j \leq r\}.$$

We conjugate r_ψ in $\widehat{G}$ so that the centralizer of $\varphi_{\psi_\infty}(\mathrm{W}_\mathbb{C})$ is $\widehat{T}$, and that $\varphi_{\psi_\infty}(z) = z^\lambda \overline{z}^{\lambda'}$ with $\lambda \in \frac{1}{2}\mathrm{X}_*(\widehat{T})$ dominant with respect to $\widehat{B}$.

There is a unique element in $\Pi_\infty(\psi)$, namely the irreducible representation with infinitesimal character z_{ψ_∞}. The character $\tau(\pi_\infty)$ is absolutely canonical here as each automorphism of $G(\mathbb{R})$ is inner and there is a unique choice of strong real form t for G_t (namely $t = 1$). By Cor. A.12, this character $\tau(\pi_\infty)$ is $(\rho^\vee)_{|\mathrm{C}_{\psi_\infty}}$, where $\rho^\vee$ denotes the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$, namely $\rho^\vee = re_1 + (r-1)e_2 + \cdots + e_r$. In particular $\rho^\vee \in \mathrm{X}^*(\widehat{T})$ and it satisfies the congruence

$$\rho^\vee \equiv e_r + e_{r-2} + e_{r-4} + \cdots \mod 2\mathrm{X}^*(\widehat{T}).$$

Observe that $\rho^\vee(-1) = 1$ as $r \equiv 0, 3 \bmod 4$, so that $\rho^\vee$ is trivial on $\mathrm{Z}(\widehat{G})$.

Consider the generators s_i of C_ψ introduced in §3.27. We shall now give an explicit formula for the $\rho^\vee(s_i)$. Fix some $i \in \{1, \ldots, k\}$ and write $n_i = r_i d_i$. Assume first that d_i and r_i are even. Then 0 is not a Hodge weight of π_i, as otherwise z_{ψ_∞} would have twice the eigenvalue $\frac{1}{2}$. The positive eigenvalues of z_{ψ_∞} associated to the summand $\pi_i[d_i]$ of ψ are thus the union of the d_i consecutive half-integers

$$\frac{w_j + d_i - 1}{2}, \frac{w_j + d_i - 3}{2}, \cdots, \frac{w_j + 1 - d_i}{2}$$

where w_j runs among the $\frac{r_i}{2}$ Hodge weights of π_i. It follows that

$$\rho^\vee(s_i) = (-1)^{\frac{d_i}{2}\frac{r_i}{2}} = (-1)^{\frac{n_i}{4}}.$$

If d_i is even and r_i is odd, the positive eigenvalues of z_{ψ_∞} coming from the summand $\pi_i[d_i]$ are of the form above, plus the $\frac{d_i}{2}$ consecutive half-integers $\frac{d_i - 1}{2}, \cdots, \frac{3}{2}, \frac{1}{2}$. One rather obtains

$$\rho^\vee(s_i) = \begin{cases} -(-1)^{[\frac{r_i}{2}]\frac{d_i}{2}} & \text{if } \frac{d_i}{2} \equiv 1, 2 \bmod 4, \\ (-1)^{[\frac{r_i}{2}]\frac{d_i}{2}} & \text{otherwise.} \end{cases}$$

If d_i is odd, in which case r_i is even, the sign $\rho^\vee(s_i)$ depends on the Hodge weights of π_i. Precisely, denote by

$$w_1 > \cdots > w_r$$

the positive odd integers w_j such that the eigenvalues of $\mathrm{St}(z_{\psi_\infty})$ in $\mathrm{SL}_{2r}(\mathbb{C})$ are the $\pm\frac{w_j}{2}$ (see formula (3.8)). There is a unique subset $J \subset \{1, \cdots, r\}$ such that the Hodge weights of π_i are the w_j for $j \in J$. Denote by J' the subset of $j \in J$ such that $j \equiv r \bmod 2$. It is then clear that

$$\rho^\vee(s_i) = (-1)^{|J'|}.$$

Although these formulas are explicit, we do not especially recommend to use them in a given particular case, as usually the determination of $\rho^\vee(s_i)$ is pretty immediate by definition from the inspection of ψ !

3.30.2. *The definite even orthogonal group* $G = \mathrm{SO}_{2r}$.

In this case $r \equiv 0 \bmod 4$ and $\widehat{G} = \mathrm{SO}_{2r}(\mathbb{C})$. Consider the standard based root datum for $(\widehat{G}, \widehat{B}, \widehat{T})$ with $\mathrm{X}^*(\widehat{T}) = \mathbb{Z}^r$ with canonical basis (e_i) and

$$\Phi^+(\widehat{G}, \widehat{T}) = \{e_i \pm e_j, 1 \leq i < j \leq r\}.$$

We conjugate r_ψ in $\widehat{G}$ as in the odd orthogonal case.

If the $\mathrm{Out}(\widehat{G})$-orbit of ψ_∞ consists of only one equivalence class, in which case the $\mathrm{Out}(\widehat{G})$-orbit of z_{ψ_∞} is a singleton, then the unique element of $\Pi_\infty(\psi)$ is the representation of $G(\mathbb{R})$ with infinitesimal character z_{ψ_∞}. Otherwise, the two elements of $\Pi_\infty(\psi)$, again two finite dimensional irreducible representations, have the property that their infinitesimal characters are exchanged by the outer automorphism of $G(\mathbb{R})$, and both in the $\mathrm{Out}(\widehat{G})$-orbit of z_{ψ_∞}. Observe that there is still the possibility that the $\mathrm{Out}(\widehat{G})$-orbit of z_{ψ_∞} is a singleton : in this case $\Pi_\infty(\pi)$ consists of two isomorphic representations. However, observe also that by definition all the members of $\Pi(\psi) \subset \Pi(G)$ have the same Archimedean component in this case.

Recall we have fixed an isomorphism between $G(\mathbb{R})$ and G_t for $t = \{\pm 1\} \in \mathrm{Z}(G)$ as in §A.1. Assume first that we actually chosen $t = 1$. It follows that the one or two elements in $\Pi_\infty(\psi)$ have the same character $\rho^\vee$ by Cor. A.12. Here we have $\rho^\vee = (r-1)e_1 + (r-2)e_2 + \cdots + e_{r-1}$, thus $\rho^\vee \in \mathrm{X}^*(\widehat{T})$ and

$$\rho^\vee \equiv e_{r-1} + e_{r-3} + e_{r-5} + \cdots \bmod 2\mathrm{X}^*(\widehat{T}).$$

Observe again that $\rho^\vee(-1) = 1$ as $r \equiv 0 \bmod 4$.

Consider the generators s_i of C_ψ introduced in §3.27. Fix some $i \in \{1, \ldots, k\}$ and write $n_i = r_i d_i$. If d_i is even, then r_i is even as well as $s(\pi_i) = -1$, and we have

$$\rho^\vee(s_i) = (-1)^{\frac{n_i}{4}}.$$

If d_i is odd, in which case r_i is even as $n_i = d_i r_i$ is even by assumption, the sign $\rho^\vee(s_i)$ depends on the Hodge weights of π_i. Precisely, denote by

$$w_1 > \cdots > w_r$$

the nonnegative even integers w_j such that the eigenvalues of $\mathrm{St}(z_{\psi_\infty})$ in $\mathrm{SL}_{2r}(\mathbb{C})$ are the $\pm\frac{w_j}{2}$ (see formula (3.8)). There is a unique subset $J \subset \{1, \cdots, r\}$ such that the Hodge weights of π_i are the w_j for $j \in J$. Denote by J' the subset of $j \in J$ such that $j \equiv r-1 \bmod 2$. It is then clear that

$$\rho^\vee(s_i) = (-1)^{|J'|}.$$

For coherence reasons, we shall check now that the multiplicity formula does not change if we choose to identify $G(\mathbb{R})$ with G_{-1} or if we modify the fixed isomorphism by the outer automorphism of $G(\mathbb{R})$. This second fact is actually trivial by what we already said, so assume that we identified $G(\mathbb{R})$ with G_{-1}. The effect of this choice is that the one or two elements of $\Pi_\infty(\psi)$ become parameterized by the character

$$\rho^\vee + \chi,$$

where χ is the generator of the group $\mathcal{N}(T)$, by Lemma A.10. As $-1 = e^{i\pi\chi}$ we have

$$\chi \equiv \sum_{i=1}^{r} e_i \bmod 2\mathrm{X}_*(\widehat{T})$$

and we claim that this character is trivial on C_ψ. Indeed, it follows from Lemma 3.23 that if n_i is even then $n_i \equiv 0 \bmod 4$, so that $\chi(s_i) = (-1)^{n_i/2} = 1$.

3.30.3. *The Chevalley groups* Sp_{2g}*,* $\mathrm{SO}_{2,2}$ *and* $\mathrm{SO}_{3,2}$*.*

The case of the symplectic groups Sp_{2g} will be treated in details in Chapter 9, especially in §9.2. We shall only consider there the multiplicity formula for a π such that π_∞ is a holomorphic discrete series.

The cases $G = \mathrm{SO}_{2,2}$ and $\mathrm{SO}_{3,2}$ will be used in Chapter 4. For $\mathrm{SO}_{2,2}$ we shall not use that Arthur's packets are the same as the ones of Adams-Johnson. For $G = \mathrm{SO}_{3,2}$ we shall need it only in §4.2, i.e. to compute $\mathrm{S}(w,v)$, for the $\psi \in \Psi_{\mathrm{alg}}(G)$ of the form $\pi \oplus [2]$. In this case this is probably not too difficult to check but due to the already substantial length of this paper we decided not to include this twisted character computation here. We hope to do so in the future.

4. Determination of $\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n)$ for $n \leq 5$

In this chapter we justify the formulas for $\mathrm{S}(w)$ and $\mathrm{S}(w,v)$ given in the introduction and prove Theorem 1.15 there. We recall that various sets

$$\Pi_{\mathrm{alg}}^{*}(\mathrm{PGL}_n) \subset \Pi_{\mathrm{cusp}}^{*}(\mathrm{PGL}_n) \subset \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$$

have been introduced in Definitions 3.10 and 3.16.

4.1. **Determination of $\Pi_{\mathrm{cusp}}^{\perp}(\mathrm{PGL}_2)$.** A representation $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)$ is necessarily self-dual as $g \mapsto {}^t g^{-1}$ is an inner automorphism of PGL_2. It is even symplectic by Theorem 3.9, so that

$$\Pi_{\mathrm{cusp}}(\mathrm{PGL}_2) = \Pi_{\mathrm{cusp}}^{\perp}(\mathrm{PGL}_2) = \Pi_{\mathrm{cusp}}^{\mathrm{s}}(\mathrm{PGL}_2).$$

If $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)$, the infinitesimal character of π_∞ has the form $\mathrm{diag}(\frac{w}{2}, -\frac{w}{2}) \in \mathfrak{sl}_2(\mathbb{C})$ for some integer $w \geq 1$ if and only if π_∞ is a discrete series representation, in which case w is odd and determines π_∞ (see e.g. §3.11 and [Kna94], or [Bum96, §2]).

Let $w \geq 1$ be an odd integer and let $\mathcal{F}_w$ be the set of

$$F = \sum_{m \geq 1} a_m q^m \in \mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z}))$$

which are eigenforms for all the Hecke operators and normalized so that $a_1 = 1$: see [Ser70]. As is well-known, and explained by Serre, $\mathcal{F}_w$ is a basis of the complex vector space $\mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z}))$. Moreover, each $F \in \mathcal{F}_w$ generates a $\pi_F \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)$, and the map $F \mapsto \pi_F$ is a bijection between $\mathcal{F}_w$ and the set of π in $\Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ such that π_∞ has Hodge weight w (see [Bum96, §3.2]). In particular

$$\mathrm{S}(w) = \dim(\mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z})))$$

as recalled in the introduction. We shall always identify an $F \in \mathcal{F}_w$ with π_F in the bijection above, and even write $F \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. For $w \in \{11, 13, 15, 17, 19, 21\}$ we shall denote by

$$\Delta_w \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$$

the unique element with Hodge weight w, as a reminiscence of the notation Δ for Jacobi's discriminant function, i.e. $\Delta = \Delta_{11}$.

4.2. **Determination of $\Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_4)$.**

Fix $w > v$ odd positive integers. Let $\mathrm{S}_{w,v}(\mathrm{Sp}_4(\mathbb{Z}))$ be the space of Siegel cusp forms of genus 2 recalled in §1.10 of the introduction. Denote also by

$$\Pi_{w,v}(\mathrm{PGSp}_4) \subset \Pi_{\mathrm{cusp}}(\mathrm{PGSp}_4)$$

the subset of $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGSp}_4)$ such that π_∞ is the holomorphic discrete series whose infinitesimal character has the eigenvalues $\pm\frac{w}{2}, \pm\frac{v}{2}$, viewed as a semisimple conjugacy class in $\mathfrak{sl}_4(\mathbb{C})$. It is well-known that to each Hecke-eigenform F in $\mathrm{S}_{w,v}(\mathrm{Sp}_4(\mathbb{Z}))$ one may associate a unique $\pi_F \in \Pi_{w,v}(\mathrm{PGSp}_4)$, and that the image of the map $F \mapsto \pi_F$ is $\Pi_{w,v}(\mathrm{PGSp}_4)$ (see e.g. [AS01]).

The semisimple $\mathbb{Z}$-group PGSp_4 is isomorphic to $\mathrm{SO}(3,2)$ hence we may view it as a classical semisimple group over $\mathbb{Z}$. It follows from Arthur's multiplicity formula (§3.29,§3.30.3) that the multiplicity of any such π_F as above is 1, so that the Hecke-eigenspace containing a given Hecke-eigenform F is actually not bigger than $\mathbb{C}F$. It follows that if we denote by $\mathcal{F}_{w,v}$ the set of these (one dimensional) Hecke-eigenspaces in $\mathrm{S}_{w,v}(\mathrm{Sp}_4(\mathbb{Z}))$, then

$$|\mathcal{F}_{w,v}| = \dim \mathrm{S}_{w,v}(\mathrm{Sp}_4(\mathbb{Z})) = |\Pi_{w,v}(\mathrm{PGSp}_4)|.$$

The following formula was claimed in the introduction.

Proposition 4.3.** *For $w > v > 0$ odd,* $\mathrm{S}(w,v) = \mathrm{S}_{w,v}(\mathrm{Sp}_4(\mathbb{Z})) - \delta_{v=1}\delta_{w\equiv 1 \bmod 4}\mathrm{S}(w)$.

Before starting the proof, recall that if φ is a discrete series Langlands parameter for $\mathrm{PGSp}_4(\mathbb{R})$, its L-packet $\Pi(\varphi)$ has two elements $\{\pi_{\mathrm{hol}}, \pi_{\mathrm{gen}}\}$ where π_{gen} is generic and π_{hol} is holomorphic. One has moreover

$$\mathrm{C}_\varphi = \mathrm{S}_\varphi \simeq (\mathbb{Z}/2\mathbb{Z})^2$$

in the notation of §A.5, and the two Shelstad characters of C_φ associated to the elements of $\Pi(\varphi)$ are the ones which are trivial on the center Z of $\mathrm{Sp}_4(\mathbb{C})$. Of course $\tau(\pi_{\mathrm{gen}}) = 1$ and so $\tau(\pi_{\mathrm{hol}})$ is the unique non-trivial character of C_φ which is trivial on the center $Z = \{\pm 1\}$ of $\mathrm{Sp}_4(\mathbb{C})$.

Fix a $\psi \in \Psi_{\mathrm{alg}}(\mathrm{PGSp}_4)$ whose infinitesimal character has the eigenvalues $\pm\frac{w}{2}, \pm\frac{v}{2}$. One has to determine if $\Pi_\infty(\psi)$ contains the holomorphic discrete series and, if it is so, to determine the multiplicity of the unique $\pi \in \Pi(\psi)$ such that π_∞ is this holomorphic discrete series. Such a π is necessarily cuspidal as π_∞ is tempered, by a result of Wallach [Wal84, Thm. 4.3] (as pointed out to us by Wallach, this discrete series case is actually significantly simpler than the general case treated there). We proceed by a case by case argument depending on the global Arthur parameter ψ :

Case (i) : (stable tempered case) $\psi = \pi_1$ where $\pi_1 \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$. In this case ψ_∞ is a discrete series Langlands parameter. It follows from Arthur's multiplicity formula that $m(\pi) = 1$, as $\mathrm{C}_\psi = Z$. The number of such π is the number $\mathrm{S}(w,v)$ that we want to compute.

Case (ii) : $\psi = [4]$. The unique $\pi \in \Pi_{\mathrm{disc}}(\mathrm{PGSp}_4)$ with $\psi(\pi) = \psi$ is the trivial representation, for which π_∞ is not a discrete series.

Case (iii) : $\psi = \pi_1 \oplus \pi_2$ where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ and π_1, π_2 have different Hodge weights. In this case one has

$$\mathrm{C}_\psi = \mathrm{C}_{\psi_\infty} = (\mathbb{Z}/2\mathbb{Z})^2.$$

Moreover, $r_\psi(\mathrm{SL}_2(\mathbb{C})) = 1$ so ε_ψ is trivial and ψ_∞ is a discrete series parameter for $\mathrm{PGSp}_4(\mathbb{R})$. If $\pi \in \Pi(\psi)$ is the unique element such that π_∞ is holomorphic, Arthur's multiplicity formula thus shows that $m(\pi) = 0$ as ε_ψ is trivial but $\tau(\pi_\infty)$ is not.

Case (iv) : $\psi = \pi_1 \oplus [2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ with Hodge weight $w \neq 1$ (which is actually automatic as $\mathrm{S}(1) = 0$). Again one has

$$\mathrm{C}_\psi = \mathrm{C}_{\psi_\infty} = (\mathbb{Z}/2\mathbb{Z})^2.$$

This time $r_\psi(\mathrm{SL}_2(\mathbb{C})) \neq 1$, and if $s = s_1$ is the generator of C_ψ/Z, then

$$\varepsilon_\psi(s) = \varepsilon(\pi_1 \times 1) = \varepsilon(\pi_1) = (-1)^{(w+1)/2}.$$

The Adams-Johnson parameter ψ_∞ has an associated complex Levi subgroup L isomorphic to $\mathrm{SO}_2(\mathbb{C}) \times \mathrm{SO}_3(\mathbb{C})$ (see §A.2 and §A.5). It follows that the set $\Pi_\infty(\psi)$, which has two elements, contains the holomorphic discrete series (associated to the order 2 element in the center of L). For more details, see Chapter 9 where the general case $\mathrm{Sp}_{2g}(\mathbb{R})$ will be studied. The character of this holomorphic discrete series relative to this ψ_∞ is again the non-trivial character of C_ψ trivial on Z by the discrete series case recalled above and Lemma A.9. It follows that if $\pi \in \Pi(\psi)$ is the unique element such that $\pi_\infty = \pi_{\mathrm{hol}}$, then by Arthur's multiplicity formula we have $m(\pi) = 0$ if $w \equiv 3 \bmod 4$, and $m(\pi) = 1$ if $w \equiv 1 \bmod 4$.

This concludes the proof of the proposition. □

Remark 4.4. By the formula for $\mathrm{S}(w)$, the first w for which a π as in case (iv) exists is for $w = 17$, for which $\psi(\pi) = \Delta_{17} \oplus [2]$. The representations π occurring in case (iv) have a long history, their existence had been conjectured by Saito and Kurokawa in 1977, and proved independently of this theory by Maass, Andrianov and Zagier. We refer to Arthur's paper [Art04] for a discussion about this (and most of the discussion of this paragraph).

When $\mathrm{S}(w, v) = 1$ we shall denote by $\Delta_{w,v}$ the unique element of $\pi \in \Pi_{w,v}(\mathrm{PGSp}_4)$ such that $\psi(\pi) \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_4)$. As recalled in §1.10, an explicit formula for dim $\mathrm{S}_{w,v}(\mathrm{Sp}_4(\mathbb{Z}))$ has been given by T. Tsushima (and by Igusa when $v = 1$). See Table 6 for a sample of values. For $w < 25$, one observes that $\mathrm{S}(w, v)$ is either 0 or 1. For those $w < 25$, there are exactly 7 forms $\Delta_{w,v}$, for the following values (w, v) :

$$(19, 7), (21, 5), (21, 9), (21, 13), (23, 7), (23, 9), (23, 13).$$

Contrary to the PGL_2 case where one has simple formulas for the $c_p(\Delta_w)$ thanks to the q-expansion of Eisenstein series or the product formula for Δ_{11}, much less seems to be known at the moment for the $c_p(\pi)$ where $\pi \in \Pi_{w,v}(\mathrm{PGSp}_4)$, even (say) for $\pi = \Delta_{w,v}$ and (w, v) in the list above. We refer to the recent work [RRST] for a survey on this important problem, as well as some implementation on SAGE.

To cite a few results especially relevant to our purposes here, let us mention first the work of Skoruppa [Sko92] computing $c_p(\pi)$ for the first 22 primes p when π is any of the 18 elements in the $\Pi_{w,1}(\mathrm{PGSp}_4)$ for $w \leq 61$. Moreover, works of Faber and Van der Geer (see [Geer08, §24, §25]) compute the trace of $c_p(\Delta_{v,w})$ in the standard 4-dimensional representations when $p \leq 37$, and even $c_p(\Delta_{w,v})$ itself when $p \leq 7$, whenever (w, v) is in the list above. In the work [CL14] of the first author and Lannes, the first 4 of these forms, namely $\Delta_{19,7}$, $\Delta_{21,5}$, $\Delta_{21,9}$ and $\Delta_{21,13}$, appeared in the study of the Kneser p-neighbors of the Niemeier lattices. Properties of the Leech lattice also allowed those authors to compute $\mathrm{Trace}(\mathrm{c}_p(\Delta_{w,v}))$ for those 4 pairs (w, v) up to $p \leq 79$.

4.5. **An elementary lifting result for isogenies.** Consider $\iota : G \to G'$ a central isogeny between semisimple Chevalley groups over $\mathbb{Z}$. The morphism ι is thus a finite flat group scheme homomorphism, $Z = \operatorname{Ker} \iota \subset \mathrm{Z}(G)$ is a central multiplicative $\mathbb{Z}$-group scheme and $G' = G/Z$. The following proposition is easy to observe for all the isogenies we shall consider later, but it is perhaps more satisfactory to give a general proof.

Proposition 4.6. *ι induces a homeomorphism $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}) \overset{\sim}{\to} G'(\mathbb{Q})\backslash G'(\mathbb{A})/G'(\widehat{\mathbb{Z}})$.*

Proof — By Prop. 3.5, it is enough to check that the map

$$G(\mathbb{Z})\backslash G(\mathbb{R}) \to G'(\mathbb{Z})\backslash G'(\mathbb{R})$$

induced by ι is a homeomorphism. As this map is continuous and open it is enough to show it is bijective. As the source and target are connected by Prop. 3.5, it is surjective. Moreover, it is injective if and only if the inverse image of $G'(\mathbb{Z})$ in $G(\mathbb{R})$ coincides with $G(\mathbb{Z})$, what we check now. The fppf exact sequence defined by ι leads to the following commutative diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Z}(\mathbb{R}) & \longrightarrow & G(\mathbb{R}) & \longrightarrow & G'(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, Z) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathrm{Z}(\mathbb{Z}) & \longrightarrow & G(\mathbb{Z}) & \longrightarrow & G'(\mathbb{Z}) & \longrightarrow & H^1(\mathbb{Z}, Z) \end{array}$$

The left vertical map is an isomorphism by Prop. 3.5. The right vertical one is an isomorphism as well, as so are the natural maps

$$\mathbb{Z}^\times/(\mathbb{Z}^\times)^n = H^1(\mathbb{Z}, \mu_n) \to H^1(\mathbb{R}, \mu_n) = \mathbb{R}^\times/(\mathbb{R}^\times)^n$$

for each integer $n \geq 1$. A simple diagram chasing concludes the proof. $\square$

Denote by $\iota^\vee : \widehat{G'} \to \widehat{G}$ the isogeny dual to ι. We now define a map[12]

$$\mathcal{R}_\iota : \Pi(G') \longrightarrow \mathcal{P}(\Pi(G))$$

associated to ι as follows. If $\pi' = \pi'_\infty \otimes \pi'_f \in \Pi(G')$ we define $\mathcal{R}_\iota(\pi')$ as the set of representations $\pi \in \Pi(G)$ such that :

(i) For each prime p the Satake parameter of π_p is $\iota^\vee(c_p(\pi'))$,

(ii) π_∞ is a constituent of the restriction to $G(\mathbb{R}) \to G'(\mathbb{R})$ of π'_∞.

Let $\pi \in \mathcal{R}_\iota(\pi')$. Observe that π_p is uniquely determined by (i). Moreover the restriction of π'_∞ to $G(\mathbb{R})$ is a direct sum of finitely many irreducible representations of same infinitesimal character as π'_∞. In particular $\mathcal{R}_\iota(\pi')$ is a finite nonempty set. We denote by $[\pi_\infty : \pi'_\infty]$ the multiplicity of π_∞ in $(\pi'_\infty)_{|G(\mathbb{R})}$. If $\pi \in \Pi(H)$ we also write $m_H(\pi)$ for $m(\pi)$ to emphasize the $\mathbb{Z}$-group H (see §3.3).

[12] We denote by $\mathcal{P}(X)$ the set of all subsets of X.

Proposition 4.7. *If $\pi \in \Pi(G)$ then*

$$m_G(\pi) = \sum_{\{\pi' \in \Pi(G') \mid \pi \in \mathcal{R}_\iota(\pi')\}} m_{G'}(\pi')[\pi_\infty, \pi'_\infty].$$

In particular, the two following properties hold :

(a) *For any $\pi \in \Pi_{\rm disc}(G)$ there exists $\pi' \in \Pi_{\rm disc}(G')$ such that $\pi \in \mathcal{R}_\iota(\pi')$.*

(b) *For any $\pi' \in \Pi_{\rm disc}(G')$ then $\mathcal{R}_\iota(\pi') \subset \Pi_{\rm disc}(G)$.*

Before giving the proof we need to recall certain properties of the Satake isomorphism. Following Satake, consider the $\mathbb{C}$-linear map

$$\iota^* : \mathcal{H}(G) \to \mathcal{H}(G')$$

sending the characteristic function of $G(\widehat{\mathbb{Z}})gG(\widehat{\mathbb{Z}})$ to the one of $G'(\widehat{\mathbb{Z}})\iota(g)G'(\widehat{\mathbb{Z}})$. It follows from [Sat63, Prop. 7.1], that ι^* is a ring homomorphism. Indeed, it is enough to check the assumptions there. Let ι_p be the morphism $G(\mathbb{Q}_p) \to G'(\mathbb{Q}_p)$ induced by ι. Then $\iota_p(G(\mathbb{Q}_p))$ is a normal open subgroup of $G'(\mathbb{Q}_p)$. Moreover $\iota_p^{-1}(G'(\mathbb{Z}_p)) = G(\mathbb{Z}_p)$ as this latter group is a maximal compact subgroup of $G(\mathbb{Q}_p)$ by [Tit79] and ι_p is proper. Last but not least, the Cartan decomposition shows that ι_p induces an injection $G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \to G'(\mathbb{Z}_p)\backslash G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p)$ (see e.g. [Gro98]).

If V is a representation of $G'(\mathbb{A}_f)$, it defines by restriction by ι a representation V_ι of $G(\mathbb{A}_f)$ as well, and $V^{G'(\widehat{\mathbb{Z}})} \subset V_\iota^{G(\widehat{\mathbb{Z}})}$. The following lemma is presumably well-known.

Lemma 4.8. *Let V be a complex representation of $G'(\mathbb{A}_f)$ and let $T \in \mathcal{H}(G)$. The diagram*

$$\begin{array}{ccc} V^{G'(\mathbb{A}_f)} & \hookrightarrow & V_\iota^{G(\mathbb{A}_f)} \\ \downarrow{\scriptstyle \iota^*(T)} & & \downarrow{\scriptstyle T} \\ V^{G'(\mathbb{A}_f)} & \hookrightarrow & V_\iota^{G(\mathbb{A}_f)} \end{array}$$

is commutative.

Proof — We have to show that if $\psi : G'(\mathbb{A}_f) \to \mathbb{C}$ is a locally constant function which is right $G'(\widehat{\mathbb{Z}})$-invariant and with support in $\iota(G(\mathbb{A}_f))G'(\widehat{\mathbb{Z}})$, then

$$\int_{G'(\mathbb{A}_f)} \psi(g)dg = \int_{G(\mathbb{A}_f)} \psi(\iota(h))dh.$$

Here the Haar measures dg and dh on $G'(\mathbb{A}_f)$ and $G(\mathbb{A}_f)$ are normalized so that $G'(\widehat{\mathbb{Z}})$ and $G(\widehat{\mathbb{Z}})$ have respective measure 1. But this follows from the already mentioned equality $\iota^{-1}(G'(\widehat{\mathbb{Z}})) = G(\widehat{\mathbb{Z}})$, and form the well-known fact that $\iota(G(\mathbb{A}_f))$ is a normal subgroup of $G'(\mathbb{A}_f)$. ☐

A finer property of ι^* is that it commutes with the Satake isomorphism. Recall that if $\mathcal{H}_p(G)$ denotes the Hecke algebra of $(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$, the Satake isomorphism is a canonical isomorphism

$$S_{G/\mathbb{Z}_p} : \mathcal{H}_p(G) \overset{\sim}{\to} R(\widehat{G})$$

where $R(\widehat{G})$ denotes the $\mathbb{C}$-algebra of polynomial class functions on $\widehat{G}$. Satake shows *loc. cit.* that the diagram

$$\begin{array}{ccc} \mathcal{H}_p(G) & \xrightarrow{S_{G/\mathbb{Z}_p}} & R(\widehat{G}) \\ \downarrow \iota^* & & \downarrow \iota^\vee \\ \mathcal{H}_p(G') & \xrightarrow[S_{G'/\mathbb{Z}_p}]{} & R(\widehat{G'}) \end{array}$$

is commutative, where $\iota^\vee : R(\widehat{G}) \to R(\widehat{G'})$ also denotes the restriction by $\iota^\vee$.

Proposition 4.6 ensures that the map $f(g) \mapsto f(\iota(g))$ defines a $\mathbb{C}$-linear isomorphism

$$\mathrm{Res}_\iota : \mathcal{L}(G') \overset{\sim}{\to} \mathcal{L}(G). \tag{4.1}$$

The homomorphism ι^* defines a natural $\mathcal{H}(G)$-module structure on $\mathcal{L}(G')$ and Lemma 4.8 ensures that Res_ι is $\mathcal{H}(G)$-equivariant for this structure on the left-hand side and the natural structure on the right-hand side. The isomorphism Res_ι is obviously $G(\mathbb{R})$-equivariant as well. As $\iota(G(\mathbb{R}))$ is open of finite index in $G'(\mathbb{R})$ we may replace the two $\mathcal{L}$'s in (4.1) by $\mathcal{L}_{\mathrm{disc}}$. We have thus proved the following proposition.

Proposition 4.9. Res_ι *induces an isomorphism* $\mathcal{L}_{\mathrm{disc}}(G') \overset{\sim}{\to} \mathcal{L}_{\mathrm{disc}}(G)$ *which commutes with the natural actions of* $G(\mathbb{R})$ *and* $\mathcal{H}(G)$ *on both sides.*

This proposition implies Proposition 4.7 thanks to formula (3.1).

Corollary 4.10. *Assume that* $m_G(\pi) = 1$ *for each* $\pi \in \Pi_{\mathrm{disc}}(G)$. *Then* $m_{G'}(\pi') = 1$ *for each* $\pi' \in \Pi_{\mathrm{disc}}(G')$ *as well. Moreover, the* $\mathcal{R}_\iota(\pi')$ *with* $\pi' \in \Pi_{\mathrm{disc}}(G')$ *form a partition of* $\Pi_{\mathrm{disc}}(G)$.

This corollary would apply for instance to the isogeny $\mathrm{Sp}_{2g} \to \mathrm{PGSp}_{2g}$ for any $g \geq 1$ by Arthur's multiplicity formula if we knew that the Archimedean Arthur packets are sets rather than multisets (see the discussion following Conjecture 3.30). It applies for $g = 1$ by the multiplicity one theorem of Labesse and Langlands [LL79].

Corollary 4.11. *If* $G = \mathrm{SO}_{2,2}$ *then* $m_G(\pi) = 1$ *for any* $\pi \in \Pi_{\mathrm{disc}}(G)$.

Proof — We just recalled that $m_H(\pi) = 1$ for any $\pi \in \Pi_{\mathrm{disc}}(H)$ when $H = \mathrm{SL}_2$, hence for $H = \mathrm{SL}_2 \times \mathrm{SL}_2$ as well. To conclude we apply Cor. 4.10 to the central isogeny

$$(\mathrm{SO}_{2,2})_{\mathrm{sc}} \simeq \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_{2,2}.$$

□

4.12. **Symmetric square functoriality and** $\Pi_{\rm cusp}^{\perp}(\mathrm{PGL}_3)$**.** It follows from Theorem 3.9 that

$$\Pi_{\rm cusp}^{\rm o}(\mathrm{PGL}_3) = \Pi_{\rm cusp}^{\perp}(\mathrm{PGL}_3).$$

Recall the $\mathbb{C}$-morphism $\mathrm{Sym}^2 : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$.

Proposition* 4.13. *There is a unique bijection* $\mathrm{Sym}^2 : \Pi_{\rm cusp}(\mathrm{PGL}_2) \to \Pi_{\rm cusp}^{\perp}(\mathrm{PGL}_3)$ *such that for each* $\pi \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$ *we have* $c(\mathrm{Sym}^2\pi) = \mathrm{Sym}^2 c(\pi)$*. It induces a bijection* $\Pi_{\rm alg}(\mathrm{PGL}_2) \overset{\sim}{\to} \Pi_{\rm alg}^{\rm o}(\mathrm{PGL}_3)$*.*

If $\pi \in \Pi_{\rm alg}(\mathrm{PGL}_2)$ has Hodge weight w, it follows that $\mathrm{Sym}^2(\pi)$ has Hodge weight $2w$. The proposition implies thus part (i) of Thm. 1.15. Observe in particular that the Hodge weight of any $\pi \in \Pi_{\rm alg}^{\rm o}(\mathrm{PGL}_3)$ is $\equiv 2 \bmod 4$, as asserted in general by Prop. 1.14.

Proof — The existence of a unique map $\mathrm{Sym}^2 : \Pi_{\rm cusp}(\mathrm{PGL}_2) \to \Pi_{\rm cusp}^{\perp}(\mathrm{PGL}_3)$ satisfying $c(\mathrm{Sym}^2\pi) = \mathrm{Sym}^2 c(\pi)$ is due to Gelbart and Jacquet [GJ78, Thm. 9.3] (the assumption in their theorem is satisfied as π as conductor 1). It is however instructive to deduce it as well from Arthur's results, as follows. Consider the isogeny $\iota : \mathrm{SL}_2 \to \mathrm{PGL}_2$. Let $\pi \in \Pi_{\rm disc}(\mathrm{PGL}_2)$ and let $\rho \in \mathrm{Res}_\iota(\pi)$. By Proposition 4.7 (b), we have $\rho \in \Pi_{\rm disc}(\mathrm{SL}_2)$. By definition, $c(\rho)$ is the image of $c(\pi)$ under the isogeny $\iota^\vee : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{SO}_3(\mathbb{C})$; observe that the composition of $\iota^\vee$ with the standard representation of $\mathrm{SO}_3(\mathbb{C})$ is nothing else than the Sym^2 representation of $\mathrm{SL}_2(\mathbb{C})$. In particular, $\psi(\rho)$ does not depend on the choice of ρ in $\mathrm{Res}_\iota(\pi)$, and it thus makes sense to consider

$$\widetilde{\psi}(\pi) = \psi(\rho) \in \Psi_{\rm glob}(\mathrm{SL}_2).$$

We have $\widetilde{\psi}(\pi) = [3]$ if and only if ρ is the trivial representation, which can happen only if π is trivial as well (see e.g. the decomposition (3.2)). Otherwise, the only remaining possibilities are that $\pi \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$ and $\widetilde{\psi}(\pi) \in \Pi_{\rm cusp}^{\rm o}(\mathrm{PGL}_3)$. If we set $\mathrm{Sym}^2\pi = \widetilde{\psi}(\pi)$, then $c(\mathrm{Sym}^2\pi) = \mathrm{Sym}^2 c(\pi)$ by construction: this is another definition of the Gelbart-Jacquet map.

The Sym^2 map is surjective. Indeed, if $\pi' \in \Pi_{\rm cusp}^{\perp}(\mathrm{PGL}_3)$ there exists $\rho \in \Pi_{\rm disc}(\mathrm{SL}_2)$ such that $\pi' = \psi(\rho)$ by Arthur's Theorem 3.9. But there exists $\pi \in \Pi_{\rm disc}(\mathrm{PGL}_2)$ such that $\rho \in \mathrm{Res}_\iota(\pi)$ by Proposition 4.7 (a). One sees as above that π is non-trivial, hence cuspidal (Selberg). It follows that $\pi' = \mathrm{Sym}^2\pi$.

It only remains to check that Sym^2 is injective. Let $\pi, \pi' \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$ be such that $\mathrm{Sym}^2\pi \simeq \mathrm{Sym}^2\pi'$. For each prime p, the one or two elements[13] in $\mathrm{Res}_\iota(\pi)$, and the one or two elements in $\mathrm{Res}_\iota(\pi')$, all have the same Satake parameters at p. By the multiplicity formula of Labesse-Langlands [LL79], this implies that all these representations are in a same *global* L*-packet*; this means here that their Archimedean components are all conjugate under $\mathrm{PGL}_2(\mathbb{R})$. It follows that $\mathrm{Res}_\iota(\pi) = \mathrm{Res}_\iota(\pi')$. But by Labesse-Langlands [LL79] again, each element in $\Pi_{\rm disc}(\mathrm{SL}_2)$ has multiplicity one. It follows that $\pi \simeq \pi'$ by Corollary 4.10. □

[13]Recall that the image of $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ has index 2.

4.14. **Tensor product functoriality and $\Pi^{\rm o}_{\rm cusp}(\mathrm{PGL}_4)$.** We consider the natural map

$$\mathcal{X}(\mathrm{SL}_2(\mathbb{C})) \times \mathcal{X}(\mathrm{SL}_2(\mathbb{C})) \to \mathcal{X}(\mathrm{SL}_4(\mathbb{C}))$$

given by the tensor product $(x, y) \mapsto x \otimes y$ of conjugacy classes. If X is a set, we denote by $\Sigma_2 X$ the set of all subsets of X with two elements.

Proposition* 4.15. *There is a unique bijection* $\Sigma_2\,\Pi_{\rm cusp}(\mathrm{PGL}_2) \overset{\sim}{\to} \Pi^{\rm o}_{\rm cusp}(\mathrm{PGL}_4)$*, that we shall denote* $\{\pi, \pi'\} \mapsto \pi \otimes \pi'$*, such that for each* $\pi \neq \pi' \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$*,*

$$c(\pi \otimes \pi') = c(\pi) \otimes c(\pi').$$

It induces a bijection $\Sigma_2\,\Pi_{\rm alg}(\mathrm{PGL}_2) \overset{\sim}{\to} \Pi^{\rm o}_{\rm alg}(\mathrm{PGL}_4)$*.*

Consider the central isogeny $\iota : \mathrm{SO}_{2,2} \to \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Let $(\pi, \pi') \in \Pi_{\rm disc}(\mathrm{PGL}_2)^2$ and let $\rho \in \mathrm{Res}_\iota((\pi, \pi'))$. By Proposition 4.7 (b), we have $\rho \in \Pi_{\rm disc}(\mathrm{SO}_{2,2})$. By definition, $c(\rho)$ is the image of $c(\pi) \times c(\pi')$ under the isogeny $\iota^\vee : \mathrm{SL}_2(\mathbb{C})^2 \to \mathrm{SO}_4(\mathbb{C})$. If we compose this latter isogeny with the standard representation of $\mathrm{SO}_4(\mathbb{C})$, we obtain nothing else than the tensor product representation $\mathrm{SL}_2(\mathbb{C})^2 \to \mathrm{SL}_4(\mathbb{C})$. In particular, $\psi(\rho)$ does not depend on the choice of ρ in $\mathrm{Res}_\iota(\pi)$, and it thus makes sense to define

$$\psi(\pi, \pi') = \psi(\rho) \in \Psi_{\rm glob}(\mathrm{SO}_{2,2}).$$

It is clear that $\psi(\pi, \pi') = \psi(\pi', \pi)$.

Proposition* 4.16. *Let* $\pi, \pi' \in \Pi_{\rm disc}(\mathrm{PGL}_2)$*.*

(i) *If* π, π' *are both the trivial representation then* $\psi(\pi, \pi') = [3] \oplus [1]$*,*
(ii) *If* π' *is the trivial representation and* π *is cuspidal then* $\psi(\pi, \pi') = \pi[2]$*,*
(iii) *If* $\pi = \pi'$ *is cuspidal, then* $\psi(\pi, \pi') = \mathrm{Sym}^2\pi \oplus [1]$*,*
(iv) *If* π, π' *are distinct and cuspidal, then* $\psi(\pi, \pi') \in \Pi^{\rm o}_{\rm cusp}(\mathrm{PGL}_4)$*. Moreover,* $\psi(\pi, \pi')$ *determines the pair* $\{\pi, \pi'\}$*.*

Note that assertion (iii) makes sense by Proposition 4.13.

Proof — Assertions (i), (ii) and (iii) follow from an immediate inspection of Satake parameters and from the uniqueness of global Arthur parameters in Theorem 3.19.

Fix distinct $\pi, \pi' \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$. The strong multiplicity one theorem for PGL_2 shows that the global Arthur parameter $\psi(\pi, \pi')$ cannot contain the symbol [1]. Moreover, Jacquet-Shalika's bound shows that $\psi(\pi, \pi')$ cannot have the form $\pi''[2]$ for $\pi'' \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$. The only remaining possibility is that $\psi(\pi, \pi') \in \Pi^{\rm o}_{\rm cusp}(\mathrm{PGL}_4)$.

Fix now $\omega \in \Pi^{\perp}_{\rm cusp}(\mathrm{PGL}_4)$ of the form $\psi(\pi, \pi')$ for some distinct $\pi, \pi' \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$. We want to show that ω determines the pair $\{\pi, \pi'\}$. Consider for this the subset $\mathcal{X}(\mathrm{SL}_4(\mathbb{C}))^\perp$ of $\mathcal{X}(\mathrm{SL}_4(\mathbb{C}))$ of all the conjugacy classes which are equal to their inverse, and consider the map

$$t : \mathcal{X}(\mathrm{SL}_4(\mathbb{C}))^\perp \longrightarrow \mathcal{X}(\mathrm{SL}_3(\mathbb{C}))^2/\mathfrak{S}_2$$

defined as follows. Start with the standard representation $\mathrm{O}_4(\mathbb{C}) \to \mathrm{SL}_4(\mathbb{C})$. An element $x \in \mathcal{X}(\mathrm{SL}_4(\mathbb{C}))^\perp$ is the image of a unique $\mathrm{O}_4(\mathbb{C})$-conjugacy class y in $\mathrm{SO}_4(\mathbb{C})$.

The image of y via the isogeny $\mathrm{SO}_4(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})^2$ is a well defined element[14] $z \in \mathcal{X}(\mathrm{PGL}_2(\mathbb{C}))^2/\mathfrak{S}_2$. Set $t(x) = \mathrm{ad}(z)$ where $\mathrm{ad} : \mathrm{PGL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$ is the adjoint representation. Observe that for each prime p, we have

$$t(c_p(\omega)) \sim (\mathrm{ad} \circ \mu(c_p(\pi)), \mathrm{ad} \circ \mu(c_p(\pi')))$$

where μ is the isogeny $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. But $\mathrm{ad} \circ \mu(c_p(\pi)) = c_p(\mathrm{Sym}^2\pi)$, and similarly for π'. It follows from Jacquet-Shalika's structure theorem for isobaric representation [JS81] that the pair $\{\mathrm{Sym}^2\pi, \mathrm{Sym}^2\pi'\}$ is uniquely determined by ω. But by Proposition 4.13 this in turn determines $\{\pi, \pi'\}$. □

Let us finally prove the first assertion of Proposition 4.15. If $\pi, \pi' \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)$ are distinct we set

$$\pi \otimes \pi' = \psi(\pi, \pi'),$$

so that $c(\pi \otimes \pi') = c(\pi) \otimes c(\pi')$ by definition. It follows from Proposition 4.16 (iv) that $\{\pi, \pi'\} \mapsto \pi \otimes \pi'$ defines an injection $\Sigma_2(\Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)) \to \Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_4)$. Let us check that it is surjective. If $\omega \in \Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_4)$, Theorem 3.9 shows the existence of $\rho \in \Pi_{\mathrm{disc}}(\mathrm{SO}_{2,2})$ such that $\omega = \psi(\rho)$. Proposition 4.7 ensures that ρ belongs to $\mathrm{Res}_\iota((\pi, \pi'))$ for some $(\pi, \pi') \in \Pi_{\mathrm{disc}}(\mathrm{PGL}_2)^2$. But then $\omega = \psi(\rho) = \psi(\pi, \pi')$, so π, π' are distinct and cuspidal by Proposition 4.16, hence $\omega = \pi \otimes \pi'$.

If $\pi, \pi' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ have respective Hodge weights $w \geq w'$, the infinitesimal character of $\pi \otimes \pi'$ has the eigenvalues $\pm\frac{w+w'}{2}, \pm\frac{w-w'}{2}$. This implies that $\pi \otimes \pi'$ is in $\Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_4)$. Indeed, this is clear if $w \neq w'$. If $w = w'$, Clozel's purity lemma 3.13 shows that $\mathrm{L}((\pi \otimes \pi')_\infty) = \mathrm{I}_{2w} \oplus \chi_1 \oplus \chi_2$ where $\chi_1, \chi_2 \in \{1, \varepsilon_{\mathbb{C}/\mathbb{R}}\}$. But as $\pi \otimes \pi'$ has a trivial central character, we have $\chi_1\chi_2 = \det \mathrm{I}_{2w} = \varepsilon_{\mathbb{C}/\mathbb{R}}$, and we are done.

This ends the proof of the proposition, and shows part (ii) of Thm. 1.15. □

4.17. Λ^* **functorality and** $\Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_5)$**.** If $\pi \in \Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathrm{PGL}_4)$, there is a unique element $\widetilde{c(\pi)} \in \mathcal{X}(\mathrm{Sp}_4(\mathbb{C}))$ such that $\mathrm{St}(\widetilde{c(\pi)}) = c(\pi)$ (see § 3.8). We denote by Λ^* the irreducible representation $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{SL}_5(\mathbb{C})$, so that $\Lambda^2\mathbb{C}^4 = \Lambda^* \oplus 1$.

Proposition 4.18.** *There is a unique map* $\Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4) \overset{\sim}{\to} \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_5)$*, denoted* $\pi \mapsto \Lambda^*\pi$*, such that for each* $\pi \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$ *we have* $\Lambda^*(\widetilde{c(\pi)}) = c(\Lambda^*\pi)$*.*

Note that if $\pi \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$ has Hodge weights $w > v$, then $\Lambda^*\pi$ has Hodge weights $w + v > w - v$. The proposition implies thus Thm. 1.15 (iii).

Consider the central isogeny $\iota : \mathrm{Sp}_4 \to \mathrm{PGSp}_4 = \mathrm{SO}_{3,2}$. Let $\pi \in \Pi_{\mathrm{disc}}(\mathrm{SO}_{3,2})$ and let $\rho \in \mathrm{Res}_\iota(\pi)$. By Proposition 4.7 (b), we have $\rho \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_4)$. By definition, $c(\rho)$ is the image of $c(\pi)$ under the isogeny $\mathrm{Sp}_4(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C})$. If we compose this latter isogeny with the standard representation of $\mathrm{SO}_5(\mathbb{C})$, we obtain nothing else than the Λ^* representation of $\mathrm{Sp}_4(\mathbb{C})$. In particular, $\psi(\rho)$ does not depend on the choice of ρ in $\mathrm{Res}_\iota(\pi)$, and it thus makes sense to set

$$\widetilde{\psi}(\pi) = \psi(\rho) \in \Psi_{\mathrm{glob}}(\mathrm{Sp}_4).$$

[14] If X is a set $X^2/\mathfrak{S}_2$ denotes the quotient of X^2 by the equivalence relation $(x, y) \sim (y, x)$.

Observe that $\widetilde{\psi}(\pi) = \widetilde{\psi}(\pi')$ if $\psi(\pi) = \psi(\pi')$.

Proposition ** **4.19.** *Let* $\pi \in \Pi_{\rm disc}(\mathrm{SO}_{3,2})$.

(i) *If* $\psi(\pi) = [4]$ *then* $\widetilde{\psi}(\pi) = [5]$,

(ii) *If* $\psi(\pi) = \pi_1 \oplus [2]$ *with* $\pi_1 \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$ *then* $\widetilde{\psi}(\pi) = \pi_1[2] \oplus [1]$,

(iii) *If* $\psi(\pi) = \pi_1 \oplus \pi_2$ *with distinct* $\pi_1, \pi_2 \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$ *then* $\widetilde{\psi}(\pi) = \pi_1 \otimes \pi_2 \oplus [1]$,

(iv) *If* $\psi(\pi) \in \Pi^{\rm s}_{\rm alg}(\mathrm{PGL}_4)$ *then* $\widetilde{\psi}(\pi) \in \Pi^{\rm o}_{\rm alg}(\mathrm{PGL}_5)$. *Moreover,* $\widetilde{\psi}(\pi)$ *determines* $\psi(\pi)$ *in this case.*

Proof — Assertions (i), (ii) and (iii) follow from an immediate inspection of Satake parameters. Assertion (iii) makes sense by Proposition 4.15. Let us check (iv). Assume that $\omega := \psi(\pi)$ is cuspidal. Jacquet-Shalika's bound shows that $\widetilde{\psi}(\pi)$ cannot have the form $[5]$ or $\pi'[2] \oplus [1]$ for $\pi' \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$. The only remaining possibility is that $\widetilde{\psi}(\pi)$ is either cuspidal or of the form $\pi_1 \otimes \pi_2 \oplus [1]$ for two distinct $\pi_1, \pi_2 \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$. To rule out this latter case and prove (iv), we shall need to known a certain property of Arthur's Archimedean packets that we have not been able to extract from [Art11], namely that if ψ an Archimedean *generic* parameter in his sense, the packet $\widetilde{\Pi}_\psi$ defined *loc. cit.* contains with multiplicity one each element of the associated Langlands packet having a Whittaker model (see [Art89]). This is why we assume from now on that $\psi(\pi) \in \Pi^{\rm s}_{\rm alg}(\mathrm{PGL}_4)$ and we shall eventually rely instead on Conjecture 3.30.

Consider first any $\psi' \in \Psi_{\rm alg}(\mathrm{SO}_{3,2})$ which is either cuspidal or of the form $\pi_1 \oplus \pi_2$ with distinct $\pi_1, \pi_2 \in \Pi_{\rm cusp}(\mathrm{PGL}_2)$. The Archimedean Arthur parameter ψ'_∞ is a discrete series Langlands parameter. The associated set of discrete series of $\mathrm{SO}_{3,2}(\mathbb{R})$ with infinitesimal character $z_{\psi'_\infty}$ contains a unique element $\pi_{\rm gen}$ having a Whittaker model; its Shelstad character $\tau(\pi_{\rm gen})$ is trivial by definition (§A.7,§4.2). Arthur's multiplicity formula 3.30 for $\mathrm{SO}_{3,2}$ shows thus that the unique element $\rho' \in \Pi_{\rm disc}(\mathrm{SO}_{3,2})$ such that $\psi(\rho') = \psi'$ and $\rho'_\infty \simeq \pi_{\rm gen}$ has multiplicity 1. This construction applies for instance to $\psi' = \psi(\pi) = \omega$ and gives a representation ρ' that we shall denote by ϖ.

Assume now that $\widetilde{\psi}(\pi)$ has the form $\pi_1 \otimes \pi_2 \oplus [1]$. Consider $\psi' = \pi_1 \oplus \pi_2 \in \Psi_{\rm glob}(\mathrm{SO}_{3,2})$. It has the same infinitesimal character as $\psi(\pi)$, so that $\psi' \in \Psi_{\rm alg}(\mathrm{SO}_{3,2})$. Let ρ' be the representation associated to ψ' as in the previous paragraph. Then

$$\widetilde{\psi}(\varpi) = \widetilde{\psi}(\rho'), \quad \varpi_\infty \simeq \rho'_\infty, \quad \text{but} \quad \omega = \psi(\varpi) \neq \psi(\rho') = \pi_1 \oplus \pi_2.$$

The first two equalities imply that $\mathrm{Res}_\iota(\varpi) = \mathrm{Res}_\iota(\rho')$ (these sets actually have two elements because the restriction of $\pi_{\rm gen}$ to $\mathrm{Sp}_4(\mathbb{R})$ has two factors). The last one and Proposition 4.7 imply then that the elements of $\mathrm{Res}_\iota(\varpi)$ have multiplicity ≥ 2 in $\mathcal{L}_{\rm disc}(\mathrm{Sp}_4)$. This contradicts Arthur's multiplicity formula 3.30 for Sp_4.

It follows that $\widetilde{\psi}(\pi)$ is cuspidal. By the exact same argument as in the previous paragraph we see that if $\pi' \in \Pi_{\rm disc}(\mathrm{SO}_{3,2})$ is such that $\psi(\pi')$ is cuspidal and satisfies $\widetilde{\psi}(\pi') = \widetilde{\psi}(\pi)$, then $\psi(\pi') = \psi(\pi)$. □

Let $\pi \in \Pi^{\rm s}_{\rm cusp}({\rm PGL}_4)$. By Arthur's Theorem 3.9, we may find a $\rho \in \Pi_{\rm disc}({\rm SO}_{3,2})$ such that $\pi = \psi(\rho)$. We set

$$\Lambda^*\pi = \widetilde{\psi}(\rho)$$

It belongs to $\Pi^{\rm o}_{\rm cusp}({\rm PGL}_5)$ by Proposition 4.19 (iv) and does not depend on the choice of ρ such that $\psi(\rho) = \pi$. The same proposition shows that $\pi \mapsto \Lambda^*\pi$ is injective. It only remains to check the surjectivity. If $\omega \in \Pi^{\rm o}_{\rm cusp}({\rm PGL}_5)$, Theorem 3.9 shows the existence of $\rho \in \Pi_{\rm disc}({\rm Sp}_4)$ such that $\omega = \psi(\rho)$. Proposition 4.7 ensures that ρ belongs to ${\rm Res}_\iota(\pi)$ for some $\pi \in \Pi_{\rm disc}({\rm SO}_{3,2})$. But then $\omega = \psi(\rho) = \widetilde{\psi}(\pi)$ so $\psi(\pi)$ is cuspidal by Proposition 4.19. This finishes the proof of Proposition 4.18.

5. $\Pi_{\rm disc}(\mathrm{SO}_7)$ AND $\Pi^{\rm s}_{\rm alg}(\mathrm{PGL}_6)$

5.1. **The semisimple $\mathbb{Z}$-group SO_7.** Consider the semisimple classical $\mathbb{Z}$-group

$$G = \mathrm{SO}_7 = \mathrm{SO}_{\mathrm{E}_7},$$

i.e. the special orthogonal group of the root lattice E_7 (§3.1). Let $\mathrm{W}(E_7)$ denote the Weyl group of the root system of E_7, let $\varepsilon : \mathrm{W}(E_7) \to \{\pm 1\}$ be the signature and $\mathrm{W}(E_7)^+ = \mathrm{Ker}\,\varepsilon$. As the Dynkin diagram of E_7 has no non-trivial automorphism one has $\mathrm{O}(\mathrm{E}_7) = \mathrm{W}(E_7)$ (see §3.1), thus

$$G(\mathbb{Z}) = \mathrm{W}(E_7)^+.$$

The group $\mathrm{W}(E_7)^+$ has order $1451520 = 7! \cdot 2^5 \cdot 3^2$, it is isomorphic via the reduction modulo 2 to the finite simple group $G(\mathbb{F}_2) \simeq \mathrm{Sp}_6(\mathbb{F}_2)$ ([Bou81, Ch. VI, Ex. 3 §4]).

The class set $\mathrm{Cl}(G) \simeq \mathrm{X}_7$ has one element as $X_7 = \{\mathrm{E}_7\}$ (§ 3.1,§ 3.4). By Arthur's multiplicity formula, each $\pi \in \Pi_{\rm disc}(G)$ has multiplicity 1. It follows from Prop. 3.6 that the number $m(V)$ of $\pi \in \Pi_{\rm disc}(G)$ such that π_∞ is a given irreducible representation of $G(\mathbb{R})$ is

$$m(V) = \dim V^{\mathrm{W}(\mathrm{E}_7)^+},$$

which is exactly the number computed in the first chapter § 2.5 Case I. We refer to Table 2 and to the url [CR] for a sample of results.

The dual group of SO_7 is $\widehat{G} = \mathrm{Sp}_6(\mathbb{C})$.

5.2. **Parameterization by the infinitesimal character.** From the point of view of Langlands parameterization, it is more natural to label the irreducible representations of $G(\mathbb{R})$ by their infinitesimal character rather than their highest weight.

Let H be a compact connected Lie group, fix $T \subset H$ a maximal torus and $\Phi^+ \subset \mathrm{X}^*(T)$ a set of positive roots as in § 2.2. Denote by $\rho \in \mathrm{X}^*(T)[1/2]$ the half sum of the elements of Φ^+. As recalled in §3.7, under the Harish-Chandra isomorphism the infinitesimal character of the irreducible representation V_λ of H of highest weight λ is the $\mathrm{W}(H,T)$-orbit of $\lambda + \rho$.

For instance if $H = \mathrm{SO}_n(\mathbb{R})$, and in terms of the standard root data defined in § 2.5,

$$\rho = \begin{cases} \frac{2l-1}{2}e_1 + \frac{2l-3}{2}e_2 + \cdots + \frac{1}{2}e_l & \text{if } n = 2l+1, \\ (l-1)e_1 + (l-2)e_2 + \cdots + e_{l-1} & \text{if } n = 2l. \end{cases}$$

The map $\lambda \mapsto \lambda + \rho = \sum_i \frac{w_i}{2} e_i$ induces thus a bijection between the dominant weights and the collection of $w_1 > w_2 > \cdots > w_l$ where the w_i are odd positive integers when $n = 2l+1$, even integers with $w_{l-1} > |w_l|$ when $n = 2l$.

Definition 5.3. *Let $n \geq 1$ be an integer, set $l = [n/2]$, and let $\underline{w} = (w_1, \cdots, w_l)$ where $w_1 > w_2 > \cdots > w_l \geq 0$ are distinct nonnegative integers all congruent to n modulo 2. We denote by*

$$U_{\underline{w}}$$

the finite dimensional irreducible representation V_λ of $\mathrm{SO}_n(\mathbb{R})$ such that $\lambda + \rho = \sum_i \frac{w_i}{2} e_i$.

As an example, observe that if $\mathrm{H}_m(\mathbb{R}^n)$ is the representation of $\mathrm{SO}_n(\mathbb{R})$ defined in § 2.7, then $\mathrm{H}_m(\mathbb{R}^n) = U_{\underline{w}}$ for

$$w = \begin{cases} (2m+n-2, n-4, n-6, \cdots, 3, 1) & \text{if } n \equiv 1 \bmod 2, \\ (2m+n, n-2, n-4, \cdots, 2, 0) & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

The infinitesimal character $\lambda + \rho$ is related to the Langlands parameterization of V_λ as follows. Assume to simplify that H is semisimple and that $-1 \in \mathrm{W}(H,T)$. This is always the case if $H = G(\mathbb{R})$ and G is semisimple over $\mathbb{Z}$, and for $H = \mathrm{SO}_n(\mathbb{R})$ this holds if and only if $n \not\equiv 2 \bmod 4$. Then the Langlands dual group of H is a connected semisimple complex group $\widehat{H}$. Recall that $\widehat{H}$ is equipped with a maximal torus $\widehat{T}$, a set of positive roots $(\Phi^\vee)^+$ for $(\widehat{H}, \widehat{T})$, and an isomorphism between the dual based root datum of $(\widehat{H}, \widehat{T}, (\Phi^\vee)^+)$ and the one of (H, T, Φ^+). In particular, $\mathrm{X}_*(\widehat{T})$ and $\mathrm{X}^*(T)$ are identified by definition. The Langlands parameter of V_λ is up to $\widehat{H}$-conjugation the unique continuous homomorphism $\mathrm{L}(V_\lambda) : \mathrm{W}_\mathbb{R} \to \widehat{H}$ with finite centralizer and such that in Langlands' notation (see §3.11)

$$\mathrm{L}(V_\lambda)(z) = (z/\overline{z})^{\lambda+\rho} \in \widehat{T} \quad \forall z \in \mathbb{C}^\times = \mathrm{W}_\mathbb{C}.$$

When $H = \mathrm{SO}_n(\mathbb{R})$ and $\underline{w} = (w_1, w_2, \cdots, w_l)$ is as in definition 5.3, it follows that in the standard representation $\mathrm{St} : \widehat{H} \to \mathrm{GL}(2l, \mathbb{C})$ of the classical group $\widehat{H}$, we have

$$\mathrm{St} \circ \mathrm{L}(U_{\underline{w}}) \simeq \bigoplus_{i=1}^{l} \mathrm{I}_{w_i}.$$

This is the reason why the normalization above will be convenient.

Definition 5.4. *Let G be the semisimple classical definite $\mathbb{Z}$-group SO_n defined in §3.1. If $\underline{w} = (w_1, \cdots, w_l)$ is as in Definition 5.3 we define*

$$\Pi_{\underline{w}}(G) = \{\pi \in \Pi_{\mathrm{disc}}(G), \pi_\infty \simeq U_{\underline{w}}\}$$

and set $m(\underline{w}) = |\Pi_{\underline{w}}(G)|$.

If $\pi \in \Pi_{\mathrm{disc}}(G)$, we shall say that π has Hodge weights $\underline{w}$ if $\pi \in \Pi_{\underline{w}}(G)$.

5.5. **Endoscopic partition of** $\Pi_{\mathrm{disc}}(\mathrm{SO}_7)$**.** Recall that if $\pi \in \Pi_{\underline{w}}(\mathrm{SO}_n)$, it has a global Arthur parameter

$$\psi(\pi) = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(\mathrm{SO}_n)$$

whose equivalence class in well-defined (§3.18). The associated collection $(k, (n_i), (d_i))$ will be called the *endoscopic type of* π. As for $\psi(\pi)$, the endoscopic type will be called *stable* if $k = 1$, and tempered if $d_i = 1$ for $i = 1, \cdots, k$. By Lemma 3.23, $\psi(\pi)$ is stable and tempered if and only it belongs to $\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}(2l))$ where $l = [n/2]$.

So far we have computed $|\Pi_{\underline{w}}(\mathrm{SO}_7)|$ for any possible Hodge weights $\underline{w}$. Our next aim will be to compute the number of elements in $\Pi_{\underline{w}}(\mathrm{SO}_7)$ of each possible endoscopic type. As we shall see, thanks to Arthur's multiplicity formula and our previous computation of $\mathrm{S}(w)$, $\mathrm{S}(w,v)$ and $\mathrm{O}^*(w)$, we will be able to compute the contribution of each endoscopic type except one, namely the stable and tempered type, which is actually $\mathrm{S}(w_1, w_2, w_3)$.

We will in turn obtain this later number from our computation of $|\Pi_w(\mathrm{SO}_7)|$. The Corollary 1.11 and Table 7 will follow form these computations.

Fix a triple $\underline{w} = (w_1, w_2, w_3)$. Fix as well once and for all a global Arthur parameter

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(\mathrm{SO}_7)$$

such that the semisimple conjugacy class $\mathrm{St}(z_{\psi_\infty})$ in $\mathfrak{sl}_6(\mathbb{C})$ has the eigenvalues

$$\pm\frac{w_1}{2}, \pm\frac{w_2}{2}, \pm\frac{w_3}{2}.$$

Let us denote by π the unique element in $\Pi(\psi)$. We shall make explicit Arthur's multiplicity formula for $m(\pi)$, which is either 0 or 1 as $m_\psi = 1$, following §3.30.1. Recall the important groups

$$\mathrm{C}_\psi \subset \mathrm{C}_{\psi_\infty} \subset \mathrm{Sp}_6(\mathbb{C}).$$

For each $1 \leq i \leq k$ one has a distinguished element $s_i \in \mathrm{C}_\psi$ (§ 3.27). Those k-elements s_i generate $\mathrm{C}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^k$ and their product generates the center $Z = \{\pm 1\}$ of $\mathrm{Sp}_6(\mathbb{C})$.

5.5.1. *The stable case.* This is the case $k = 1$, i.e. $\mathrm{C}_\psi = Z$, for which the multiplicity formula trivially gives $m(\pi) = 1$. Let us describe the different possibilities for ψ. One has $\psi(\pi) = \pi_1[d_1]$ with $d_1|6$, $\pi_1 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}(6/d_1))$ and $(-1)^{d_1-1}s(\pi_1) = -1$.

Case (i) : $\psi = \pi_1$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_6)$, this is the unknown we want to count.

Case (ii) : $\psi = \pi_1[2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_3)$, say of Hodge weight $u > 2$ (so $u \equiv 2 \bmod 4$). This occurs if and only if $\underline{w}$ has the form $(u+1, u-1, 1)$. Recall that $\pi_1 = \mathrm{Sym}^2\pi'$ for a unique $\pi' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ with Hodge weight $u/2$.

Case (iii) : $\psi = \pi_1[3]$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$, say of Hodge weight $u > 1$ (an odd integer). This occurs if and only if w has the form $(u+2, u, u-2)$.

Case (iv) : $\psi = [6]$. This occurs if and only if $w = (5, 3, 1)$, and π is then the trivial representation of G.

5.5.2. *Endoscopic cases of type* $(n_1, n_2) = (4, 2)$. In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $\mathrm{C}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z. One will have to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\mathrm{Min}(d_1,d_2)}$ in each case. Recall that $\rho^\vee : \mathrm{C}_{\psi_\infty} \to \{\pm 1\}$ is the fundamental character defined in § 3.30.1. There are three cases.

Case (v) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_4)$ and $\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. Denote by $a > b$ the Hodge weights of π_1 and by c the Hodge weight of π_2. One has $\{a, b, c\} = \{w_1, w_2, w_3\}$. One sees that

$$\rho^\vee(s_1) = 1 \text{ iff } a > c > b$$

But $\varepsilon_\psi(s_1) = 1$ as all the d_i are 1 (tempered case). It follows that

$$m(\pi) = 1 \quad \Leftrightarrow a > c > b$$

or which is the same, $m(\pi) = 1$ if and only if $(w_1, w_2, w_3) = (a, c, b)$.

Case (vi) : $d_1 = 1$, $d_2 = 2$, i.e. $\psi = \pi_1 \oplus [2]$ where $\pi_1 \in \Pi_{\rm alg}^{\rm s}({\rm PGL}_4)$ has Hodge weights $w_1 > w_2$ with $w_2 > 1$. One sees that $\rho^\vee(s_1) = -1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_1) = (-1)^{(w_1+w_2+2)/2}$, it follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad w_1 + w_2 \equiv 0 \bmod 4.$$

Case (vii) : $d_1 = 4$, $d_2 = 1$, i.e. $\psi = [4] \oplus \pi_2$ where $\pi_2 \in \Pi_{\rm alg}({\rm PGL}_2)$ has Hodge weight w_1 with $w_1 > 3$. One sees that $\rho^\vee(s_1) = -1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_2) = (-1)^{(w_1+1)/2}$, it follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad w_1 \equiv 1 \bmod 4.$$

5.5.3. *Endoscopic cases of type* $(n_1, n_2, n_3) = (2, 2, 2)$. In this case $k = 3$, and ${\rm C}_\psi$ is generated by Z and s_1, s_2. There are two cases.

Case (viii) : (*tempered case*) $d_i = 1$ for each i, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3$ where each $\pi_i \in \Pi_{\rm alg}({\rm PGL}_2)$ and π_i has Hodge weight w_i. and $\pi_2 \in \Pi_{\rm alg}({\rm PGL}_2)$. Of course ε_ψ is trivial here, so $m(\pi) = 1$ if and only if $\rho^\vee$ is trivial on ${\rm C}_\psi$. But ${\rm C}_\psi = {\rm C}_{\psi_\infty}$ and $\rho^\vee$ is a non-trivial character, so

$$m(\pi) = 0$$

in all the cases.

Case (ix) : $d_1 = d_2 = 1$ and $d_3 = 2$, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus [2]$ where $\pi_1, \pi_2 \in \Pi_{\rm alg}({\rm PGL}_2)$ have respective Hodge weights $w_1 > w_2$, with $w_2 > 1$. One has thus $\rho^\vee(s_1) = -1$ and $\rho^\vee(s_2) = 1$. On the other hand for $i = 1, 2$ one has $\varepsilon_\psi(s_i) = \varepsilon(\pi_i) = (-1)^{(w_i+1)/2}$. It follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad (w_1, w_2) \equiv (1, 3) \bmod 4.$$

5.6. **Conclusions.** First, one obtains the value of ${\rm S}(w_1, w_2, w_3)$ as the difference between $m(w_1, w_2, w_3)$ and the sum of the eight last contributions above. For instance, one sees that if $w_1 - 2 > w_2 > w_3 + 2 > 3$ then

$${\rm S}(w_1, w_2, w_3) = m(w_1, w_2, w_3) - {\rm S}(w_1, w_3) \cdot {\rm S}(w_2).$$

It turns out that all the formulas for the nine cases considered above perfectly fit our computations, in the sense that ${\rm S}(w_1, w_2, w_3)$ always returned to us a positive integer. This is again a substantial confirmation for both our computer program and for the remarkable precision of Arthur's results. This also gives some mysterious significance for the first non-trivial invariants of the group ${\rm W}(E_7)^+$. One deduces in particular Table 7, and from this table Corollary 1.11 of the introduction (see also [CR]).

Corollary 5.7.** *If* $w_1 < 23$ *then* ${\rm S}(w_1, w_2, w_3) = 0$. *There are exactly* 7 *triples* $(23, w_2, w_3)$ *such that* ${\rm S}(23, w_2, w_3) \neq 0$, *and for each of them* ${\rm S}(23, w_2, w_3) = 1$.

As far as we know, none of these 7 automorphic representations (symplectic of rank 6) had been discovered before. As explained in the introduction, they are related to the 121 Borcherds even lattices of rank 25 and covolume $\sqrt{2}$, in the same way as the 4 Tsushima's forms $\Delta_{19,7}, \Delta_{21,5}, \Delta_{21,9}$ and $\Delta_{21,13}$ are related to Niemeier lattices, as discovered in [CL14]. It would be interesting to know more about those forms, e.g. some of their Satake parameters. Our tables actually reveals a number of triples (w_1, w_2, w_3) such that $\mathrm{S}(w_1, w_2, w_3) = 1$.

One obtains as well a complete endoscopic description of each $\Pi_{\underline{w}}(\mathrm{SO}_7)$. For instance Tables 12 and 13 describe entirely the set $\Pi_{w_1,w_2,w_3}(\mathrm{SO}_7)$ for $w_1 \leq 25$ whenever it is non-empty. Recall the following notation already introduced in §1.21: when $\mathrm{S}(w_1, \cdots, w_r) = 1$ we denote by $\Delta_{w_1,\cdots,w_r}$ the unique $\pi \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_{2r})$ with Hodge weights $w_1 > \cdots > w_r$. When $\mathrm{S}(w_1, \cdots, w_r) = k$ we also denote by $\Delta^k_{w_1,\cdots,w_r}$ any of the k elements of $\Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_{2r})$ with Hodge weights $w_1 > \cdots > w_r$.

Let us explore some examples. It follows from case (iii) above that the number of $\pi \in \Pi_{\underline{w}}(\mathrm{SO}_7)$ such that $\psi(\pi)$ has the form $\pi_1[3]$ is $\delta_{w_1=w_3+4} \cdot \mathrm{S}(w_2)$. For instance the first such π is $\Delta_{11}[3]$ which thus belongs to $\Pi_{13,11,9}(\mathrm{SO}_7)$. Our computations gives $m(13, 11, 9) = 1$ (hence nonzero!) which is not only in accord with Arthur's result but also says that

$$\Pi_{13,11,9}(G) = \{\Delta_{11}[3]\}.$$

The triple $\underline{w} = (13, 11, 9)$ turns out to be the first triple $\neq (5, 3, 1)$ such that $m(\underline{w}) \neq 0$. Our table even shows that

$$\forall\ 3 \leq u \leq 25,\ \ m(u + 2, u, u - 2) = \mathrm{S}(u),$$

which describes entirely $\Pi_{u+2,u,u-2}(\mathrm{SO}_7)$ for those u. One actually has

$$m(29, 27, 25) = 4 > \mathrm{S}(27) = 2.$$

Let us determine $\Pi_{29,27,25}(\mathrm{SO}_7)$. We already found two forms $\Delta^2_{27}[3]$ (there are two elements in $\Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ of Hodge weight 27). On the other hand, one checks from Tsushima's formula that $\mathrm{S}(29, 25) = 1$, so that there is a unique element in $\Delta_{29,25} \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$ with Hodge weights $29 > 25$. The missing two elements are thus the two $\Delta_{29,25} \oplus \Delta^2_{27}$. Indeed, we are here in the endoscopic case (v): 27 is between 25 and 29.

As another example, consider now the $\pi \in \Pi_{\underline{w}}(G)$ such that $\psi(\pi)$ has the form $\pi_1[2]$ (endoscopic case (ii)). There are exactly

$$\delta_{w_3=1} \cdot \delta_{w_1-w_2=2} \cdot \delta_{w_1\equiv 1 \bmod 4} \cdot \mathrm{S}(\frac{w_1+1}{2})$$

such π's. The first one is thus $\mathrm{Sym}^2\Delta_{11}[2]$ which belongs to $\Pi_{23,21,1}(\mathrm{SO}_7)$. Our computations gives $m(23, 21, 1) = 1$, which is not only in accord with Arthur's result but also says that

$$\Pi_{23,21,1}(\mathrm{SO}_7) = \{\mathrm{Sym}^2\Delta_{11}[2]\}.$$

6. Description of $\Pi_{\mathrm{disc}}(\mathrm{SO}_9)$ and $\Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_8)$

6.1. **The semisimple $\mathbb{Z}$-group** SO_9**.** Consider the semisimple classical $\mathbb{Z}$-group

$$G = \mathrm{SO}_9,$$

i.e. the special orthogonal group of the root lattice $L = \mathrm{A}_1 \oplus \mathrm{E}_8$ (§3.1). Let $\mathrm{W}(E_8)$ denote the Weyl group of the root system of E_8 and let $\varepsilon : \mathrm{W}(E_8) \to \{\pm 1\}$ be the signature. There is a natural homomorphism $\mathrm{W}(E_8) \to G(\mathbb{Z})$, if we let $\mathrm{W}(E_8)$ act on $\mathrm{A}_1 \oplus \mathrm{E}_8$ as $w \mapsto (w, \varepsilon(w))$ (§ 2.5 case III). One has $\mathrm{O}(L) = \{\pm 1\} \times \mathrm{W}(E_8)$ by §3.1, thus a natural isomorphism

$$\mathrm{W}(E_8) \xrightarrow{\sim} G(\mathbb{Z}).$$

The group W has order

$$|\mathrm{W}(E_8)| = 8! \cdot 2^5 \cdot 3^3 \cdot 5 = 696729600$$

and the natural map $\mathrm{W}(E_8) \to \mathrm{SO}_{\mathrm{E}_8}(\mathbb{F}_2)$ is surjective with kernel $\{\pm 1\}$ ([Bou81, Ch. VI, Ex. 1 §4]).

The class set $\mathrm{Cl}(G) \simeq \mathrm{X}_9$ has one element as $X_9 = \{\mathrm{A}_1 \oplus \mathrm{E}_8\}$ (§ 3.1,§ 3.4). By Arthur's multiplicity formula, each $\pi \in \Pi_{\mathrm{disc}}(G)$ has multiplicity 1. It follows from Prop. 3.6 that the number $m(V)$ of $\pi \in \Pi_{\mathrm{disc}}(G)$ such that π_∞ is a given irreducible representation of $G(\mathbb{R})$ is

$$m(V) = \dim V^{\mathrm{W}(E_8)^+},$$

which is exactly the number computed in the first chapter § 2.5 Case III. We refer to Table 4 and to [CR] for a sample of results.

The dual group of SO_9 is $\widehat{G} = \mathrm{Sp}_8(\mathbb{C})$.

6.2. **Endoscopic partition of** Π_w**.** We proceed in a similar way as in § 5.5.

Fix $\underline{w} = (w_1, w_2, w_3, w_4)$ with $w_1 > w_2 > w_3 > w_4$ odd positive integers. Fix as well once and for all a global Arthur parameter

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(G)$$

such that the semisimple conjugacy class $\mathrm{St}(z_{\psi_\infty})$ in $\mathfrak{sl}_8(\mathbb{C})$ has the eigenvalues

$$\{\pm \frac{w_i}{2}, 1 \leq i \leq 4\}.$$

Let us denote by π the unique element in $\Pi(\psi)$. We shall make explicit Arthur's multiplicity formula for $m(\pi)$, which is either 0 or 1, as in §3.30.1. Recall the groups

$$\mathrm{C}_\psi \subset \mathrm{C}_{\psi_\infty} \subset \widehat{G} = \mathrm{Sp}_8(\mathbb{C}).$$

6.2.1. *The stable cases.* This is the case $k=1$, i.e. $\mathrm{C}_\psi = Z$, for which the multiplicity formula trivially gives $m(\pi)=1$. One has $\psi(\pi)=\pi_1[d_1]$ with $d_1|8$, $\pi_1 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}(8/d_1))$, and $(-1)^{d_1-1}s(\pi_1)=-1$.

Case (i) : (*tempered case*) $\psi=\pi_1$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_8)$, this is the unknown we want to count.

Case (ii) : $\psi=\pi_1[2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_4)$, say of Hodge weights $u>v$ (recall u,v even and $u+v \equiv 2 \bmod 4$). This occurs if and only if w has the form $(u+1,u-1,v+1,v-1)$. Recall from Proposition 4.15 that $\pi_1=\pi'\otimes\pi''$ for a unique pair $\pi',\pi'' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ with respective Hodge weights $(u+v)/2$ and $(u-v)/2$.

Case (iii) : $\psi=[8]$. This occurs if and only if $\underline{w}=(7,5,3,1)$, and π is then the trivial representation of G.

6.2.2. *Endoscopic cases of type* $(n_1,n_2)=(6,2)$. In this case $k=2$,

$$\psi=\pi_1[d_1]\oplus\pi_2[d_2]$$

and $\mathrm{C}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z. One will have to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1)=\varepsilon(\pi_1\times\pi_2)^{\mathrm{Min}(d_1,d_2)}$ in each case. Recall that $\rho^\vee : \mathrm{C}_{\psi_\infty} \to \{\pm1\}$ is the fundamental character defined in § 3.30.1. There are 6 cases.

Case (iv) : (*tempered case*) $d_1=d_2=1$, i.e. $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_6)$ and $\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. Denote by $a>b>c$ the Hodge weights of π_1 and by d the Hodge weight of π_2. One has $\{a,b,c,d\}=\{w_1,w_2,w_3,w_4\}$. Moreover $\varepsilon_\psi(s_1)=1$ as all the d_i are 1 (tempered case), so $m(\pi)=1$ if, and only if, $\rho^\vee(s_1)=1$, i.e. if $d\in\{w_1,w_3\}$. In other words,

$$m(\pi)=1 \quad\Leftrightarrow\quad d>a>b>c \ \text{ or } \ a>b>d>c.$$

Case (v) : $d_1=1$, $d_2=2$, i.e. $\psi=\pi_1\oplus[2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_6)$ has Hodge weights $w_1>w_2>w_3$, with $w_3>1$. One sees that $\rho^\vee(s_1)=-1$. On the other hand $\varepsilon_\psi(s_1)=\varepsilon(\pi_1)=(-1)^{(w_1+w_2+w_3+3)/2}$, it follows that

$$m(\pi)=1 \quad\Leftrightarrow\quad w_1+w_2+w_3 \equiv 3 \bmod 4.$$

Case (vi) : $d_1=2$, $d_2=1$, i.e. $\psi=\pi_1[2]\oplus\pi_2$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_3)$ and $\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. Denote by a and b the respective Hodge weights of π_1 and π_2, so that $\{w_1,w_2,w_3,w_4\}=\{a+1,a-1,b,1\}$. There are two cases: either $b>a$ or $b<a$. One sees that $\rho^\vee(s_1)=1$ in both cases. On the other hand

$$\varepsilon_\psi(s_1)=\varepsilon(\pi_1\times\pi_2)=-(-1)^{\frac{b+1}{2}+\mathrm{Max}(a,b)}.$$

It follows that

$$m(\pi)=1 \quad\Leftrightarrow\quad \begin{cases} b\equiv 3 \bmod 4, & \text{if } b>a+1,\\ b\equiv 1 \bmod 4, & \text{if } b<a-1.\end{cases}$$

Case (vii) : $d_1=3$, $d_2=1$, i.e. $\psi=\pi_1[3]\oplus\pi_2$ where $\pi_1,\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. Denote by a and b the respective Hodge weights of π_1 and π_2, so that $\{w_1,w_2,w_3,w_4\}=$

$\{a+1, a, a-1, b\}$. One sees that $\rho^\vee(s_1) = 1$ if $b > a+1$, -1 otherwise. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = 1$. It follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad b > a+1.$$

Case (viii) : $d_1 = 3$, $d_2 = 2$, i.e. $\psi = \pi_1[3] \oplus [2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. The Hodge weight of π_1 is thus $w_2 > 3$. We have $\rho^\vee(s_1) = -1$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1)^2 = 1$. It follows that

$$m(\pi) = 0$$

Case (ix) : $d_1 = 6$, $d_2 = 1$, i.e. $\psi = [6] \oplus \pi_2$ where $\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ has Hodge weight w_1 with $w_1 > 5$. One sees that $\rho^\vee(s_1) = 1$. On the other hand $\varepsilon_\psi(s_1) = \varepsilon(\pi_2) = (-1)^{(w_1+1)/2}$, it follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad w_1 \equiv 3 \bmod 4.$$

Remark 6.3. Observe that the case $d_1 = d_2 = 2$, i.e. $\psi = \pi_1[2] \oplus [2]$ where $\pi_1 \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_3)$, is impossible as it implies $w_2 = w_1 = 1$.

6.3.1. *Endoscopic cases of type* $(n_1, n_2, n_3) = (4, 2, 2)$. In this case $k = 3$, and C_ψ is generated by Z (or s_3) and s_1, s_2. There are three cases.

Case (x) : (*tempered case*) $d_i = 1$ for each i, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3$ where $\pi_1 \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$ and $\pi_2, \pi_3 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. Denote by $a > b$ the Hodge weights of π_1 and by c and d the ones of π_2, π_3, assuming $c > d$. Of course ε_ψ is trivial here, so $m(\pi) = 1$ if and only if $\rho^\vee$ is trivial on $\mathrm{C}_\psi = \mathrm{C}_{\psi_\infty}$. One thus obtains

$$m(\pi) = 1 \quad \Leftrightarrow \quad c > a > d > b.$$

Case (xi) : $d_1 = d_2 = 1$ and $d_3 = 2$, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus [2]$ where $\pi_1 \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$, $\pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ have respective Hodge weights $a > b$ and c, with $\{w_1, w_2, w_3, w_4\} = \{a, b, c, 1\}$. If $a > c > b$ then $\rho^\vee(s_1) = 1$ and $\rho^\vee(s_2) = -1$, otherwise $\rho^\vee(s_1) = -1$ and $\rho^\vee(s_2) = 1$. On the other hand for $i = 1, 2$ one has $\varepsilon_\psi(s_i) = \varepsilon(\pi_i)$. It follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad \begin{cases} (a+b, c) \equiv (2,1) \bmod 4, & \text{if } a > c > b \\ (a+b, c) \equiv (0,3) \bmod 4, & \text{otherwise.} \end{cases}$$

Case (xii) : $d_1 = 4$, $d_2 = d_3 = 1$, i.e. $\psi = [4] \oplus \pi_2 \oplus \pi_3$ where $\pi_2, \pi_3 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ with respective Hodge weights w_1 and w_2, with $w_2 > 3$. One has $\rho^\vee(s_2) = 1$, $\rho^\vee(s_3) = -1$, $\varepsilon_\psi(s_i) = \varepsilon(\pi_i)$ for $i = 2, 3$, thus

$$m(\pi) = 1 \quad \Leftrightarrow \quad (w_1, w_2) \equiv (3, 1) \bmod 4.$$

6.3.2. *Endoscopic cases of type* $(n_1, n_2, n_3, n_4) = (2, 2, 2, 2)$. In this case $k = 4$, and C_ψ is generated by Z (or s_4) and s_1, s_2, s_3. There are two cases.

Case (xiii) : (*tempered case*) $d_i = 1$ for each i, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4$ where $\pi_i \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ has Hodge weight w_i. As ε_ψ is trivial but not $\rho^\vee$ on $\mathrm{C}_\psi = \mathrm{C}_{\psi_\infty}$ we have in all cases

$$m(\pi) = 0.$$

Case (xiv) : $d_4 = 2$ and $d_1 = d_2 = d_3 = 1$, i.e. $\psi = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus [2]$ where $\pi_i \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ has Hodge weight w_i, and $w_3 > 1$. One has $\rho^\vee(s_1) = \rho^\vee(s_3) = 1$ and $\rho^\vee(s_2) = -1$. On the other hand $\varepsilon_\psi(s_i) = \varepsilon(\pi_i)$ for $i = 1, 2, 3$. It follows that

$$m(\pi) = 1 \quad \Leftrightarrow \quad (w_1, w_2, w_3) \equiv (3, 1, 3) \bmod 4.$$

6.3.3. *Endoscopic cases of type* $(n_1, n_2) = (4, 4)$. In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $\mathrm{C}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z. One only has to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\mathrm{Min}(d_1, d_2)}$ in each case.

Case (xv) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\psi = \pi_1 \oplus \pi_2$ with $\pi_1, \pi_2 \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$. Let $a > b$ be the Hodge weight of π_1 and $c > d$ the ones of π_2, one may assume that $a > c$, i.e. $a = w_1$. As $\varepsilon_\psi = 1$, one sees that

$$m(\pi) = 1 \quad \Leftrightarrow \quad a > c > b > d.$$

Case (xvi) : $d_1 = 1$ and $d_2 = 4$, i.e. $\psi = \pi_1 \oplus [4]$ where $\pi_1 \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathrm{PGL}_4)$ has Hodge weights $w_1 > w_2$ with $w_2 > 3$. It follows that $\rho^\vee(s_1) = -1$, and as $\varepsilon_\psi(s_1) = \varepsilon(\pi_1)$ one obtains

$$m(\pi) = 1 \quad \Leftrightarrow \quad w_1 + w_2 \equiv 0 \bmod 4.$$

6.4. **Conclusions.** The inspection of each case above, and our previous computation of $\mathrm{S}(w)$, $\mathrm{S}(w, v)$, $\mathrm{S}(w, v, u)$, $\mathrm{O}^*(w)$ and $\mathrm{O}(w, v)$, allow to compute the contribution of each endoscopic type except one, namely the stable and tempered type, which is actually $\mathrm{S}(w_1, w_2, w_3, w_4)$, that we thus deduce from our computation of $m(w_1, w_2, w_3, w_4)$. The Corollary 1.12 and Table 8 follow form these computations (see also [CR]).

Corollary** **6.5.** *If* $w_1 < 25$ *then* $\mathrm{S}(w_1, w_2, w_3, w_4) = 0$. *There are* 33 *triples* (w_2, w_2, w_4) *such that* $\mathrm{S}(25, w_2, w_3, w_4) \neq 0$, *and in each case* $\mathrm{S}(25, w_2, w_3, w_4) = 1$.

We refer to Table 14 for the description of all the nonempty $\Pi_{\underline{w}}(\mathrm{SO}_9)$ when $w_1 \leq 23$.

For the application to Theorem 1.25, consider for instance the problem of describing $\Pi_{27,23,9,1}(\mathrm{SO}_9)$. Our program tells us that

$$m(27, 23, 9, 1) = 5,$$

so that $|\Pi_{27,23,9,1}(\mathrm{SO}_9)| = 5$. Fix $\pi \in \Pi_{27,23,9,1}(\mathrm{SO}_9)$ and let $\psi(\pi) = (k, (n_i), (d_i), (\pi_i))$.

Assume first that $\psi(\pi)$ is not tempered, i.e. that some $d_i \neq 1$. We may assume that $d_k > 1$. One sees that $k > 1$, $d_k = 2$ and $d_i = 1$ for $i < k$. As $\mathrm{S}(9) = 0$ we have $k \leq 3$. If $k = 2$ then we are in case (v). As $27 + 23 + 9 \equiv 3 \bmod 4$ one really has to compute

S(27, 23, 9). Our computer program tells us that $m(27, 23, 9) = 4$. On the other hand one has

$$\mathrm{S}(27, 23, 9) = m(27, 23, 9) - \mathrm{S}(27, 9) \cdot \mathrm{S}(23)$$

by § 5.6. By Tsushima's formula we have $\mathrm{S}(27, 9) = 1$. As $\mathrm{S}(23) = 2$ we obtain

$$\mathrm{S}(27, 23, 9) = 2.$$

There are thus two representations $\Delta^2_{27,23,9} \oplus [2]$ in $\Pi_{27,23,9,1}(\mathrm{SO}_9)$.

Assume now that $k = 3$, so we are in case (xi) and the Hodge weights of π_1 are a and 9. As $a + 9 \equiv 0 \bmod 4$, the multiplicity formula forces thus $a = 23$. Tsushima's formula shows that $\mathrm{S}(23, 9) = 1$. As $\mathrm{S}(27) = 2$ there are indeed two parameters $\Delta^2_{27} \oplus \Delta_{23,9} \oplus [2]$ in case (xi), whose associated π each have multiplicity 1 by the multiplicity formula.

Suppose now that π is tempered, i.e. $d_i = 1$ for all i. The multiplicity formula shows that 1 and 23 are Hodge weights of a same π_i, say π_{i_0}. But we already checked that $\mathrm{S}(23, 1) = 0$ and $\mathrm{S}(23, 9, 1) = 0$, and $\mathrm{S}(9) = 0$, it follows that $k = 1$, i.e. π is stable.

Corollary 6.6.** $\Pi_{27,23,9,1}(\mathrm{SO}_9) = \{\Delta^2_{27,23,9} \oplus [2], \Delta^2_{27} \oplus \Delta_{23,9} \oplus [2], \Delta_{27,23,9,1}\}$.

7. Description of $\Pi_{\rm disc}(\mathrm{SO}_8)$ and $\Pi^{\rm o}_{\rm alg}(\mathrm{PGL}_8)$

7.1. **The semisimple $\mathbb{Z}$-group** SO_8**.** Consider the semisimple classical $\mathbb{Z}$-group

$$G = \mathrm{SO}_8 = \mathrm{SO}_{\mathrm{E}_8},$$

i.e. the special orthogonal group of the root lattice E_8. Recall that $\mathrm{W}(E_8)$ denote the Weyl group of the root system of E_8, that $\varepsilon : \mathrm{W}(E_8) \to \{\pm 1\}$ is the signature and that $\mathrm{W}(E_8)^+ = \mathrm{Ker}\,\varepsilon$. As the Dynkin diagram of E_8 has no non-trivial automorphism one has $\mathrm{O}(\mathrm{E}_8) = \mathrm{W}(E_8)$ (§3.1), thus

$$G(\mathbb{Z}) = \mathrm{W}(E_8)^+.$$

The class set $\mathrm{Cl}(G) \simeq \widetilde{X}_8$ has one element as $X_8 = \{\mathrm{E}_8\}$ and of course $\mathrm{O}(\mathrm{E}_8) \neq \mathrm{SO}(\mathrm{E}_8)$ (§ 3.1,§ 3.4).

We shall consider quadruples $\underline{w} = (w_1, w_2, w_3, w_4)$ where $w_1 > w_2 > w_3 > w_4 \geq 0$ are even integers. It is not necessary to consider the (w_1, w_2, w_3, w_4) with $w_4 < 0$ as $\mathrm{O}(\mathrm{E}_8) = \mathrm{W}(E_8)$ contains root reflexions. Indeed, fix such a reflexion s. Then s acts by conjugation on $\mathcal{L}(G)$, hence on $\Pi_{\rm disc}(G)$, with the following property : if π_∞ has the highest weight (n_1, n_2, n_3, n_4), then $s(\pi)_\infty$ has the highest weight $(n_1, n_2, n_3, -n_4)$. Moreover $m(s(\pi)) = m(\pi)$.

Consider the number $m'(\underline{w}) :== \sum_{\pi \in \Pi_{\underline{w}}(G)} m(\pi)$. It follows from Prop. 3.6 that

$$m'(\underline{w}) = \dim U_{\underline{w}}^{\mathrm{W}(E_8)^+},$$

which is exactly the number computed in the first chapter § 2.5 Case II. We refer to Table 3 and to the url [CR] for a sample of results.

By Arthur's multiplicity formula, for each $\pi \in \Pi_{\underline{w}}(G)$ we have $m(\pi) + m(s(\pi)) \leq 2$. In particular, if $\underline{w} = (w_1, w_2, w_3, w_4)$ is such that $w_4 \neq 0$, then $m(\pi) = 1$. In this case, it follows that

$$m(\underline{w}) = m'(\underline{w}) = \dim U_{\underline{w}}^{\mathrm{W}(\mathrm{E}_8)}.$$

(Recall that $m(\underline{w}) = |\Pi_{\underline{w}}(G)|$).

The dual group of SO_8 is $\widehat{G} = \mathrm{SO}_8(\mathbb{C})$.

7.2. **Endoscopic partition of** Π_w**.** We proceed again in a similar way as in § 5.5.

Fix $\underline{w} = (w_1, w_2, w_3, w_4)$ with $w_1 > w_2 > w_3 > w_4 \geq 0$ even integers. Fix as well once and for all a global Arthur parameter

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\rm glob}(G)$$

such that the semisimple conjugacy class $\mathrm{St}(z_{\psi_\infty})$ in $\mathfrak{sl}_8(\mathbb{C})$ has the eigenvalues

$$\{\pm \frac{w_i}{2}, 1 \leq i \leq 4\}.$$

We shall make explicit Arthur's multiplicity formula for the number

$$m'(\psi) = \sum_{\pi \in \Pi(\psi) \cap \Pi_{\underline{w}}(G)} m(\pi),$$

following §3.30.2. It will be convenient to introduce the number

$$e(\underline{w}) = \begin{cases} 1 & \text{if } w_4 > 0, \\ 2 & \text{otherwise.} \end{cases}$$

Recall also the groups

$$\mathrm{C}_\psi \subset \mathrm{C}_{\psi_\infty} \subset \widehat{G} = \mathrm{SO}_8(\mathbb{C}).$$

Denote by $J \subset \{1, \cdots, k\}$ the set of integers j such that $n_j \equiv 1 \bmod 2$. It follows from Lemma 3.23 that:

(i) If $j \notin J$ then $n_j \equiv 0 \bmod 4$.

(ii) $|J| = 0$ or 2, and in this latter case $\sum_{j \in J} n_j \equiv 0 \bmod 4$.

We will say that ψ is *even-stable* if $k = 1$, and *odd-stable* if $k = 2$ and $J = \{1, 2\}$.

7.2.1. *The even-stable cases.* We have $\mathrm{C}_\psi = Z$ so the multiplicity formula trivially gives $m'(\psi) = e(\underline{w})$. One has $\psi(\pi) = \pi_1[d_1]$ with $d_1 | 8$, $\pi_1 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}(8/d_1))$, and $(-1)^{d_1 - 1} s(\pi_1) = -1$.

Case (i) : (*tempered case*) $\psi = \pi_1$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_8)$, this is the first unknown we want to count.

Case (ii) : $\psi = \pi_1[2]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_4)$. This occurs if and only if $w_1 - w_2 = w_3 - w_4 = 2$ and π_1 has Hodge weights $w_1 - 1, w_3 - 1$.

Case (iii) : $\psi = \pi_1[4]$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$. This occurs if and only if $w_1 = w_4 + 6$ and π_1 has Hodge weight $w_1 - 3$.

7.2.2. *The odd-stable cases.* We have again $\mathrm{C}_\psi = Z$ so the multiplicity formula trivially gives $m'(\psi) = 1$. One has $\psi(\pi) = \pi_1[d_1] \oplus \pi_2[d_2]$ with n_1, n_2, d_1 and d_2 odd. These cases only occur when $w_4 = 0$.

Case (iv) : $d_1 = n_2 = 1$, i.e $\psi = \pi_1 \oplus [1]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_7)$, which is the second unknown we want to count.

Case (v) : $d_1 = d_2 = 1$, $n_1 = 5$, i.e. $\psi = \pi_1 \oplus \pi_2$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_5)$ and $\pi_2 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_3)$.

Case (vi) : $d_1 = n_1 = 5$, $d_2 = 1$, i.e. $\psi = [5] \oplus \pi_2$ where $\pi_2 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_3)$. In this case $w_2 = 4$.

Case (vii) : $d_1 = 1$, $n_1 = 5$, $d_2 = 3$, i.e. $\psi = \pi_1 \oplus [3]$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_5)$. In this case $w_3 = 2$.

Case (viii) : $d_1 = 7$, $n_2 = 1$, i.e. $\psi = [7] \oplus [1]$. This occurs if and only if $\underline{w} = (6, 4, 2, 0)$ and π is then the trivial representation of G.

7.2.3. *Endoscopic cases of type* $(n_1, n_2) = (4, 4)$. In this case $k = 2$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2]$$

and $\mathrm{C}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 (or s_2) and the center Z. One will have to describe $\rho^\vee(s_1)$ and $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\mathrm{Min}(d_1,d_2)}$ in each case. Recall that $\rho^\vee : \mathrm{C}_{\psi_\infty} \to \{\pm 1\}$ is the fundamental character defined in § 3.30.2.

Case (ix) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1, \pi_2 \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_4)$. Denote by $a > b$ the Hodge weights of π_1 and by $c > d$ the ones of π_2. We may assume $a > c$. One has $\{a, b, c, d\} = \{w_1, w_2, w_3, w_4\}$. Moreover $\varepsilon_\psi(s_1) = 1$ as all the d_i are 1 (tempered case), so $m'(\psi) \neq 0$ if, and only if, $\rho^\vee(s_1) = 1$, i.e. if $a > c > b > d$. In other words,

$$m'(\psi) = \begin{cases} e(\underline{w}) & \text{if } a > c > b > d, \\ 0 & \text{otherwise.} \end{cases}$$

Case (x) : $d_1 = 2$, $d_2 = 1$, i.e. $\psi = \pi_1[2] \oplus \pi_2$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ and $\pi_2 \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_4)$. If a is the Hodge weight of π_1 and $b > c$ are the Hodge weights of π_2 then $\{w_1, w_2, w_3, w_4\} = \{a + 1, a - 1, b, c\}$. One has

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = (-1)^{\mathrm{Max}(a,b)+\mathrm{Max}(a,c)}.$$

On the other hand $\rho^\vee(s_1) = -1$. It follows that

$$m'(\psi) = \begin{cases} e(\underline{w}) & \text{if } b > a > c, \\ 0 & \text{otherwise.} \end{cases}$$

Case (xi) : $d_1 = d_2 = 2$, i.e. $\psi = \pi_1[2] \oplus \pi_2[2]$ where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ have respective Hodge weights $w_1 - 1$ and $w_3 - 1$. One has $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2) = 1$ and $\rho^\vee(s_1) = -1$. It follows that

$$m'(\psi) = 0$$

in all cases.

7.2.4. *Endoscopic cases of type* $(n_1, n_2, n_3) = (4, 3, 1)$. In this case $k = 3$, $w_4 = 0$,

$$\psi = \pi_1[d_1] \oplus \pi_2[d_2] \oplus [1]$$

and $\mathrm{C}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^2$. It follows that C_ψ is generated by s_1 and the center Z. We have

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)^{\mathrm{Min}(d_1,d_2)}\varepsilon(\pi_1).$$

Case (xii) : (*tempered case*) $d_1 = d_2 = 1$, i.e. $\pi_1 \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_4)$ and $\pi_2 \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_3)$. Denote by $a > b$ the Hodge weights of π_1 and c the one of π_2. One has $\{a, b, c\} = \{w_1, w_2, w_3\}$ and $\varepsilon_\psi = 1$. The multiplicity is thus nonzero if and only if $\rho^\vee(s_1) = 1$, i.e. $a > c > b$:

$$m'(\psi) = \begin{cases} 1 & \text{if } a > c > b, \\ 0 & \text{otherwise.} \end{cases}$$

Case (xiii) : $d_1 = 2$, $d_2 = 1$, i.e. $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ and $\pi_2 \in \Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_3)$. If a is the Hodge weight of π_1 and if b is the one of π_2, then $\{a + 1, a - 1, b\} = \{w_1, w_2, w_3\}$.

One has $\rho^\vee(s_1) = -1$. On the other hand, $\varepsilon_\psi(s_1) = \varepsilon(\pi_1 \times \pi_2)\varepsilon(\pi_1) = (-1)^{\mathrm{Max}(a,b)+1}$. It follows that

$$m'(\psi) = \begin{cases} 1 & \text{if } b > a, \\ 0 & \text{otherwise.} \end{cases}$$

Case (xiv) : $d_1 = 1$, $d_2 = 3$, i.e. $\psi = \pi_1 \oplus [3] \oplus [1]$ with $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_4)$ of Hodge weights $w_1 > w_2$ (here $w_3 = 2$). We have $\varepsilon_\psi = 1$ and $\rho^\vee(s_1) = -1$, so

$$m'(\psi) = 0$$

in all cases.

Case (xv) : $d_1 = 2$, $d_2 = 3$, i.e. $\psi = \pi_1[2] \oplus [3] \oplus [1]$ with $\pi \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ of Hodge weight $a = w_1 - 1$. We have $\varepsilon_\psi(s_1) = \varepsilon(\pi_1) = (-1)^{\frac{a+1}{2}}$ and $\rho^\vee(s_1) = -1$, so

$$m'(\psi) = \begin{cases} 1 & \text{if } a \equiv 1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

7.3. **Conclusions.** The inspection of each case above, and our previous computation of $\mathrm{S}(w)$, $\mathrm{S}(w,v)$, $\mathrm{S}(w,v,u)$, $\mathrm{O}^*(w)$, $\mathrm{O}(w,v)$ and $\mathrm{O}^*(w,v)$, allow to compute the contribution of each endoscopic type except two, namely the even and odd stable and tempered types. The contribution of the even-stable tempered type is exactly

$$\mathrm{O}(w_1, w_2, w_3, w_4)$$

when $w_4 \neq 0$, and $2 \cdot \mathrm{O}(w_1, w_2, w_3, w_4)$ when $w_4 = 0$. The contribution of the odd-stable tempered type is

$$\mathrm{O}^*(w_1, w_2, w_3).$$

This concludes the proof of Theorem 1.5. The Corollary 1.18 and Tables 9 and 10 follow form these computations.

Let us mention that we also have in our database the computation of the number of discrete automorphic representations of the non-connected group O_8 of any given infinitesimal character. We shall not say more about this in this paper however.

8. Description of $\Pi_{\mathrm{disc}}(\mathrm{G}_2)$

8.1. **The semisimple definite G_2 over $\mathbb{Z}$.** Consider the unique semisimple $\mathbb{Z}$-group G of type G_2 such that $G(\mathbb{R})$ is compact, namely the automorphism group scheme over $\mathbb{Z}$ of "the" ring of Coxeter octonions (see [Cox46],[BS59],[Gro96, §4]). We shall simply write G_2 for this $\mathbb{Z}$-group G. The reduction map $\mathrm{G}_2(\mathbb{Z}) \to \mathrm{G}_2(\mathbb{F}_2)$ is an isomorphism and

$$|\mathrm{G}_2(\mathbb{Z})| = 2^6 \cdot 3^3 \cdot 7 = 12096.$$

The $\mathbb{Z}$-group G_2 admits a natural homomorphism into the $\mathbb{Z}$-group SO_7 by its action on the lattice $L \simeq \mathrm{E}_7$ of pure Coxeter octonions. For a well-chosen basis of $L[1/2]$, it follows from [CNP96, §4] that the group $\mathrm{G}_2(\mathbb{Z})$ becomes the subgroup of $\mathrm{GL}_7(\mathbb{Z}[1/2])$ generated by the two elements

$$\frac{1}{2}\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

This allows not only to enumerate (with the computer) all the elements of $\mathrm{G}_2(\mathbb{Z}) \subset \mathrm{GL}_7(\mathbb{Z}[1/2])$ but to compute as well their characteristic polynomials (see [CR]). The list of the twelve obtained characteristic polynomials, together with the number of elements with that characteristic polynomial, is given in Table 1 (we denote by Φ_d the d-th cyclotomic polynomial). One easily checks for instance with this table that if $\chi(g,t)$ denotes the characteristic polynomial of g then

$$\frac{1}{12096} \sum_{g \in \mathrm{G}_2(\mathbb{Z})} \chi(g,t) = t^7 - t^4 + t^3 - 1,$$

which is compatible with well-known fact that $\dim(\Lambda^3 L \otimes \mathbb{C})^{\mathrm{G}_2(\mathbb{C})} = 1$.

Table 1. Characteristic polynomials of the elements of $\mathrm{G}_2(\mathbb{Z}) \subset \mathrm{SO}_7(\mathbb{R})$.

Char. Poly.	$\sharp$	Char. Poly.	$\sharp$
$\Phi_1\Phi_3^3$	56	$\Phi_1\Phi_3\Phi_6^2$	504
$\Phi_1\Phi_2^2\Phi_4^2$	378	$\Phi_1^3\Phi_4^2$	378
$\Phi_1^3\Phi_2^4$	315	Φ_1^7	1
$\Phi_1\Phi_2^2\Phi_3\Phi_6$	2016	$\Phi_1\Phi_3\Phi_{12}$	3024
$\Phi_1\Phi_2^2\Phi_8$	1512	$\Phi_1\Phi_4\Phi_8$	1512
$\Phi_1\Phi_7$	1728	$\Phi_1^3\Phi_3^2$	672

8.2. **Polynomial invariants for** $\mathrm{G}_2(\mathbb{Z}) \subset \mathrm{G}_2(\mathbb{R})$**.** To describe the finite dimensional representations of $\mathrm{G}_2(\mathbb{R})$ we fix a maximal torus T and a system of positive roots Φ^+ of $(\mathrm{G}_2(\mathbb{R}), T)$. Let $X = \mathrm{X}^*(T)$, $X^\vee = \mathrm{X}_*(T)$ and denote by $\langle \cdot, \cdot \rangle$ the canonical perfect pairing between them.

Let $\alpha, \beta \in X$ the simple roots in Φ^+ where α is short and β is long. The positive roots are thus

$$\alpha, \beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha, 2\beta + 3\alpha,$$

where $\alpha, \beta + \alpha$ and $\beta + 2\alpha$ are short, and $X = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. The inverse root system is again of type G_2, with simple positive roots $\alpha^\vee, \beta^\vee \in X^\vee$ with $\alpha^\vee$ long, and where

$$\langle \alpha, \beta^\vee \rangle = -1 \text{ and } \langle \beta, \alpha^\vee \rangle = -3.$$

It follows that the dominant weights are the $a\alpha + b\beta$ where $a, b \in \mathbb{Z}$ satisfy $2b \geq a \geq 3b/2$. The fundamental representations with respective fundamental weights

$$\omega_1 = 2\alpha + \beta, \quad \omega_2 = 3\alpha + 2\beta$$

will be denoted by V_7 and V_{14}, because of their respective dimension 7 and 14. One easily checks that $V_7 = L \otimes \mathbb{C}$ and V_{14} is the adjoint representation. The half-sum of positive roots is $\rho = 5\alpha + 3\beta = \omega_1 + \omega_2$.

Definition 8.3. *If $w > v$ are even non-negative integers, we denote by $U_{w,v}$ the irreducible representation of $\mathrm{G}_2(\mathbb{R})$ with highest weight*

$$\frac{w - v - 2}{2}\,\omega_1 + \frac{v - 2}{2}\,\omega_2.$$

We also denote by $\Pi_{w,v}(\mathrm{G}_2)$ the subset of $\pi \in \Pi_{\mathrm{disc}}(\mathrm{G}_2)$ such that $\pi_\infty \simeq U_{w,v}$, and set $m(w, v) = \sum_{\pi \in \Pi_{w,v}} m(\pi)$.

This curious looking numbering has the following property. If

$$\varphi : \mathrm{W}_\mathbb{R} \longrightarrow \widehat{\mathrm{G}_2}$$

is the Langlands parameter of $U_{w,v}$, and if $\rho_7 : \widehat{\mathrm{G}_2} \to \mathrm{SO}_7(\mathbb{C})$ is the 7-dimensional irreducible representation of $\widehat{\mathrm{G}_2}$, then $\rho_7 \circ \varphi$ is the representation

$$\mathrm{I}_{w+v} \oplus \mathrm{I}_w \oplus \mathrm{I}_v \oplus \varepsilon.$$

Indeed, the weights of ρ_7 are $0, \pm\beta^\vee, \pm(\alpha^\vee + \beta^\vee), \pm(\alpha^\vee + 2\beta^\vee)$.

Observe that $\rho_7 \circ \varphi$ determines the equivalence class of φ. This is a special case of the fact that the conjugacy class of any element $g \in \mathrm{G}_2(\mathbb{R})$ (resp. of any semisimple element in $\mathrm{G}_2(\mathbb{C})$) is uniquely determined by its characteristic polynomial in V_7. Indeed, this follows from the identity

$$V_{14} \oplus V_7 \simeq \Lambda^2 V_7.$$

This property makes the embedding $\mathrm{G}_2(\mathbb{C}) \subset \mathrm{SO}_7(\mathbb{C})$ quite suitable to study G_2 and its subgroups. In particular, Table 1 leads to a complete determination of the semisimple conjugacy classes in $\mathrm{G}_2(\mathbb{R})$ of the elements of $\mathrm{G}_2(\mathbb{Z})$, which is the ingredient we need to apply the method of § 2.5.

Van der Blij and Springer have shown in [BS59] that $|\mathrm{Cl}(\mathrm{G}_2)| = 1$ (se also [Gro96, §5]), it follows that

$$m(w,v) = \dim U_{w,v}^{\mathrm{G}_2(\mathbb{Z})}.$$

See Table 5 and the url [CR] for a sample of computations. As we shall see below, one should have $m(\pi) = 1$ for each $\pi \in \Pi_{\mathrm{disc}}(G)$, and we thus expect that $m(w,v) = |\Pi_{w,v}(\mathrm{G}_2)|$.

Automorphic forms for the $\mathbb{Q}$-group G_2 have been previously studied by Gross, Lansky, Pollack and Savin: see [GS98], [GP05], [LP02] and [Pol98]. Although most of the automorphic forms studied by those authors are Steinberg at one finite place, they may be trivial at the infinite place. Pollack and Lansky are also able to compute some Hecke eigenvalues in some cases.

8.4. **Endoscopic classification of $\Pi_{\mathrm{disc}}(\mathrm{G}_2)$.** We recall Arthur's conjectural description of $\Pi_{\mathrm{disc}}(\mathrm{G}_2)$, following his general conjecture in [Art89]. *Most of the results here will thus be conditional to the existence of the group $\mathcal{L}_{\mathbb{Z}}$ discussed in Appendix B and to these conjectures*, that we will make explicit. All the facts stated below about the structure of G_2 can be simply checked on its root system. We refer to [GG05] for a complete analysis of Arthur's conjectures for the split groups of type G_2, in a much greater generality than we actually need here, and for a survey of the known results.

A global discrete Arthur parameter for the $\mathbb{Z}$-group G_2 is a $\widehat{\mathrm{G}_2}$-conjugacy class of morphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{\mathrm{G}_2}$$

such that:

(a) $\mathrm{Im}\,\psi$ has a finite centralizer in $\widehat{\mathrm{G}_2}$,

(b) $\psi_\infty = \psi_{|\mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})}$ is an Adams-Johnson parameter for $\mathrm{G}_2(\mathbb{R})$ (see Appendix A).

Observe that by property (b) the centralizer C_{ψ_∞} of $\mathrm{Im}\,\psi_\infty$ in $\widehat{\mathrm{G}_2}$ is an elementary abelian 2-group, hence so is the centralizer $\mathrm{C}_\psi \subset \mathrm{C}_{\psi_\infty}$ of ψ. As $\mathcal{L}_{\mathbb{Z}}$ is connected, observe that the Zariski-closure of $\mathrm{Im}\psi$ is a connected complex reductive subgroup of $\widehat{\mathrm{G}_2}$.

This severely limits the possibilities for $\mathrm{Im}\,\psi$. Up to conjugacy there are exactly 3 connected complex reductive subgroups of $\widehat{\mathrm{G}_2}$ whose centralizers are elementary abelian 2-groups:

(i) the group $\widehat{\mathrm{G}_2}$ itself, with trivial centralizer,

(ii) a principal $\mathrm{PGL}_2(\mathbb{C})$ homomorphism, again with trivial centralizer,

(iii) the centralizer $H_s \simeq \mathrm{SO}_4(\mathbb{C})$ of an element s of order 2, whose centralizer is the center $\langle s \rangle$ of H_s.

Recall that up to conjugacy there is a unique element s of order 2 in $\widehat{\mathrm{G}_2}$. The isomorphism $H_s \simeq \mathrm{SO}_4(\mathbb{C})$ in (iii) is actually canonical up to inner automorphisms as H_s is its own normalizer in $\widehat{\mathrm{G}_2}$. Indeed, one has two distinguished injective homomorphisms $\mathrm{SL}_2(\mathbb{C}) \to H_s$, one of which being a short radicial $\mathrm{SL}_2(\mathbb{C})$ and the other one being a long radicial $\mathrm{SL}_2(\mathbb{C})$ (the long and short roots being orthogonal)

We shall need some facts about the restrictions of V_7 and V_{14} to these groups. We denote by $\nu_{\rm long}$ and $\nu_{\rm short}$ the two 2-dimensional irreducible representations of H_s which are respectively non-trivial on the long and short $\mathrm{SL}_2(\mathbb{C})$ inside H_s.

Lemma 8.5. *Let* $s \in \widehat{\mathrm{G}_2}$ *be an element of order* 2.

(i) $(V_7)_{|H_s} = \nu_{\rm long} \otimes \nu_{\rm short} \oplus \mathrm{Sym}^2 \nu_{\rm short}$,

(ii) $(V_{14})_{|H_s} = \mathrm{Sym}^2 \nu_{\rm long} \oplus \mathrm{Sym}^2 \nu_{\rm short} \oplus \mathrm{Sym}^3 \nu_{\rm short} \otimes \nu_{\rm long}$.

Moreover, the restriction of V_7 *to a principal* $\mathrm{PGL}_2(\mathbb{C})$ *is isomorphic to* $\nu_7 = \mathrm{Sym}^6(\mathbb{C}^2)$.

If ψ is a global Arthur parameter for G_2, then $\rho_7 \circ \psi$ actually defines a global Arthur parameter for Sp_6, that we shall denote $\psi^{\rm SO}$. The previous lemma and discussion show that the equivalent class of $\psi^{\rm SO}$ determines the conjugacy class of ψ.

Fix a global Arthur parameter ψ as above. We denote by

$$\pi(\psi) \in \Pi(\mathrm{G}_2)$$

the unique representation π such that $c(\pi)$ is associated to ψ by the standard Arthur recipe. Explicitly, for each prime p we have $c_p(\pi) = \psi(\mathrm{Frob}_p \times e_p)$ (see § 3.18), and π_∞ is the unique representation of $\mathrm{G}_2(\mathbb{R})$ whose infinitesimal character is the one of the Langlands parameter φ_{ψ_∞} (assumption (b) on ψ). Arthur's conjectures describe $\Pi_{\rm disc}(\mathrm{G}_2)$ as follows. First, any $\pi \in \Pi_{\rm disc}(\mathrm{G}_2)$ should be of the form $\pi(\psi)$ for a unique ψ satisfying (a) and (b). Second, they describe $m(\pi(\psi))$ for each π as follows.

Case (i) : (*stable tempered cases*) $\psi^{\rm SO} \in \Pi_{\rm alg}^{\rm o}(\mathrm{PGL}_7)$. This is when $\mathrm{C}_\psi = 1$ and $\psi(\mathrm{SL}_2(\mathbb{C})) = 1$. In this case

$$m(\pi) = 1.$$

By Prop.B.5, a $\pi \in \Pi_{\rm alg}^{\rm o}(\mathrm{PGL}_7)$ has the form $\psi^{\rm SO}$ for a stable tempered ψ if and only if $c(\pi) \in \rho_7(\mathcal{X}(\widehat{G}_2(\mathbb{C})))$. It is equivalent to ask that $c(\pi) \times 1$, viewed as an element in $\mathcal{X}(\mathrm{SO}_8(\mathbb{C}))$ is invariant by a triality automorphism. Moreover, $\mathrm{Im}\,\psi$ is either isomorphic to the compact group G_2 or to $\mathrm{SO}(3)$. The latter case occurs if and only if $\pi(\psi)_\infty \simeq U_{w,v}$ where $v \equiv 2 \bmod 4$ and $w = 2v$, in which case it occurs exactly $\mathrm{S}(v/2)$ times.

Case (ii) : (*stable non-tempered case*) $\psi^{\rm SO} = [7]$. Then π is the trivial representation, the unique element in $\Pi_{4,2}(\mathrm{G}_2)$.

There are three other cases for which $\mathrm{Im}(\psi) = H_s$. In those cases we have

$$\mathrm{C}_\psi = \langle s \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

Arthur's multiplicity formula requires two ingredients. The first one is the character

$$\varepsilon_\psi : \mathrm{C}_\psi \to \mathbb{C}^\times$$

given by Arthur's general recipe [Art89]. This character is trivial if $\psi(\mathrm{SL}_2(\mathbb{C})) = 1$. Otherwise there are two distinct cases:

(i) $\nu_{\rm short}\circ\psi_{|{\rm SL}_2(\mathbb{C})} = \nu_2$ and $\nu_{\rm long}\circ\psi_{|\mathcal{L}_\mathbb{Z}} = r(\pi)$ for some $\pi \in \Pi_{\rm alg}({\rm PGL}_2)$ (see Appendix B). Then
$$\varepsilon_\psi(s) = \varepsilon(\pi) = (-1)^{(w+1)/2}$$
where w is the Hodge weight of π.

(ii) $\nu_{\rm long}\circ\psi_{|{\rm SL}_2(\mathbb{C})} = \nu_2$ and $\nu_{\rm short}\circ\psi_{|\mathcal{L}_\mathbb{Z}} = r(\pi)$ for some $\pi \in \Pi_{\rm alg}({\rm PGL}_2)$. Then $\varepsilon_\psi(s) = \varepsilon({\rm Sym}^3\pi)$. If w is the Hodge weight of π, observe that
$$\varepsilon({\rm Sym}^3\pi) = (-1)^{(w+1)/2+(3w+1)/2} = -1$$
for each w, thus ε_ψ is the non-trivial character in this case.

Observe that the a priori remaining case $\psi(\mathcal{L}_\mathbb{Z}) = 1$ does not occur as property (b) is not satisfied for such a ψ (the infinitesimal character z_{ψ_∞} is not regular).

The second ingredient is the restriction to ${\rm C}_\psi$ of the character $\rho^\vee : {\rm C}_{\psi_\infty} \to \mathbb{C}^\times$. The multiplicity formula will then take the form: $m(\pi) = 1$ if $\rho^\vee(s) = \varepsilon_\psi(s)$ and $m(\pi) = 0$ otherwise.

In order to compute $\rho^\vee(s)$ we fix $\widehat{T}$ a maximal torus in $\widehat{{\rm G}_2}$ such that ${\rm X}^*(\widehat{T}) = X^\vee$. Observe that the centralizer T' of $\rho_7(\widehat{T})$ in ${\rm SO}_7(\mathbb{C})$ is a maximal torus of the latter group. We consider the standard root system Φ' for $({\rm SO}_7(\mathbb{C}), T')$ recalled in § 2.5, in particular ${\rm X}^*(T') = \oplus_{i=1}^3 \mathbb{Z}e_i$. Then $\Phi^\vee = \Phi'_{|\widehat{T}}$ is a root system for $(\widehat{T}, \widehat{{\rm G}_2})$ with positive roots $(\Phi^\vee)^+ = (\Phi')^+_{|\widehat{T}}$: up to conjugating ρ_7 we may thus assume that $(\Phi^\vee)^+$ is the positive root system of § 8.2.

Lemma 8.6. *Under the assumptions above, we have $\rho^\vee(s) = e_2(\rho_7(s))$.*

Proof — Under the assumptions above, if $\lambda \in {\rm X}_*(\widehat{T})$ is such that $\rho_7(\lambda)$ is $(\Phi')^+$-dominant then λ is $(\Phi^\vee)^+$-dominant. We have already seen that the respective restriction to $\widehat{T}$ of e_1, e_2, e_3 are the elements $2\beta^\vee+\alpha^\vee$, $\beta^\vee+\alpha^\vee$ and $\beta^\vee$. The lemma follows from the identity
$$\rho^\vee = 5\beta^\vee + 3\alpha^\vee \equiv (e_2)_{|\widehat{T}} \bmod 2{\rm X}^*(\widehat{T}).$$
□

We can now make explicit the three remaining multiplicity formulae.

Case (iii) : (*tempered endoscopic case*) $\psi^{\rm SO} = \pi_{\rm long}\otimes\pi_{\rm short} \oplus {\rm Sym}^2\pi_{\rm short}$ where $\pi_{\rm short}, \pi_{\rm long} \in \Pi_{\rm alg}({\rm PGL}_2)$ have respective Hodge weights $w_{\rm short}, w_{\rm long}$. Of course, ${\rm Sym}^2\pi_{\rm short} \in \Pi^{\rm o}_{\rm alg}({\rm PGL}_3)$ has Hodge weight $2w_{\rm short}$ and $\pi_{\rm long}\otimes\pi_{\rm short} \in \Pi^{\rm o}_{\rm alg}({\rm PGL}_4)$ has Hodge weights $w_{\rm short}+w_{\rm long}$ and $|w_{\rm short}-w_{\rm long}|$. We also have $\varepsilon_\psi(s) = 1$. But by Lemma 8.5 (i) we have $e_2(\rho_7(s)) = 1$ if and only if
$$w_{\rm long}+w_{\rm short} > 2w_{\rm short} > w_{\rm long}-w_{\rm short},$$
thus $m(\pi) = 1$ in this case and $m(\pi) = 0$ otherwise.

Case (iv) : (*non-tempered endoscopic case* 1) $\psi^{\rm SO} = \pi[2] \oplus {\rm Sym}^2\pi$ where $\pi \in \Pi_{\rm alg}({\rm PGL}_2)$, say with Hodge weight w. We have seen that in this case $\varepsilon_\psi(s) = -1$.

On the other hand $e_2(\rho_7(s)) = -1$ if and only if $w - 1 < 2w$, which is always satisfied as $w > 1$. Arthur's multiplicity formula tells us that

$$m(\pi) = 1$$

in all cases.

Case (v) : (*non-tempered endoscopic case* 2) $\psi^{\mathrm{SO}} = \pi[2] \oplus [3]$ where $\pi \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$, say with Hodge weight w. We have seen that $\varepsilon_\psi(s) = \varepsilon(\pi) = (-1)^{(w+1)/2}$. Observe that $e_2(\rho_7(s)) = -1$ as $w - 1 > 3$. Arthur's multiplicity formula tells us then that

$$m(\pi) = 1 \Leftrightarrow w \equiv 1 \bmod 4.$$

Let us mention that the multiplicity formula for the Arthur's packets appearing in case (v) has been established for the split groups of type G_2 in [GG06].

8.7. **Conclusions.** The inspection of each case above and the well-known formula for $\mathrm{S}(w)$ allow to compute the conjectural number $\mathrm{G}_2(w,v)$ of $\pi \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathrm{PGL}_7)$ such that $c(\pi) \in \rho_7(\mathcal{X}(\widehat{\mathrm{G}_2}))$ and with Hodge weights $w + v > w > v$. Concretely,

$$\mathrm{G}_2(w,v) = m(w,v) - \delta_{w=4} - \mathrm{O}^*(w) \cdot \mathrm{O}(w+v,v)$$
$$-\delta_{w-v=2} \cdot \mathrm{S}(w-1) - \delta_{v=2} \cdot \delta_{w \equiv 0 \bmod 4} \cdot \mathrm{S}(w+1).$$

See Table 11 for a sample of results when $w + v \leq 58$. The Sato-Tate group of each of the associated π is conjecturally the compact group of type $\mathrm{G}_2 \subset \mathrm{SO}_7(\mathbb{R})$ (rather than $\mathrm{SO}_3(\mathbb{R})$ principally embedded in the latter) : this follows from Prop. B.5 as the motivic weight of $\mathrm{Sym}^6\pi$ for $\pi \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ is at least $66 > 58$.

9. Application to Siegel modular forms

9.1. **Vector valued Siegel modular forms of level 1.** We consider in this chapter the classical Chevalley $\mathbb{Z}$-group Sp_{2g}, whose dual group is $\mathrm{SO}_{2g+1}(\mathbb{C})$. Let

$$\underline{w} = (w_1, w_2, \cdots, w_g)$$

where the w_i are even positive integers such that $w_1 > w_2 > \cdots > w_g$. To such a $\underline{w}$ we may associate a semisimple conjugacy class $z_{\underline{w}}$ in $\mathfrak{so}_{2g+1}(\mathbb{C})$, namely the class with eigenvalues $\pm\frac{w_i}{2}$ for $i = 1, \cdots, g$, and 0. Recall that for any such $\underline{w}$, there is an L-packet of discrete series with infinitesimal characters $z_{\underline{w}}$, and that this L-packet contains two "holomorphic" discrete series which are outer conjugate by $\mathrm{PGSp}_{2g}(\mathbb{R})$. We make once and for all a choice for the holomorphic ones (hence for the anti-holomorphic as well).

Recall the space

$$\mathrm{S}_{\underline{w}}(\mathrm{Sp}_{2g}(\mathbb{Z}))$$

of holomorphic vector valued Siegel modular forms with infinitesimal character $\mathfrak{z}_{\underline{w}}$. If (ρ, V) is the irreducible representation of $\mathrm{GL}_g(\mathbb{C})$ with standard highest weight $m_1 \geq m_2 \geq \cdots \geq m_g$, and if $m_g > g$, recall that a (ρ, V)-valued Siegel modular form has infinitesimal character $z_{\underline{w}}$ where $\underline{w} = (w_i)$ and $w_i = 2(m_i - i)$ for each $i = 1, \cdots, g$ (see e.g. [AS01, §4.5]). Denote also

$$\Pi_{\underline{w}}(\mathrm{Sp}_{2g})$$

the set of $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2g})$ such that π_∞ is the holomorphic discrete series with infinitesimal character $z_{\underline{w}}$. Such a π is tempered at the infinite place, thus it follows from a result of Wallach [Wal84, Thm. 4.3] that $\Pi_{\underline{w}}(\mathrm{Sp}_{2g}) \subset \Pi_{\mathrm{cusp}}(\mathrm{Sp}_{2g})$. By Arthur's multiplicity formula, $m(\pi) = 1$ for each $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2g})$, it is well-known that this implies

$$\dim \mathrm{S}_{\underline{w}}(\mathrm{Sp}_{2g}(\mathbb{Z})) = |\Pi_{\underline{w}}(\mathrm{Sp}_{2g})|.$$

By Lemma 3.23, Arthur's multiplicity formula allows to express $|\Pi_{\underline{w}}(\mathrm{Sp}_{2g})|$ in terms of various $\mathrm{S}(-)$, $\mathrm{O}(-)$ and $\mathrm{O}^*(-)$. We shall give now the two ingredients needed to make this computation in general and we shall apply them later in the special case $g = 3$.

9.2. **Two lemmas on holomorphic discrete series.** Let

$$\varphi_{\underline{w}} : \mathrm{W}_{\mathbb{R}} \to \mathrm{SO}_{2g+1}(\mathbb{C})$$

be the discrete series Langlands parameter with infinitesimal character $z_{\underline{w}}$, and let

$$\Pi(\varphi_{\underline{w}})$$

be the associated L-packet of discrete series representations of $\mathrm{Sp}_{2g}(\mathbb{R})$ with infinitesimal character $z_{\underline{w}}$. Recall that the centralizer of $\varphi_{\underline{w}}(\mathrm{W}_{\mathbb{C}})$ in $\mathrm{SO}_{2g+1}(\mathbb{C})$ is a maximal torus $\widehat{T}$ in $\mathrm{SO}_{2g+1}(\mathbb{C})$ and that the centralizer $\mathrm{C}_{\varphi_{\underline{w}}}$ of $\varphi_{\underline{w}}(\mathrm{W}_{\mathbb{R}})$ is the 2-torsion subgroup of $\widehat{T}$. There is also a unique Borel subgroup $\widehat{B} \supset \widehat{T}$ for which the element $\lambda \in \mathrm{X}_*(\widehat{T})[1/2]$ such that $\varphi_{\underline{w}}(z) = (z/\overline{z})^\lambda$ for all $z \in \mathrm{W}_{\mathbb{C}}$ is dominant with respect to $\widehat{B}$.

We consider the setting and notations of §A.1 with $G = \mathrm{Sp}_{2g}(\mathbb{C})$. The strong forms $t \in \mathcal{X}_1(T)$ such that $G_t \simeq \mathrm{Sp}_{2g}(\mathbb{R})$ are the ones such that

$$t^2 = -1,$$

and they form a single W-orbit. Let us fix an isomorphism between $\mathrm{Sp}_{2g}(\mathbb{R})$ and $G_{[t]}$ for t in this W-orbit, which thus identify $\Pi(\varphi_{\underline{w}})$ with $\Pi(\varphi_{\underline{w}}, G_{[t]})$ (§A.5). This being done, Shelstad's parameterization gives a canonical injective map (see §A.7)

$$\tau : \Pi(\varphi_{\underline{w}}) \to \mathrm{Hom}(\mathrm{C}_{\varphi_{\underline{w}}}, \mathbb{C}^\times).$$

(We may replace $\mathrm{S}_{\varphi_{\underline{w}}}$ by $\mathrm{C}_{\varphi_{\underline{w}}}$ in the range as $\mathrm{Sp}_{2g}(\mathbb{R})$ is split, see Cor.A.14). Our first aim is to determine the image of π_{hol} and π_{ahol}, namely the holomorphic and anti-holomorphic discrete series in $\Pi(\varphi_{\underline{w}})$.

For a well-chosen $\mathbb{Z}$-basis (e_i) of $\mathrm{X}^*(\widehat{T})$, the positive roots of $(\mathrm{SO}_{2g+1}(\mathbb{C}), \widehat{B}, \widehat{T})$ are $\{e_i, i = 1, \cdots, g\} \cup \{e_i \pm e_j, 1 \leq i < j \leq g\}$ as in § 2.5. Let $(e_i^*) \in \mathrm{X}^*(T)$ denote the dual basis of (e_i). The set of positive roots of $(\mathrm{Sp}_{2g}(\mathbb{C}), B, T)$ dual to the positive root system above is the set $\{2e_i^*, i = 1, \cdots, g\} \cup \{e_i^* \pm e_j^*, 1 \leq i < j \leq g\}$. If $t \in T$ we also write $t = (t_i)$ where $t_i = e_i^*(t)$ for each $i = 1, \cdots, g$.

Lemma 9.3. *The Shelstad characters of* π_{hol} *and* π_{ahol} *are the restrictions to* $\mathrm{C}_{\varphi_{\underline{w}}}$ *of the following elements of* $\mathrm{X}^*(\widehat{T})$*:*

$$e_1 + e_3 + e_5 + \cdots + e_{2[(g-1)/2]+1} \text{ and } e_2 + e_4 + e_6 + \cdots + e_{2[g/2]}.$$

Proof — Let $t \in \mathcal{X}_1(T)$ such that $t^2 = -1$. Recall that $\mathrm{Int}(t)$ is a Cartan involution of G_t and that K_t is the associated maximal compact subgroup of $G_t \simeq \mathrm{Sp}_{2g}(\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{k}_t \oplus \mathfrak{p}$ the Cartan decomposition relative to $\mathrm{Int}(t)$. We have $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where $\mathfrak{p}_\pm \subset \mathfrak{p}$ are two distinct irreducible K_t-submodules for the adjoint action. As a general fact, the representation $\pi_t(\lambda)$ is a holomorphic or anti-holomorphic discrete series of G_t if and only if $\mathfrak{b}$ is included in either $\mathfrak{k}_t \oplus \mathfrak{p}_+$ or in $\mathfrak{k}_t \oplus \mathfrak{p}_-$. In those cases, $\mathfrak{k}_t$ is thus a standard Levi subalgebra of $(\mathfrak{g}, \mathfrak{b}, \mathfrak{t})$ isomorphic to $\mathfrak{gl}_g$. There is a unique such Lie algebra, namely the one with positive roots the $e_i^* - e_j^*$ for $i < j$. It follows that $\pi_t(\lambda)$ is a holomorphic discrete series if and only if the positive roots of T in $\mathfrak{k}_t$ are the $e_i^* - e_j^*$ for each $1 \leq i < j \leq g$, i.e. if $t_i = t_j$ for $j \neq i$. As $t^2 = -1$, the two possibilities are thus the elements

$$t_+ = (i, i, \ldots, i) \text{ and } t_- = (-i, -i, \ldots, -i).$$

We have $t_b = e^{i\pi\rho^\vee} = (i^{2g-1}, \ldots, i, -i, i)$ (see §A.7), so $t_\pm t_b^{-1} = \pm(\ldots, -1, 1, -1, 1)$. Let $\mu = e_1 + e_3 + \ldots$ and $\mu' = e_2 + e_4 + \ldots$ be the two elements of $\mathrm{X}_*(T) = \mathrm{X}^*(\widehat{T})$ given in the statement. One concludes as

$$e^{i\pi\mu} = (-1, 1, -1, \ldots) \text{ and } e^{i\pi\mu'} = (1, -1, 1, \ldots).$$

□

The second ingredient is to determine which Adams-Johnson packets $\Pi(\psi)$ of $\mathrm{Sp}_{2g}(\mathbb{R})$ contains a holomorphic discrete series.

Lemma 9.4. *Let $\psi : \mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2g+1}(\mathbb{C})$ be an Adams-Johnson parameter for $\mathrm{Sp}_{2g}(\mathbb{C})$. Then $\widetilde{\Pi}(\psi)$ contains discrete series of $\mathrm{Sp}_{2g}(\mathbb{R})$ if and only if the underlying representation of $\mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^{2g+1}$ does not contain any $1 \otimes \nu_q$ or $\varepsilon \otimes \nu_q$ where $q > 1$.*

Furthermore, if ψ has this property then the holomorphic and anti-holomorphic discrete series of $\mathrm{Sp}_{2g}(\mathbb{R})$ belong to $\Pi(\psi)$.

Proof — Let T, B, L and λ be attached to ψ as in §A.2 and §A.5, recall that $L \subset \mathrm{Sp}_{2g}(\mathbb{C})$ is a Levi factor of a parabolic subgroup. From the last example of §A.2, from which we take the notations, we have

$$L \simeq \mathrm{Sp}_{d-1}(\mathbb{C}) \times \prod_{i \neq 0} \mathrm{GL}_{d_i}(\mathbb{C}).$$

Moreover, the underlying representation of $\mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^{2g+1}$ does not contain any $1 \otimes \nu_q$ or $\varepsilon \otimes \nu_q$ where $q > 1$ if and only if $d = 1$. On the other hand, $\Pi(\psi)$ contains a discrete series $\pi_t(\lambda)$ of $\mathrm{Sp}_{2g}(\mathbb{R})$ if and only if there is a $t \in \mathfrak{X}_1(T) \cap \mathrm{Z}(L)$ such that $t^2 = -1$, by Lemma A.6. As $\mathrm{Sp}(d-1, \mathbb{C})$ does not contain any element of square -1 in its center for $d > 1$, the first assertion follows.

Assume now that L is a product of general complex linear groups. It is equivalent to ask that the positive roots of L with respect to (B, T) are among the $e_i^* - e_j^*$ for $i < j$. In particular, the element $t_0 = \pm(i, i, \cdots, i) \in \mathfrak{X}_1(T)$ is in the center of L, thus $\pi_{t_0,B}(\lambda) \in \widetilde{\Pi}(\psi)$ by Lemma A.6 and Lemma A.9. But we have seen in the proof of Lemma 9.3 that this is a holomorphic/anti-holomorphic discrete series. □

The difference between π_{hol} and π_{ahol} is not really meaningful four our purposes, and we will not need to say exactly which of the two characters in Lemma 9.3 corresponds to e.g. π_{hol} (of course this would be possible if we had defined π_{hol} more carefully). More importantly, let

$$\chi = \sum_{i=1}^{g} e_i$$

be the sum of the two elements of the statement of Lemma 9.3. Fix $\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\mathrm{glob}}(\mathrm{Sp}_{2g})$ with infinitesimal character $z_{\underline{w}}$, one has canonical embeddings

$$\mathrm{C}_\psi \subset \mathrm{C}_{\psi_\infty} \subset \mathrm{C}_{\varphi_{\underline{w}}}$$

by §3.21 and §A.7, as $(\psi_\infty)_{\mathrm{disc}} = \varphi_{\underline{w}}$. Recall that C_ψ is generated by elements s_i as in §3.27.

Lemma 9.5. *For each $i = 1, \cdots, k$ such that n_i is even we have $\chi(s_i) = 1$.*

Proof — Indeed, it follows from Lemma 3.23 that if n_i is even then $n_i \equiv 0 \bmod 4$. □

Assume now that $\Pi_\infty(\psi)$ contains holomorphic discrete series, i.e. that $n_i \neq d_i$ for each i such that $n_i > 1$ by Lemma 9.4. In this case it follows from Lemma A.9 that the characters of $\pi_{\rm hol}$ and $\pi_{\rm ahol}$ viewed as elements of $\Pi_\infty(\psi)$ are again the two characters of Lemma 9.3. But it follows then from the Lemma 9.5 above that the multiplicity formula is the same for the two $\pi \in \Pi(\psi)$ such that π_∞ is either holomorphic or anti-holomorphic.

To conclude this paragraph let us say a word about the choice of the isomorphism that we fixed between $\mathrm{Sp}_{2g}(\mathbb{R})$ and $G_{[t]}$ (for $t^2 = -1$), which allowed to fix the parameterization τ. Consider for this the order 2 outer automorphism of $\mathrm{Sp}_{2g}(\mathbb{R})$ obtained as the conjugation by any element of $\mathrm{GSp}_{2g}(\mathbb{R})$ with similitude factor -1. It defines in particular element defines an involution of $\Pi(\varphi_{\underline{w}}, G_{[t]})$ and we want to check the effect of this involution on Shelstad's parameterization. The next lemma shows that it is quite benign.

Lemma 9.6. *If* $\pi \in \Pi(\varphi_{\underline{w}})$*, then* $\tau(\pi \circ \theta) = \tau(\pi) + \chi$.

Proof — Fix some t such that $t^2 = -1$ and view θ as an outer automorphism of G_t. A suitable representant of θ in $\mathrm{Aut}(G_t)$ preserves (K_t, T_c), and the automorphism of T_c obtained this way is well-defined up to $\mathrm{W}(K_t, T_c)$. It is a simple exercise to check that it coincides here with the class of the inversion $t \mapsto t^{-1}$ of T_c. As $-1 \in \mathrm{W}(G, T)$, it follows that $\pi_t(\lambda) \circ \theta = \pi_{t^{-1}}(\lambda)$. In other words, $\tau_0(\pi \circ \theta) = -\tau_0(\pi)$. As $\tau(\pi) = \tau_0(\pi) - \rho^\vee$, it follows that

$$\tau(\pi \circ \theta) = -\tau(\pi) - 2\rho^\vee.$$

But observe that $2\rho^\vee = \chi \bmod 2\mathrm{X}^*(\widehat{T})$. As $\mathrm{C}_{\varphi_{\underline{w}}}$ is an elementary abelian 2-group, the lemma follows. □

It follows then from Lemma 9.5 that the choice of our isomorphism has no effect on the multiplicity formula for the $\pi \in \Pi(\psi)$ such that π_∞ is either holomorphic or anti-holomorphic.

9.7. **An example: the case of genus** 3. We shall now describe the endoscopic classification of $\Pi_{\underline{w}}(\mathrm{Sp}_6)$ for any $\underline{w} = (w_1, w_2, w_3)$. As an application, we will deduce in particular the following proposition stated in the introduction.

Proposition 9.8.** $\dim \mathrm{S}_{w_1,w_2,w_3}(\mathrm{Sp}_6(\mathbb{Z})) = \mathrm{O}^*(w_1, w_2, w_3) + \mathrm{O}(w_1, w_3) \cdot \mathrm{O}^*(w_2)$

$$+\delta_{w_2 \equiv 0 \bmod 4} \cdot (\delta_{w_2 = w_3+2} \cdot \mathrm{S}(w_2 - 1) \cdot \mathrm{O}^*(w_1) + \delta_{w_1 = w_2+2} \cdot \mathrm{S}(w_2 + 1) \cdot \mathrm{O}^*(w_3)).$$

Let us fix a

$$\psi = (k, (n_i), (d_i), (\pi_i)) \in \Psi_{\rm alg}(\mathrm{Sp}_6)$$

with infinitesimal character $\mathfrak{z}_{\underline{w}}$. We have to determine first whether or not $\Pi(\psi_\infty)$ contains a holomorphic discrete series. Lemma 9.4 ensures that it is the case if and only if for each i such that $\pi_i = 1$ then $d_i = 1$. We thus assume that this property is satisfied and we denote by π the unique element in $\Pi(\psi)$ such that $\pi_\infty \simeq \pi_{\rm hol}$. We want then to determine $m(\pi)$. By Lemma 9.3 and the remark that follows, we have

$$\tau(\pi)_{|C_\psi} = e_{2|\mathrm{C}_\psi}.$$

By Lemma 3.23 (iii), if some n_i is even then $n_i \equiv 0 \bmod 4$, and there is exactly one integer i such that n_i is odd. It follows that either $k=1$ (stable case) or $k=2$ and (up to equivalence) $(n_1,n_2)=(4,3)$. In this latter case $\mathrm{C}_\psi = \langle s_1\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and $\pi_2 \neq 1$.

Case (i) : (*stable tempered case*) $\psi = \pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_7)$. Then ψ_∞ is a discrete series Langlands parameter (hence indeed $\pi_{\mathrm{hol}} \in \Pi(\psi_\infty)$) and $m(\pi)=1$ by the multiplicity formula. The number of such π is thus $\mathrm{O}^*(w_1,w_2,w_3)$.

Case (ii) : (*endoscopic tempered case*) $k=2$, $d_1=d_2=1$, $\psi=\pi_1\oplus\pi_2$ where $\pi_1 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_4)$ and $\pi_2 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_3)$. Say π_1 has Hodge weights $a>b$ and π_2 has Hodge weight c. Then again ψ_∞ is a discrete Langlands parameters (hence contains π_{hol}). In particular $\varepsilon_\psi(s_1)=1$. But $e_2(s_1)=1$ if and only if $a>c>b$, thus $m(\pi)=1$ if and only if $a>c>b$. The number of such π is thus $\mathrm{O}^*(w_2)\cdot\mathrm{O}(w_1,w_3)$.

Case (iii) : (*endoscopic non-tempered case*) $k=2$, $d_1=2$ and $d_2=1$, i.e. $\psi = \pi_1[2]\oplus\pi_2$ where $\pi_1 \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ and $\pi_2 \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_3)$. Say π_1 has Hodge weight a and π_2 has Hodge weight b. This time ψ_∞ is not tempered and

$$\varepsilon_\psi(s_1) = \varepsilon(\pi_1\times\pi_2) = (-1)^{1+\mathrm{Max}(a,b)+\frac{a+1}{2}}.$$

But $e_2(s_1)=-1$, so

$$m(\pi) = \begin{cases} 1 & \text{if} \quad b>a \quad \text{and} \quad a\equiv 3 \bmod 4, \\ 1 & \text{if} \quad a>b \quad \text{and} \quad a\equiv 1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

This concludes the proof of the proposition above.

We remark that we excluded three kinds of parameters thanks to Lemma 9.4, namely $[7]$, $\pi_1\oplus[3]$ and $\pi_1[2]\oplus[3]$. Alternatively, we could also have argued directly using only Lemma 9.3. Indeed, in those three cases we obviously have $\varepsilon_\psi=1$, and we see also that $e_2(s_1)=-1$.

Appendix A. Adams-Johnson packets

A.1. **Strong inner forms of compact connected real Lie groups.** Let K be a compact connected semisimple Lie group and let G be its complexification. It is a complex semisimple algebraic group equipped with an anti-holomorphic group involution $\sigma : g \mapsto \overline{g}$ such that $K = \{g \in G, \overline{g} = g\}$. As is well-known, K is a maximal compact subgroup of G.

Let T_c be a maximal torus of K and denote by $T \subset G$ the unique maximal torus of G with maximal compact subgroup T_c. Following J. Adams in [Ada11], consider the group

$$\mathcal{X}_1(T) = \{t \in T, t^2 \in \mathrm{Z}(G)\}.$$

An element of $\mathcal{X}_1(T)$ will be called a *strong inner form* of K (relative to (G,T)). As K is a maximal compact subgroup of G, we have $\mathrm{Z}(G) = \mathrm{Z}(K) \subset T_c$ and thus $\mathcal{X}_1(T) \subset T_c$. A strong inner form $t \in \mathcal{X}_1(T)$ of K is said *pure* if $t^2 = 1$.

If $t \in \mathcal{X}_1(T)$ we denote by σ_t the group automorphism $\mathrm{Int}(t) \circ \sigma$ of G. We have $\sigma_t^2 = \mathrm{Id}$. It follows that the real linear algebraic Lie group

$$G_t = \{g \in G, \sigma_t(g) = g\}$$

is an inner form of $G_1 = K$ in the usual sense. Observe that $T_c \subset G_t$ and that G_t is stable by σ. The polar decomposition of G relative to K shows then that the group

$$K_t = K \cap G_t,$$

which is also the centralizer of t in K, is a maximal compact subgroup of G_t. The torus T_c is thus a common maximal torus of all the G_t. Any involution of G of the form $\mathrm{Int}(g) \circ \sigma$ with $g \in G$ is actually of the form $\mathrm{Int}(h) \circ \sigma_t \circ \mathrm{Int}(h)^{-1}$ for some $t \in \mathcal{X}_1(T)$ and some $h \in G$ by [Ser97, §4.5]. In particular, every inner form of K inside G is G-conjugate to some G_t.

Consider the Weyl group

$$W = \mathrm{W}(G,T) = \mathrm{W}(K,T_c).$$

It obviously acts on the group $\mathcal{X}_1(T)$, and two strong real forms $t, t' \in \mathcal{X}_1(T)$ are said *equivalent* if they are in a same W-orbit. If $w \in W$, observe that $\mathrm{Int}(w)$ defines an isomorphism $G_t \to G_{w(t)}$ which is well-defined up to inner isomorphisms by T_c, so that the group G_t is canonically defined up to inner isomorphisms by the equivalence class of t. This is however not the unique kind of redundancy among the groups G_t in general, as for instance $G_t = G_{tz}$ whenever $z \in \mathrm{Z}(G)$. We shall denote by $[t] \in W\backslash\mathcal{X}_1(T)$ the equivalence class of $t \in \mathcal{X}_1(T)$ and by $G_{[t]}$ the group G_t "up to inner automorphisms". It makes sense in particular to talk about representations of $G_{[t]}$.

As a classical example, consider the case of the even special orthogonal group

$$G = \mathrm{SO}_{2r}(\mathbb{C}) = \{g \in \mathrm{SL}_{2r}(\mathbb{C}), {}^t g g = \mathrm{Id}\}$$

with the coordinate-wise complex conjugation σ, i.e. $K = \mathrm{SO}_{2r}(\mathbb{R}) = G \cap \mathrm{SL}_{2r}(\mathbb{R})$. Consider the maximal torus

$$T = \mathrm{SO}_2(\mathbb{C})^r \subset G$$

preserving each plane $P_i = \mathbb{C}e_{2i-1} \oplus \mathbb{C}e_{2i}$ for $i = 1, \cdots, r$. Here (e_i) is the canonical basis of $\mathbb{C}^{2r}$. Any $t \in \mathfrak{X}_1(T)$ is $\mathrm{W}(G,T)$-equivalent to either a unique element t_j, $0 \leq j \leq r$, where t_j acts by -1 on P_i if $i \leq j$, and by $+1$ otherwise, or to exactly one of the two element $t^*_\pm \in T_c$ sending each e_{2i} on $-e_{2i-1}$ for $i < r$ and e_{2r} on $\pm e_{2r-1}$. We have $t_j^2 = 1$ (pure inner forms) and $(t^*_\pm)^2 = -1$. We see that

$$K_{t_j} = S(\mathrm{O}(2j) \times \mathrm{O}(2r-2j))$$

and $G_{t_j} \simeq \mathrm{SO}(2j, 2r-2j)$. In particular, $G_{t_j} \simeq G_{t_{j'}}$ if and only if $j = j'$ or $j + j' = r$. Moreover, $K_{t^*_\pm}$ is isomorphic to the unitary group in r variables and $G_{t^*_\pm}$ is the real Lie group sometimes denoted by SO^*_{2r}. Observe that the only quasi-split group among the G_{t_j} and $G_{t^*_\pm}$ is $\mathrm{SO}(r+1, r-1)$ if r is odd, $\mathrm{SO}(r,r)$ if r is even. In particular, the split group $\mathrm{SO}(r,r)$ is a pure inner form of K if and only if r is even.

We leave as an exercise to the reader to treat the similar cases $G = \mathrm{Sp}_{2g}(\mathbb{C})$ and $G = \mathrm{SO}_{2r+1}(\mathbb{C})$ which are only easier. When $G = \mathrm{Sp}_{2g}(\mathbb{C})$, each twisted form of K is actually inner as $\mathrm{Out}(G) = 1$. In this case the (inner) split form $\mathrm{Sp}_{2g}(\mathbb{R})$ is not a pure inner form of K, it corresponds to the single equivalence class of t such that $t^2 = -1$. When $G = \mathrm{SO}_{2r+1}(\mathbb{C})$, then $\mathrm{Z}(G) = \mathrm{Out}(G) = 1$, and the equivalence classes of strong inner forms of K are in bijection with the isomorphism classes of inner forms of K, namely the real special orthogonal groups $\mathrm{SO}(2j, 2r+1-2j)$ of signature $(2j, 2r+1-2j)$ for $j = 0, \cdots, r$.

A.2. **Adams-Johnson parameters.** We refer to Kottwitz' exposition in [Kot88, p. 195] and to Adams paper [Ada11], from which the presentation below is very much inspired.

We keep the assumptions of §A.1 and we assume from now on that the set of strong real forms of K contains a split real group. It is equivalent to ask that the center of the simply connected covering G_{sc} of G is an elementary abelian 2-group, i.e. G has no factor of type E_6, or type A_n or D_{2n-1} for $n > 1$. We may view the Langlands dual group of G as a complex connected semisimple algebraic group $\widehat{G}$, omitting the trivial Galois action.

Denote by $\Psi(G)$ the set of Arthur parameters of the inner forms of K. This is the set of continuous homomorphisms

$$\mathrm{W}_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}$$

which are $\mathbb{C}$-algebraic on the $\mathrm{SL}_2(\mathbb{C})$-factor and such that the image of any element of $\mathrm{W}_\mathbb{R}$ is semisimple. Two such parameters are said equivalent if they are conjugate under $\widehat{G}$. Fix $\psi \in \Psi(G)$. Let $\widehat{L}$ be the centralizer in $\widehat{G}$ of $\psi(W_\mathbb{C})$, which is a Levi subgroup of some parabolic subgroup of $\widehat{G}$; as $\mathrm{W}_\mathbb{C}$ is commutative

$$\psi(W_\mathbb{C} \times \mathrm{SL}_2(\mathbb{C})) \subset \widehat{L}.$$

Let us denote by C_ψ the centralizer of $\mathrm{Im}(\psi)$ in $\widehat{G}$ and consider the following two properties of a $\psi \in \Psi(G)$.

(a) $\psi(\mathrm{SL}_2(\mathbb{C}))$ contains a regular unipotent element of $\widehat{L}$.

(b) C_ψ is finite.

Property (a) forces in particular the centralizer of $\psi(\mathrm{SL}_2(\mathbb{C}))$ in $\widehat{L}$ to be $\mathrm{Z}(\widehat{L})$, thus under (a) we have

$$\mathrm{C}_\psi = \mathrm{Z}(\widehat{L})^\theta$$

where $\theta = \mathrm{Int}(\psi(j))$ (recall that $j \in \mathrm{W}_\mathbb{R}\backslash\mathrm{W}_\mathbb{C}$ satisfies $j^2 = -1$, see §3.11). Moreover, if one assumes (a) then property (b) is equivalent to the assertion that the involution θ acts as the inversion on $\mathrm{Z}(\widehat{L})^0$. If A is an abelian group, we denote by $A[2]$ the subgroup of elements $a \in A$ such that $a^2 = 1$.

Lemma A.3. *If $\psi \in \Psi(G)$ satisfies (a) and (b) then $\mathrm{C}_\psi = \mathrm{Z}(\widehat{L})[2]$.*

Proof — Indeed, as a general fact one has $\mathrm{Z}(\widehat{L}) = \mathrm{Z}(\widehat{G})\mathrm{Z}(\widehat{L})^0$, because the character group of the diagonalizable group $\mathrm{Z}(\widehat{L})/\mathrm{Z}(\widehat{G})$ is free, being the quotient of the root lattice of $\widehat{G}$ by the root lattice of $\widehat{L}$. We also obviously have $\mathrm{Z}(\widehat{G}) \subset \mathrm{C}_\psi$ (θ acts trivially on $\mathrm{Z}(\widehat{G})$), thus $\mathrm{C}_\psi = \mathrm{Z}(\widehat{G})(\mathrm{Z}(\widehat{L})^0)^\theta$. By assumption on G one has $\mathrm{Z}(\widehat{G}) = \mathrm{Z}(\widehat{G})[2]$. As θ acts as the inversion on $\mathrm{Z}(\widehat{L})^0$ one obtains $\mathrm{Z}(\widehat{L})[2] = \mathrm{Z}(\widehat{G})(\mathrm{Z}(\widehat{L})^0[2]) = \mathrm{C}_\psi$. □

To any $\psi \in \Psi(G)$ one may attach following Arthur a Langlands parameter

$$\varphi_\psi : \mathrm{W}_\mathbb{R} \to \widehat{G}$$

defined by restricting ψ along the homomorphism

$$\mathrm{W}_\mathbb{R} \to \mathrm{W}_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C})$$

which is the identity on the first factor and the representation $|\cdot|^{1/2} \oplus |\cdot|^{-1/2}$ on the second factor. Here, $|\cdot| : \mathrm{W}_\mathbb{R} \to \mathbb{R}_{>0}$ is the norm homomorphism, sending j to 1 and $z \in W_\mathbb{C}$ to $z\overline{z}$. There is a maximal torus $\widehat{T}$ of $\widehat{G}$ such that

$$\varphi_\psi(\mathrm{W}_\mathbb{C}) \subset \widehat{T}.$$

We follow Langlands notation[15] and write

$$\varphi_\psi(z) = z^\lambda \overline{z}^\mu \tag{1.1}$$

for any $z \in \mathrm{W}_\mathbb{C}$, where $\lambda, \mu \in \mathrm{X}_*(\widehat{T}) \otimes \mathbb{C} = \mathrm{Lie}_\mathbb{C}(\widehat{T})$ and $\lambda - \mu \in \mathrm{X}_*(\widehat{T})$. The $\widehat{G}$-conjugacy class of λ in $\widehat{\mathfrak{g}}$ is called the infinitesimal character of ψ (and φ_ψ) and will be denoted by z_ψ. The last condition we shall consider is :

(c) z_ψ is the infinitesimal character of a finite dimensional $\mathbb{C}$-algebraic representation of G.

Under assumption (c), it follows that $\widehat{T}$ is the centralizer of $\varphi_\psi(\mathrm{W}_\mathbb{C})$ in $\widehat{G}$, and that there is a unique Borel subgroup $\widehat{B}$ of $\widehat{G}$ containing $\widehat{T}$ for which λ is dominant.

[15]Recall that if $z \in \mathbb{C}^\times$, and if $a, b \in \mathbb{C}$ satisfy $a - b \in \mathbb{Z}$, we set $z^a\overline{z}^b = e^{ax+b\overline{x}}$ where $x \in \mathbb{C}$ is any element such that $z = e^x$. The element $z^\lambda\overline{z}^\mu \in \widehat{T}$ is uniquely defined by the formula $\eta(z^\lambda\overline{z}^\mu) = z^{\langle\eta,\lambda\rangle}\overline{z}^{\langle\eta,\mu\rangle}$ for all $\eta \in \mathrm{X}^*(\widehat{T})$.

Definition A.4. *The subset of $\psi \in \Psi(G)$ satisfying (a), (b) and (c) will be denoted by $\Psi_{\mathrm{AJ}}(G)$.*

When G is classical group, that is either $\mathrm{SO}_r(\mathbb{C})$ or $\mathrm{Sp}_{2g}(\mathbb{C})$, then so is $\widehat{G}$. A parameter $\psi \in \Psi(G)$ is an Adams-Johnson parameter if and only if it satisfies (c) and $\mathrm{St} \circ \psi$ is a multiplicity free representation of $\mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$, where St denotes the standard representation of $\widehat{G}$.

As an example, consider the group $G = \mathrm{Sp}_{2g}(\mathbb{C})$, so that $\widehat{G} = \mathrm{SO}_{2g+1}(\mathbb{C})$. Let $\mathrm{St} : \widehat{G} \to \mathrm{GL}_{2r+1}(\mathbb{C})$ be the standard representation of $\widehat{G}$. Let $\psi \in \Psi(G)$. Then $\psi \in \Psi_{\mathrm{AJ}}(G)$ if and only if

$$\mathrm{St} \circ \psi \simeq \varepsilon^s \otimes \nu_{d_0} \oplus \bigoplus_{i \neq 0} \mathrm{I}_{w_i} \otimes \nu_{d_i}$$

for some positive integers w_i and d_i with $(-1)^{w_i+d_i-1} = 1$ for each i, with the convention $w_0 = 0$ and where $w_i > 0$ if $i \neq 0$, such that the $2g+1$ even integers

$$\pm w_i + d_i - 1, \pm w_i + d_i - 3, \cdots, \pm w_i - d_i + 1$$

are distinct. The integer s is congruent mod 2 to the number of $i \neq 0$ such that w_i is even. Moreover, the equivalence class of ψ is uniquely determined by the isomorphism class of $\mathrm{St} \circ \psi$. If ψ is as above, then $\widehat{L} \simeq \mathrm{SO}_{d_0}(\mathbb{C}) \times \prod_{i \neq 0} \mathrm{GL}_{d_i}(\mathbb{C})$ and $\mathrm{C}_\psi \simeq \prod_{i \neq 0} \{\pm 1\}$.

The case $G = \mathrm{SO}_{2r+1}(\mathbb{C})$ is quite similar, one simply has to replace the condition $(-1)^{w_i+d_i-1} = 1$ by $(-1)^{w_i+d_i-1} = -1$, and there is no more restriction on s mod 2. The case $G = \widehat{G} = \mathrm{SO}_{2r}(\mathbb{C})$ is slightly different but left as an exercise to the reader.

A.5. **Adams-Johnson packets.** In the paper [AJ87], J. Adams and J. Johnson associate to any $\psi \in \Psi_{\mathrm{AJ}}(G)$, and to any equivalence class of strong inner forms of K, a finite set of (usually non-tempered) irreducible unitary representations satisfying certain predictions of Arthur. Let us recall briefly their definition.

Fix $\psi \in \Psi_{\mathrm{AJ}}(G)$. It determines subgroups $\widehat{T}, \widehat{B}, \widehat{L}$ of $\widehat{G}$ as in the previous section. Fix a Borel subgroup B of G containing T. This choice provides an isomorphism between the based root datum of (G, T, B) and the dual of the based root datum of $(\widehat{G}, \widehat{T}, \widehat{B})$. In particular, there is a unique Levi subgroup L of G containing T whose roots correspond to the coroots of $(\widehat{L}, \widehat{T})$ via this isomorphism, as well as a privileged parabolic subgroup $P = BL$ of G. Let $t \in T$. As T_c is a compact maximal torus of L_t, it follows that the real Lie group

$$L_t = G_t \cap L$$

is a real form of L containing T_c. The real group L_t is even an inner form of $K \cap L$ (a maximal compact subgroup of L). Moreover, the Cartan involution $\mathrm{Int}(t)$ of G_t preserves P as $T \subset P$, and defines a Cartan involution of L_t as well. Assume that L_t is connected to simplify (see loc. cit. for the general case). There is a unique one-dimensional unitary character χ_λ of L_t whose restriction to T_c is $\lambda - \rho$ where ρ denotes the half-sum of the positive roots of (G, T) with respect to B. Adams and Johnson define $\pi_t(\lambda)$ as the cohomological induction relative to P from the $(\mathfrak{l}, K_t \cap L_t)$-module χ_λ to $(\mathfrak{g}, K_t)$. To emphasize the dependence on P in this construction, we shall sometimes write

$$\pi_{t,P}(\lambda)$$

rather than $\pi_t(\lambda)$.

The isomorphisms $\mathrm{Int}(w) : G_t \to G_{w(t)}$, for $w \in W$, allow to consider the collection of representations $\pi_{w(t)}(\lambda)$ as a representation of G_t (or of $G_{[t]}$). The set of such representations is the Adams-Johnson packet of $G_{[t]}$ attached to ψ, and we shall denote it by

$$\Pi(\psi, G_{[t]}).$$

It turns out that for $t, t' \in \mathfrak{X}_1(T)$ in a same W-orbit, then $\pi_t(\lambda) \simeq \pi_{t'}(\lambda)$ if and only if t and t' are in a same $\mathrm{W}(L, T)$-orbit. Observe also that for $t \in \mathfrak{X}_1(T)$ we have

$$\{w \in W, w(t) = t\} = \mathrm{W}(K_t, T_c).$$

It follows that $\Pi(\psi, G_{[t]}))$ is in natural bijection with $\mathrm{W}(L, T)\backslash W/\mathrm{W}(K_t, T_c)$ and in particular that $|\Pi(\psi, G_{[t]})|$ is the number of such double cosets.

Lemma A.6. *The representation $\pi_{t,P}(\lambda)$ is a discrete series representation if and only if $t \in \mathrm{Z}(L)$.*

Proof — Indeed, as recalled loc. cit., $\pi_{t,P}(\lambda)$ is a discrete series representation if and only if L_t is compact. The result follows as $\mathrm{Int}(t)$ is a Cartan involution of L_t. Note that for such a t the group L_t is of course always connected as so is L. □

In the special case $t \in \mathrm{Z}(G)$, i.e. G_t is compact, it follows that $\Pi(\psi, G_{[t]})$ is the singleton made of the unique irreducible representations of highest weight $\lambda - \rho$ relative to B. A more important special case is the one with $\psi(\mathrm{SL}_2(\mathbb{C})) = \{1\}$. In this case ψ is nothing more than a discrete series parameter in the sense of Langlands. Here (a) is automatic, (b) implies (c), $\varphi_\psi = \psi$ and $\widehat{L} = \widehat{T}$. Then $\pi_\lambda(t)$ is the discrete series representation with Harish-Chandra parameter λ, and $\Pi(\psi, G_{[t]})$ is simply the set of isomorphism classes of discrete series representations of $G_{[t]}$ with infinitesimal character z_ψ.

A.7. **Shelstad's parameterization map.** What follows is again much inspired from [Kot88, p. 195] and [Ada11]. We fix a $\psi \in \Psi_{\mathrm{AJ}}(G)$ and keep the assumptions and notations of the previous paragraphs. We denote by

$$\widetilde{\Pi}(\psi)$$

the disjoint union of the sets $\Pi(\psi, G_{[t]})$ where $[t]$ runs over the equivalence classes of strong real forms of K. As already explained in the previous paragraph, the map $\mathfrak{X}_1(T) \to \widetilde{\Pi}(\psi), t \mapsto \pi_t(\lambda)$, induces a bijection

$$\mathrm{W}(L, T)\backslash\mathfrak{X}_1(T) \xrightarrow{\sim} \widetilde{\Pi}(\psi). \tag{1.2}$$

Define S_ψ as the inverse image of C_ψ under the simply connected covering

$$p : \widehat{G}_{\mathrm{sc}} \to \widehat{G}.$$

Following [She82], [She08], Langlands, Arthur, [AJ87], [ABV92] and [Kot88], the set $\widetilde{\Pi}(\psi)$ is equipped with a natural map

$$\tau_0 : \widetilde{\Pi}(\psi) \to \mathrm{Hom}(\mathrm{S}_\psi, \mathbb{C}^\times)$$

that we shall now describe in the style of Adams in [Ada11]. Observe first that S_ψ is the inverse image of C_ψ in $\widehat{T}_{\mathrm{sc}} = p^{-1}(\widehat{T})$, hence it is an abelian group.

Lemma A.8. $\mathrm{S}_\psi \subset (p^{-1}(\widehat{T}[2]))^{\mathrm{W}(L,T)}$.

Proof — By Lemma A.3 and the inclusion $\mathrm{C}_\psi \subset \mathrm{C}_{\varphi_\psi} = \widehat{T}$ described in §A.2, one obtains a canonical inclusion $\mathrm{C}_\psi \subset \widehat{T}[2]$. Moreover, $\widehat{L}_{\mathrm{sc}} := p^{-1}(\widehat{L})$ is a Levi subgroup of $\widehat{G}$ containing $\widehat{T}_{\mathrm{sc}}$ and thus $p^{-1}(\mathrm{Z}(\widehat{L})) = \mathrm{Z}(\widehat{L}_{\mathrm{sc}})$ and $\mathrm{W}(L,T) = \mathrm{W}(\widehat{L},\widehat{T}) = \mathrm{W}(\widehat{L}_{\mathrm{sc}},\widehat{T}_{\mathrm{sc}})$. In particular, $\mathrm{W}(L,T)$ acts trivially on $p^{-1}(\mathrm{Z}(\widehat{L}))$, hence trivially on S_ψ. □

On the other hand, there is a natural perfect W-equivariant pairing

$$\mathcal{X}_1(T) \times p^{-1}(\widehat{T}[2]) \to \mathbb{C}^\times.$$

Indeed, if $P^\vee(T)$ denotes the co-weight lattice of T we have natural identifications

$$\mathcal{X}_1(T) = \frac{1}{2}P^\vee(T)/\mathrm{X}_*(T) \;\text{ and }\; p^{-1}(\widehat{T}[2]) = \frac{1}{2}\mathrm{X}_*(\widehat{T})/\mathrm{X}_*(\widehat{T}_{\mathrm{sc}})$$

via $\mu \mapsto e^{2i\pi\mu}$. The pairing alluded above is then $(\mu,\mu') \mapsto e^{i\pi\langle\mu,\mu'\rangle}$, where $\langle,\rangle$ is the canonical perfect pairing $\mathrm{X}_*(T)\otimes\mathbb{Q} \times \mathrm{X}_*(\widehat{T})\otimes\mathbb{Q} \to \mathbb{Q}$. The resulting pairing is perfect as well as $\mathrm{X}_*(\widehat{T}_{\mathrm{sc}})$ is canonically identified by $\langle,\rangle$ with the root lattice of T.

One then defines τ_0 as follows. Fix $\pi \in \widetilde{\Pi}(\psi)$. By the bijection (1.2), there is an element $t \in \mathcal{X}_1(T)$, whose $W(L,T)$-orbit is canonically defined, such that $\pi \simeq \pi_t(\lambda)$. The perfect pairing above associates to t a unique character $p^{-1}(\widehat{T}[2]) \to \mathbb{C}^*$, whose restriction to S_ψ only depends on the $\mathrm{W}(L,T) = \mathrm{W}(\widehat{L},\widehat{T})$-orbit of t by Lemma A.8 : define $\tau_0(\pi)$ as this character of S_ψ.

This parameterization is discussed in details in [Ada11] in the discrete series case, i.e. when $\widehat{L} = \widehat{T}$. It follows from the previous discussion that τ_0 is a bijection in this case, as $\mathrm{S}_\psi = p^{-1}(\widehat{T}[2])$. The following simple lemma shows that the determination of the parameterization of discrete series in $\widetilde{\Pi}(\psi)$ for general ψ reduces to this latter case.

Observe following [Kot88] that for any $\psi \in \Psi_{\mathrm{AJ}}(G)$ there is a unique discrete series parameter $\psi_{\mathrm{disc}} \in \Psi_{\mathrm{AJ}}(G)$ such that the centralizers of $\varphi_\psi(\mathrm{W}_\mathbb{C})$ and $\psi_{\mathrm{disc}}(\mathrm{W}_\mathbb{C})$ coincide, and such that the parameters λ for ψ and ψ_{disc} defined by (1.1) coincide as well. In particular, ψ and ψ_{disc} have the same infinitesimal character. Of course, $\psi_{\mathrm{disc}} \neq \varphi_\psi$ if $\psi \neq \psi_{\mathrm{disc}}$. If ψ is normalized as before, we have canonical inclusions

$$\mathrm{C}_\psi \subset \mathrm{C}_{\psi_{\mathrm{disc}}} = \widehat{T}[2] \;\text{ and }\; \mathrm{S}_\psi \subset \mathrm{S}_{\psi_{\mathrm{disc}}} = p^{-1}(\widehat{T}[2]).$$

The discrete series representations belonging to $\widetilde{\Pi}(\psi)$ are exactly the elements of $\widetilde{\Pi}(\psi) \cap \widetilde{\Pi}(\psi_{\mathrm{disc}})$. It will be important to distinguish in the next lemma the parameterization maps

τ_0 of $\widetilde{\Pi}(\psi)$ and $\widetilde{\Pi}(\psi_{\rm disc})$, so we shall denote them respectively by $\tau_{0,\psi}$ and $\tau_{0,\psi_{\rm disc}}$. Recall form Lemma A.6 that $\pi_t(\lambda)$ is a discrete series representation if and only if $t \in \mathrm{Z}(L)$. The following lemma is a variant of an observation by Kottwitz in [Kot88].

Lemma A.9. *Let $\psi \in \Psi_{\rm AJ}(G)$ and let $\pi \in \widetilde{\Pi}(\psi) \cap \widetilde{\Pi}(\psi_{\rm disc})$. Then*

$$\tau_{0,\psi}(\pi) = \tau_{0,\psi_{\rm disc}}(\pi)_{|\mathrm{S}_\psi}.$$

Proof — We have $\pi \simeq \pi_{t,B}(\lambda) \in \widetilde{\Pi}(\psi_{\rm disc})$ for a unique $t \in \mathfrak{X}_1(T)$ and we also have $\pi \simeq \pi_{t',P}(\lambda) \in \widetilde{\Pi}(\psi)$ for a unique element $t' \in \mathfrak{X}_1(T) \cap \mathrm{Z}(L)$ by Lemma A.6 (note that t' is fixed by $W(L,T)$). Applying the "transitivity of cohomological induction" via the compact connected group L_t (use e.g. [KV95, Cor. 11.86 (b)], here $q_0 = 0$), we have $\pi_{t,B}(\lambda) \simeq \pi_{t,P}(\lambda)$. It follows that $t = t'$, which concludes the proof. □

Observe that for $t \in \mathfrak{X}_1(T)$, G_t is compact if and only if $t \in \mathrm{Z}(G)$, in which case it coincides with its equivalence class (and it is fixed by $\mathrm{W}(L,T)$). The associated representation $\pi_{t,P}(\lambda)$ is the unique finite dimensional representation of G_t with infinitesimal character z_ψ. It occurs $|\mathrm{Z}(G)|$ times in $\widetilde{\Pi}(\psi)$, once for each $t \in \mathrm{Z}(G)$, and these representations are perhaps the most obvious elements in $\widetilde{\Pi}(\psi) \cap \widetilde{\Pi}(\psi_{\rm disc})$. To understand their characters we have to describe the image

$$\mathcal{N}(T)$$

of $\mathrm{Z}(G)$ under the homomorphism $\mathfrak{X}_1(T) \to \mathrm{Hom}(p^{-1}(\widehat{T}[2]), \mathbb{C}^*)$ induced by the canonical pairing. Observe that $\mathrm{Z}(G) = \{t^2, t \in \mathfrak{X}_1(T)\}$. The following lemma follows.

Lemma A.10. *The subgroup $\mathcal{N}(T) \subset \mathrm{Hom}(p^{-1}(\widehat{T}[2]), \mathbb{C}^*)$ is the subgroup of squares, or equivalently of characters which are trivial on $\widehat{T}_{\rm sc}[2]$.*

The parameterization τ_0 of $\widetilde{\Pi}(\psi)$ introduced so far is the one we shall need up to a translation by a certain character b_ψ of S_ψ (or "base point of ψ"). Write again temporarily $\tau_{0,\psi}$ for τ_0 in order to emphasize its dependence on ψ and we write character groups additively. The map

$$\tau_\psi = \tau_{0,\psi} - b_\psi$$

has to satisfy the following two conditions :

(i) Lemma A.9 holds with $\tau_{0,\psi}$ and $\tau_{0,\psi_{\rm disc}}$ replaced respectively by τ_ψ and $\tau_{\psi_{\rm disc}}$.

(ii) If ψ is a discrete series parameter, and if $\pi = \pi_t(\lambda) \in \widetilde{\Pi}(\psi)$ satisfies $\tau_\psi(\pi) = 1$, i.e. $\tau_{0,\psi}(\pi) = b_\psi$, then G_t is a split real group and π is generic with respect to some Whittaker functional.

Normalize ψ as in §A.2. Following [Ada11], consider the element

$$t_b = e^{i\pi\rho^\vee} \in \mathfrak{X}(T)$$

where $\rho^\vee \in \mathrm{X}_*(T)$ is the half-sum of the positive coroots with respect to (G,B,T). Under the identification $\mathfrak{X}_1(T) = \frac{1}{2}P^\vee(T)/\mathrm{X}_*(T)$, t_b is the class of $\frac{1}{2}\rho^\vee$. In particular, under the canonical pairing between $\mathrm{S}_{\psi_{\rm disc}}$ and $\mathfrak{X}_1(T)$ the element t_b corresponds to the restriction to $\mathrm{S}_{\psi_{\mathrm{disc}}}$ of the character $\rho^\vee \in \mathrm{X}^*(\widehat{T}_{\mathrm{sc}})$. The characteristic property of t_b is that for any t in the coset $\mathrm{Z}(G)t_b \subset \mathcal{X}_1(T)$, then $\pi_t(\lambda)$ is a generic (or "large" in the sense of Vogan) discrete series of the split group G_t. To fulfill the conditions (i) and (ii) one simply set $b_\psi = \rho^\vee$.

Definition A.11. *If $\psi \in \Psi_{\mathrm{AJ}}(G)$, the canonical parameterization*

$$\tau : \widetilde{\Pi}(\psi) \to \mathrm{Hom}(\mathrm{S}_\psi, \mathbb{C}^*)$$

is defined by $\tau = \tau_0 - \rho^\vee{}_{|\mathrm{S}_\psi}$ where $\widehat{T}$ is the centralizer of $\varphi_\psi(\mathrm{W}_\mathbb{C})$, $\widehat{B}$ is the unique Borel subgroup of $\widehat{G}$ containing $\widehat{T}$ with respect to which the element λ defined by (1.1) *is dominant, and $\rho^\vee$ is the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$.*

Corollary A.12. *If $\pi \in \widetilde{\Pi}(\psi)$ is a finite dimensional representation, then $\tau(\pi) \in \mathcal{N}(T) - \rho^\vee$.*

We end this paragraph by collecting a couple of well-known and simple facts we used in the paper. For $t, t' \in \mathcal{X}_1(T)$, G_t and $G_{t'}$ are pure inner forms if and only if $t^2 = (t')^2$.

Corollary A.13. *K is a pure inner form of a split group if and only if $\rho^\vee \in \mathrm{X}^*(\widehat{T})$.*

Indeed, G_t is a pure inner form of a split group if and only if $t^2 = t_b^2 = (-1)^{2\rho^\vee}$.

Corollary A.14. *Let $t \in \mathcal{X}_1(T)$. Then G_t is a pure inner form of a split group if and only if the character of $p^{-1}(\widehat{T}[2])$ associated to tt_b^{-1} under the canonical pairing factors through $\widehat{T}[2]$.*

Appendix B. The Langlands group of $\mathbb{Z}$ and Sato-Tate groups

In this brief appendix, we discuss a conjectural topological group that might be called the *Langlands group of* $\mathbb{Z}$, and that we shall denote $\mathcal{L}_\mathbb{Z}$. We will define $\mathcal{L}_\mathbb{Z}$ as a suitable quotient of the conjectural Langlands group $\mathcal{L}_\mathbb{Q}$ of the field of rational numbers $\mathbb{Q}$, originally introduced by Langlands in [Lan79]. The group $\mathcal{L}_\mathbb{Z}$ is especially relevant to understand the level 1 automorphic representations of reductive groups over $\mathbb{Z}$. As we shall explain, and following [Ch13, Ch. II §3.6], it also offers a plausible point of view on the Sato-Tate groups of automorphic representations and motives. Let us stress once and for all that *most of this appendix is purely hypothetical.* Nevertheless, we hope it might be a useful and rather precise guide to the understand the philosophy, due to Langlands and Arthur, behind the results of this paper.

We shall view $\mathcal{L}_\mathbb{Q}$ as a (Hausdorff) locally compact topological group following Kottwitz' point of view in [Kot84, §12]. We refer to Arthur's paper [Art02] for a thorough discussion of the expected properties of $\mathcal{L}_\mathbb{Q}$ and for a description of a candidate for this group as well.

B.1. **The locally compact group $\mathcal{L}_\mathbb{Z}$.** If p is a prime, recall that the group $\mathcal{L}_\mathbb{Q}$ is equipped with a conjugacy class of continuous homomorphisms $\eta_p : I_p \times \mathrm{SU}(2) \to \mathcal{L}_\mathbb{Q}$, where I_p is the inertia group of the absolute Galois group of $\mathbb{Q}_p$. *We define $\mathcal{L}_\mathbb{Z}$ as the quotient of $\mathcal{L}_\mathbb{Q}$ by the closed normal subgroup generated by the union, over all primes p, of* $\mathrm{Im}\ \eta_p$. It is naturally equipped with :

- (Frobenius elements) a conjugacy class $\mathrm{Frob_p} \subset \mathcal{L}_\mathbb{Z}$ for each prime p,
- (Hodge morphism) a conjugacy class of continuous group homomorphisms

$$h : \mathrm{W}_\mathbb{R} \to \mathcal{L}_\mathbb{Z},$$

which inherit from $\mathcal{L}_\mathbb{Q}$ a collection of axioms that we partly describe below.

As $\widehat{\mathrm{GL}_n} = \mathrm{GL}_n(\mathbb{C})$ we have a parameterization map $c : \Pi(\mathrm{GL}_n) \to \mathcal{X}(\mathrm{GL}_n(\mathbb{C}))$ as in §3.7. Denote by $\mathrm{Irr}_n(\mathbb{Z})$ the set of isomorphism classes of irreducible continuous representations $\mathcal{L}_\mathbb{Z} \to \mathrm{GL}_n(\mathbb{C})$.

(L1) (*Langlands conjecture*) For any $n \geq 1$ and any $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, there exists $r_\pi \in \mathrm{Irr}_n(\mathbb{Z})$ such that $c_p(\pi)$ is conjugate to $r_\pi(\mathrm{Frob}_p)$ for each prime p, and $\mathrm{L}(\pi_\infty) \simeq r_\pi \circ \mu$ (see §3.11). Moreover, $\pi \mapsto r_\pi$ defines a bijection $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n) \overset{\sim}{\to} \mathrm{Irr}_n(\mathbb{Z})$.

Let $\mathbb{R}_{>0} \subset \mathbb{R}^\times$ be the multiplicative subgroup of positive numbers. The adelic norm

$$|\cdot| : \mathbb{Q}^\times\backslash\mathbb{A}^\times/\widehat{\mathbb{Z}}^\times \to \mathbb{R}_{>0}$$

is an isomorphism, thus $\Pi_{\mathrm{cusp}}(\mathrm{GL}_1) = \{|\cdot|^s, s \in \mathbb{C}\}$. Set $|\cdot|_\mathbb{Z} = r_{|\cdot|} \in \mathrm{Irr}_1(\mathbb{Z})$. We have $|\mathrm{Frob}_p|_\mathbb{Z} = p^{-1}$ for any prime p and $|\cdot|_\mathbb{Z} \circ h : \mathrm{W}_\mathbb{R} \to \mathbb{R}_{>0}$ coincides with the homomorphism recalled in §3.11. If $\mathcal{D}H \subset H$ denotes the closed subgroup generated by the commutators of the topological group H, and $H^{\mathrm{ab}} = H/\mathcal{D}H$, it is natural to ask that :

(L2) (*Class field theory*) $|\cdot|_{\mathbb{Z}}$ induces a topological isomorphism $\mathcal{L}_{\mathbb{Z}}^{\mathrm{ab}} \overset{\sim}{\to} \mathbb{R}_{>0}$.

Let $\mathcal{L}_{\mathbb{Z}}^1 = \mathcal{D}\mathcal{L}_{\mathbb{Z}}$ be the kernel of $|\cdot|_{\mathbb{Z}}$.

(L3) (*Ramanujan conjecture*) $\mathcal{L}_{\mathbb{Z}}^1$ is compact.

Properties (L2) and (L3) have the following consequence on the structure of $\mathcal{L}_{\mathbb{Z}}$.

Fact 1 : (*Polar decomposition*) If $\mathcal{C} \subset \mathcal{L}_{\mathbb{Z}}$ denotes the neutral component of the center of $\mathcal{L}_{\mathbb{Z}}$, then $\mathcal{L}_{\mathbb{Z}} = \mathcal{C} \times \mathcal{L}_{\mathbb{Z}}^1$ and $|\cdot|_{\mathbb{Z}}$ induces an isomorphism $\mathcal{C} \overset{\sim}{\to} \mathbb{R}_{>0}$. In particular, $\mathcal{L}_{\mathbb{Z}}^1 = \mathcal{D}\mathcal{L}_{\mathbb{Z}}^1$.

Proof — Let $\mathcal{Z} \subset \mathcal{L}_{\mathbb{Z}}$ be the centralizer of the compact normal subgroup $\mathcal{L}_{\mathbb{Z}}^1$. As $\mathcal{L}_{\mathbb{Z}}^1$ is a compact normal subgroup of $\mathcal{L}_{\mathbb{Z}}$, and as $\mathcal{L}_{\mathbb{Z}}/\mathcal{L}_{\mathbb{Z}}^1$ is connected, a classical result of Iwasawa [Iwa49, §1] ensures that $\mathcal{L}_{\mathbb{Z}} = \mathcal{Z}\,\mathcal{L}_{\mathbb{Z}}^1$. The subgroup $\mathcal{Z}^1 = \mathcal{Z} \cap \mathcal{L}_{\mathbb{Z}}^1$ is central in $\mathcal{Z}$, and $|\cdot|_{\mathbb{Z}}$ induces an isomorphism $\mathcal{Z}/\mathcal{Z}^1 \overset{\sim}{\to} \mathbb{R}_{>0}$ by (L2) and the open mapping theorem. As any central extension of $\mathbb{Q}$ is abelian, the Hausdorff topological group $\mathcal{Z}$ is abelian : it thus coincides with the center of $\mathcal{L}_{\mathbb{Z}}$. In particular, $\mathcal{L}_{\mathbb{Z}}^1 = \mathcal{D}\mathcal{L}_{\mathbb{Z}}^1$. The center of a compact Lie group H such that $\mathcal{D}H = H$ is finite, so the center $\mathcal{Z}^1$ of $\mathcal{L}_{\mathbb{Z}}^1$ is profinite. The structure theorem of locally compact abelian groups concludes $\mathcal{Z} = \mathcal{C} \times \mathcal{Z}^1$. □

It will be convenient to identify once and for all $\mathcal{C}$ and $\mathbb{R}_{>0}$ via $|\cdot|_{\mathbb{Z}}$. In other words, we view $|\cdot|_{\mathbb{Z}}$ has a homomorphism $\mathcal{L}_{\mathbb{Z}} \to \mathcal{C}$, and write $g = |g| \cdot (g/|g|)$ for the polar decomposition of an element $g \in \mathcal{L}_{\mathbb{Z}}$. Observe that for any continuous representation $r : \mathcal{L}_{\mathbb{Z}} \to \mathrm{GL}_n(\mathbb{C})$, the elements of the compact group $r(\mathcal{L}_{\mathbb{Z}}^1)$ are semisimple and all their eigenvalues have norm 1. Moreover, if r is irreducible then $r(\mathcal{C})$ acts by scalars by Schur's lemma, so there exists $s \in \mathbb{C}$ such that $r(x) = |x|_{\mathbb{Z}}^s$ for all $x \in \mathcal{C}$. If $r = r_\pi$ for $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, observe that $\det(r_\pi) = |.|_{\mathbb{Z}}^{ns} = r_{\omega_\pi}$, where $\omega_\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_1)$ denotes the central character of π. If we consider the image of an element in the conjugacy class Frob_p and property (L1) we recover the classical Ramanujan conjecture on the $c_p(\pi)$ for $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$. This "explains" as well Clozel's purity lemma 3.13.

Fact 2 : (*Generalized Minkowski theorem*) $\mathcal{L}_{\mathbb{Z}}^1$ is connected. The conjugacy class of $h(U(1))$, where $U(1)$ is the maximal compact subgroup of $\mathrm{W}_{\mathbb{C}}$, generates a dense subgroup of $\mathcal{L}_{\mathbb{Z}}^1$.

Proof — One of the axioms on $\mathcal{L}_{\mathbb{Q}}$ is that its group of connected components is naturally isomorphic to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The first part of Fact 2 follows then from Minkowski's theorem asserting that any non-trivial number field admits at least a ramified prime. Here is another proof. Assume that $\mathcal{L}_{\mathbb{Z}}^1$ admits a non-trivial finite quotient Γ and choose a non trivial irreducible representation of Γ, say of dimension $n \geq 1$, that we view as an element $r \in \mathrm{Irr}_n(\mathbb{Z})$ trivial on $\mathcal{C}$, of finite image. Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ be such that $r = r_\pi$. As $\mathrm{Im}\, r$ is finite, $\mathrm{L}(\pi_\infty) = r \circ h$ is trivial on the connected subgroup $\mathrm{W}_{\mathbb{C}}$. To conclude the proof (of the second statement as well) it is thus enough to show that such a π is necessarily the trivial representation of GL_1. But it follows indeed from Weil's explicit formulas that the L-function of a non-trivial such π, which is entire and of conductor 1, does not exist : see [Mes86, §3]. □

We end this paragraph by a definition of the *motivic Langlands group of* $\mathbb{Z}$, that we shall define as a certain quotient $\mathcal{L}_{\mathbb{Z}}^{\mathrm{mot}}$ of $\mathcal{L}_{\mathbb{Z}}$.

Recall that a $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ is said *algebraic* if the restriction of $\mathrm{L}(\pi_\infty)$ to $\mathrm{W}_{\mathbb{C}}$ is a direct sum of characters of the form $z \mapsto z^{a_i}\overline{z}^{b_i}$ where $a_i, b_i \in \mathbb{Z}$, for $i = 1, \ldots, n$: see the footnote 1 of the introduction for references about this notion. Clozel's purity lemma ensures that a_i+b_i is independent of i, or which is the same, that $\mathrm{Z}(\mathrm{W}_\mathbb{R})$ acts as scalars in $\mathrm{L}(\pi_\infty)$. Here $\mathrm{Z}(\mathrm{W}_\mathbb{R})$ denotes the center of $\mathrm{W}_\mathbb{R}$, namely the subgroup $\mathbb{R}^\times \subset \mathrm{W}_\mathbb{C}$. For any $n \geq 1$ define $\Pi_{\mathrm{mot}}(\mathrm{GL}_n)$ as the subset of $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ such that $\mathrm{Z}(\mathrm{W}_\mathbb{R})$ acts as scalars in $\mathrm{L}(\pi_\infty)$. Let $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$. It is now a simple exercise to check that $\pi \in \Pi_{\mathrm{mot}}(\mathrm{GL}_n)$ if and only if there exists $s \in \mathbb{C}$ such that $\pi \otimes |\cdot|^s$ is algebraic. For instance,

$$\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n) \subset \Pi_{\mathrm{mot}}(\mathrm{GL}_n)$$

(see Definition 3.16).

We define $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$ as the quotient of $\mathcal{L}_\mathbb{Z}$ by the closed normal subgroup generated by the $xyx^{-1}y^{-1}$ where $x \in h(\mathrm{Z}(\mathrm{W}_\mathbb{R}))$ and $y \in \mathcal{L}_\mathbb{Z}$. By definition, if $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ then $\pi \in \Pi_{\mathrm{mot}}(\mathrm{GL}_n)$ if and only if r_π factors through $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$. The locally compact group $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$ inherits from $\mathcal{L}_\mathbb{Z}$ all the properties considered so far. Better, the subgroup $h(\mathrm{Z}(\mathrm{W}_\mathbb{R}))$ is a central subgroup of $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$, so that the polar decomposition is even simpler to understand for $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$ as $h_{|\mathbb{R}_{>0}}$ defines a central section of $|\cdot|_\mathbb{Z} : \mathcal{L}_\mathbb{Z}^{\mathrm{mot}} \to \mathbb{R}_{>0}$.

B.2. **Sato-Tate groups.** Serre's point of view in [Ser68, Ch. 1, appendix] and [Ser94, §13] suggests the following universal form of the Sato-Tate conjecture (here, in the level 1 case).

(L4) (*General Sato-Tate conjecture*) The conjugacy classes $\frac{\mathrm{Frob}_p}{|\mathrm{Frob}_p|} \subset \mathcal{L}_\mathbb{Z}^1$ are equidistributed in the compact group $\mathcal{L}_\mathbb{Z}^1$ equipped with its Haar measure of mass 1.

Note in particular that the union of the conjugacy classes $\frac{\mathrm{Frob}_p}{|\mathrm{Frob}_p|}$ is dense in $\mathcal{L}_\mathbb{Z}^1$ (*Cebotarev property*), which "explains" the strong multiplicity one theorem for GL_n by (L1).

Proposition-Definition B.3. *If $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, define its Sato-Tate group as*

$$\mathcal{L}_\pi := r_\pi(\mathcal{L}_\mathbb{Z}^1).$$

It is a compact connected subgroup of $\mathrm{SL}_n(\mathbb{C})$ well-defined up to $\mathrm{SL}_n(\mathbb{C})$-conjugacy, which acts irreducibly on $\mathbb{C}^n$, and such that $\mathcal{L}_\pi^{\mathrm{ab}} = 1$. The Satake parameters of the π_p have well-defined representatives in $\mathcal{L}_\pi$, namely the conjugacy classes $r_\pi(\mathrm{Frob}_p/|\mathrm{Frob}_p|) \subset \mathcal{L}_\pi$, which are equidistributed for a Haar measure of $\mathcal{L}_\pi$.

Remark that if $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ then $r(\mathcal{C}) = 1$ and so $\mathcal{L}_\pi = r(\mathcal{L}_\mathbb{Z})$. The last property of $\mathcal{L}_\mathbb{Z}$ we would like to discuss is the general Arthur-Langlands conjecture. This is first especially helpful in order to understand the results of Arthur recalled in §3 (see also [Art89] and the introduction of [Art11]). This will also give another way to think about $\mathcal{L}_\pi$ when $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$.

Fix G a semisimple group scheme over $\mathbb{Z}$. Following Arthur, define a *global Arthur parameter for G* as a $\widehat{G}$-conjugacy class of continuous group homomorphisms

$$\psi : \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$$

such that $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ is algebraic, and such that the centralizer C_ψ of $\mathrm{Im}\,\psi$ in $\widehat{G}$ is finite. As an example, suppose that G is a classical group and let $\mathrm{St} : \widehat{G} \to \mathrm{GL}_n(\mathbb{C})$ denote the standard representation. If ψ is a global Arthur parameter for G, the finiteness of C_ψ ensures that the representation $\mathrm{St} \circ \psi$ is a direct sum of pairwise non-isomorphic irreducible representations of $\mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})$, say of dimension n_i, hence of the form $r_i \otimes \mathrm{Sym}^{d_i-1}\,\mathbb{C}^2$ where $d_i | n_i$ and $r_i \simeq r_i^* \in \mathrm{Irr}_{n_i/d_i}(\mathbb{Z})$: via (L1) this "explains" the definition of a global Arthur parameter in §3.18, except property (ii) *loc. cit.* at the moment. (Observe that for any $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$, we have $r_\pi^* = r_{\pi^\vee}$ by (L1)).

Recall Arthur's morphism $a : \mathcal{L}_\mathbb{Z} \to \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})$, $g \mapsto (\,g\,,\,\mathrm{diag}(\,|g|_\mathbb{Z}^{1/2}\,,\,|g|_\mathbb{Z}^{-1/2}\,))$. If ψ is a global Arthur parameter for G, then $\varphi_\psi := \psi \circ a$ is a well-defined conjugacy class of continuous homomorphisms $\mathcal{L}_\mathbb{Z} \to \widehat{G}$. Moreover $\psi_\infty := \psi \circ h$ is also an Arthur parameter in the sense of §A.2. In particular, it possesses an infinitesimal character $z_{\psi_\infty} \subset \widehat{\mathfrak{g}}$ as defined *loc. cit.*

(L5) (*Arthur-Langlands conjecture*) For any $\pi \in \Pi_{\mathrm{disc}}(G)$, there is a global Arthur parameter ψ for G *associated to* π in the following sense : $\varphi_\psi(\mathrm{Frob}_p)$ is conjugate to $c_p(\pi)$ for each prime p and z_{ψ_∞} is conjugate to $c_\infty(\pi)$. Conversely, if ψ is a global Arthur parameter for G, and if $\Pi(\psi)$ is the finite set of $\pi \in \Pi(G)$ associated to ψ, then there is a formula for $\sum_{\pi\in\Pi(\psi)} m(\pi)$.

Recall that $m(\pi)$ denotes the multiplicity of π in $\mathcal{L}_{\mathrm{disc}}(G)$. Let us warn that there may be in general several ψ associated to a given π, because there are examples of continuous morphisms $H_1 \to H_2$, say between two compact connected Lie groups, which are point-wise conjugate but non conjugate (try for instance $H_1 = \mathrm{SU}(3)$ and $H_2 = \mathrm{SO}(8)$) : see [Art89] and [Art02] for more about this problem. Let us also mention that Langlands originally considered only the tempered $\pi \in \Pi_{\mathrm{cusp}}(G)$ and conjectured the existence of a ψ as in (L5) but trivial on the $\mathrm{SL}_2(\mathbb{C})$ factor.

If we consider again the example of classical groups, the first part of (L5) "explains" Arthur's Theorem 3.19. Of course, its second part is too vague as stated here : see [Art02] and [Art89] for more informations about this quite delicate point called *Arthur's multiplicity formula*. See also §1.20.2 for an explicit formula when $G(\mathbb{R})$ is compact, and to §3.29 for certain explicit special cases for classical groups. Observe also that in this latter case, the group C_ψ defined there following Arthur fortunately coïncides with the group C_ψ defined here.

An important situation where we can say more about the second part of (L5) is when C_ψ coincides with the center of $\widehat{G}$. In this case, and if $\pi \in \Pi(\psi)$, we have $m(\pi) \neq 0$ if π_∞ belongs to Arthur's conjectural set (or "packet") of unitary representations of $G(\mathbb{R})$ associated to ψ_∞. When $\psi(\mathrm{SL}_2(\mathbb{C})) = 1$, this packet is the set of unitary representations of $G(\mathbb{R})$ associated by Langlands to $\varphi_\psi \circ h$ in [Lan73]; it is never empty if G is a Chevalley group.

This last paragraph "explains" for instance Theorem 3.9 (as well as the results in §3.11). Indeed, a finite-dimensional selfdual irreducible representation of any group preserves a *unique* non-degenerate pairing up to scalars, either symmetric or anti-symmetric. This explains as well condition (ii) in the definition of a global Arthur parameter in §3.18.

This also leads to another way of thinking about $\mathcal{L}_\pi$ when $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, which involves all the semisimple groups over $\mathbb{Z}$. Indeed, fix $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ and let G_π be the Chevalley group such that $\widehat{G}_\pi$ is a complexification of the compact connected semisimple Lie group $\mathcal{L}_\pi$. By definition, we may factor r_π through a homomorphism

$$\widetilde{r_\pi} : \mathcal{L}_\mathbb{Z} \to \widehat{G_\pi}$$

such that $\mathrm{C}_{\widetilde{r_\pi}}$ is the center of $\widehat{G_\pi}$. We thus obtain *à la Langlands* a non-empty finite set of representations $\pi' \in \Pi_{\mathrm{disc}}(G_\pi)$ associated to $\widetilde{r_\pi}$. This explains for instance the discussion that we had about the group G_2 in the introduction. From this point of view, the results that we proved in §4.5 imply the following:

Fact 3 : $\mathcal{L}^1_\mathbb{Z}$ is simply connected.

Arthur has a similar prediction for $\mathcal{L}_\mathbb{Q}$ in [Art02], although this does not seem to directly imply that $\mathcal{L}_\mathbb{Z}$ should be simply connected as well. As a consequence of Fact 3, it follows that $\mathcal{L}_\mathbb{Z}$ is a direct product of $\mathbb{R}_{>0}$ and of countably many semisimple, connected, simply connected, compact Lie groups. The same property holds for $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$ by construction. It is a natural question to ask which semisimple, connected, simply connected compact Lie group appear as a direct factor of $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$ or of $\mathcal{L}_\mathbb{Z}$. The results of this paper show that this is indeed the case (for $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$) for each such group whose simple factors are of type A_1, B_2, G_2, B_3, C_3, C_4 or D_4. Let us mention that in their work [AP08], Ash and Pollack did search factors of $\mathcal{L}_\mathbb{Z}^{\mathrm{mot}}$ of type A_2 by computing cuspidal cohomology of $\mathrm{SL}_3(\mathbb{Z})$ for a quite large number of coefficients : they did not find any.

B.4. **A list in rank $n \leq 8$.** Our goal in this last paragraph is to determine the possible Sato-Tate groups of a $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n)$ when $n \leq 8$. For such a π, define $\mathcal{A}_\pi$ as the compact symplectic group of rank $n/2$ if $s(\pi) = -1$, the compact special orthogonal group $\mathrm{SO}(n)$ otherwise. By Arthur's theorem 3.9, $\mathcal{L}_\pi$ is isomorphic to a subgroup of $\mathcal{A}_\pi$.

Proposition B.5. *Assume the existence of $\mathcal{L}_\mathbb{Z}$ satisfying the axioms (L1)–(L5). Let $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n)$ and assume $n \leq 8$. Then $\mathcal{L}_\pi \simeq \mathcal{A}_\pi$ unless :*

(i) *$s(\pi) = (-1)^{n+1}$ and there exists a $\pi' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ such that $r_\pi \simeq \mathrm{Sym}^{n-1} r_{\pi'}$. In this case $\mathcal{L}_\pi \simeq \mathrm{SU}(2)$ if n is even, $\mathrm{SO}(3)$ if n is odd.*

(ii) *$n = 6$, $s(\pi) = -1$, and there exists two distinct $\pi', \pi'' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ such that $r_\pi \simeq r_{\pi'} \otimes \mathrm{Sym}^2 r_{\pi''}$. In this case $\mathcal{L}_\pi \simeq \mathrm{SU}(2) \times \mathrm{SO}(3)$.*

(iii) *$n = 7$ and $\mathcal{L}_\pi$ is the compact simple group of type G_2.*

(iv) *$n = 8$, $s(\pi) = 1$ and there exists $\pi' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$, $\pi'' \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_4)$, such that $r_\pi \simeq r_{\pi'} \otimes r_{\pi''}$. In this case $\mathcal{L}_\pi$ is the quotient of $\mathrm{SU}(2) \times \mathrm{Spin}(5)$ by the diagonal central $\{\pm 1\}$, and $\mathcal{L}_{\pi''} \simeq \mathrm{Spin}(5)$.*

(v) *$n = 8$, $s(\pi) = 1$ and there exists two distinct $\pi', \pi'' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ such that $r_\pi \simeq r_{\pi'} \otimes \mathrm{Sym}^3 r_{\pi''}$. In this case $\mathcal{L}_\pi$ is the quotient of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ by the diagonal central $\{\pm 1\}$.*

(vi) $n = 8$, $s(\pi) = 1$ *and* $\mathcal{L}_\pi \simeq \mathrm{Spin}(7)$. *This occurs if and only if there exists* $\pi' \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_7)$ *such that*

$$\mathcal{L}_{\pi'} \simeq \mathrm{SO}(7) \text{ and } \rho \circ \xi \simeq r_\pi,$$

where $\xi : \mathcal{L}_{\mathbb{Z}} \to \mathrm{Spin}(7)$ *denotes the unique lift of* $\widetilde{r_{\pi'}} : \mathcal{L}_{\mathbb{Z}} \to \mathrm{SO}(7)$, *and where* ρ *denotes the Spin representation of* $\mathrm{Spin}(7)$.

(vii) $n = 8$, $s(\pi) = -1$ *and there exists distinct* $\pi', \pi'', \pi''' \in \Pi_{\mathrm{alg}}(\mathrm{PGL}_2)$ *such that* $r_\pi \simeq r_{\pi'} \otimes r_{\pi''} \otimes r_{\pi'''}$. *In this case* $\mathcal{L}_\pi$ *is the quotient of* $\mathrm{SU}(2)^3$ *by the central subgroup* $\{(\epsilon_i) \in \{\pm 1\}^3, \epsilon_1\epsilon_2\epsilon_3 = 1\}$.

Proof — We first observe that the only simply connected quasi-simple compact Lie groups having a self-dual finite dimensional irreducible representation of dimension ≤ 8 are in types : A_1 in each dimension, $B_2 = C_2$ in dimensions 4 and 5, C_3 in dimension 6, G_2 and B_3 in dimension 7, A_2 and B_3 in dimension 8, and C_4 in dimension 8 (three representations permuted by triality).

The case $\mathcal{L}_\pi \simeq \mathrm{SU}(3)$ (type A_2), equipped with its 8-dimensional adjoint representation, does not occur. Indeed, if $r : \mathrm{W}_{\mathbb{R}} \to \mathrm{SU}(3)$ is a continuous 3-dimensional representation trivial on $\mathbb{R}_{>0} \subset \mathrm{W}_{\mathbb{C}}$, then the adjoint representation of r on $\mathrm{Lie}(\mathrm{SU}(3))$ is never multiplicity free, which contradicts $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_8)$.

We conclude the proof by a case-by-case inspection. □

Appendix C. Tables

$G = \mathrm{SO}_7(\mathbb{R})$, $\Gamma = \mathrm{W}^+(\mathrm{E}_7)$.

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0, 0, 0)	1	(9, 6, 3)	2	(10, 7, 2)	1	(10, 10, 10)	2	(11, 9, 0)	2
(4, 4, 4)	1	(9, 6, 4)	1	(10, 7, 3)	3	(11, 3, 0)	1	(11, 9, 1)	1
(6, 0, 0)	1	(9, 6, 6)	1	(10, 7, 4)	2	(11, 3, 2)	1	(11, 9, 2)	4
(6, 4, 0)	1	(9, 7, 2)	1	(10, 7, 5)	2	(11, 4, 1)	1	(11, 9, 3)	4
(6, 6, 0)	1	(9, 7, 3)	1	(10, 7, 6)	2	(11, 4, 3)	2	(11, 9, 4)	5
(6, 6, 6)	1	(9, 7, 4)	2	(10, 7, 7)	1	(11, 4, 4)	1	(11, 9, 5)	4
(7, 4, 3)	1	(9, 7, 6)	1	(10, 8, 0)	3	(11, 5, 0)	2	(11, 9, 6)	5
(7, 6, 3)	1	(9, 8, 1)	1	(10, 8, 2)	3	(11, 5, 2)	2	(11, 9, 7)	3
(7, 7, 3)	1	(9, 8, 3)	1	(10, 8, 3)	1	(11, 5, 3)	1	(11, 9, 8)	2
(7, 7, 7)	1	(9, 8, 4)	1	(10, 8, 4)	4	(11, 5, 4)	1	(11, 9, 9)	1
(8, 0, 0)	1	(9, 8, 5)	1	(10, 8, 5)	1	(11, 6, 1)	2	(11, 10, 1)	3
(8, 4, 0)	1	(9, 8, 6)	1	(10, 8, 6)	3	(11, 6, 2)	1	(11, 10, 2)	3
(8, 4, 2)	1	(9, 9, 0)	1	(10, 8, 7)	1	(11, 6, 3)	4	(11, 10, 3)	5
(8, 4, 4)	1	(9, 9, 3)	1	(10, 8, 8)	1	(11, 6, 4)	2	(11, 10, 4)	4
(8, 6, 0)	1	(9, 9, 4)	1	(10, 9, 1)	2	(11, 6, 5)	2	(11, 10, 5)	6
(8, 6, 2)	1	(9, 9, 6)	1	(10, 9, 2)	1	(11, 6, 6)	2	(11, 10, 6)	5
(8, 6, 4)	1	(9, 9, 9)	1	(10, 9, 3)	3	(11, 7, 0)	1	(11, 10, 7)	5
(8, 6, 6)	1	(10, 0, 0)	1	(10, 9, 4)	2	(11, 7, 1)	1	(11, 10, 8)	3
(8, 7, 2)	1	(10, 2, 0)	1	(10, 9, 5)	3	(11, 7, 2)	4	(11, 10, 9)	2
(8, 7, 4)	1	(10, 4, 0)	2	(10, 9, 6)	2	(11, 7, 3)	3	(11, 10, 10)	2
(8, 7, 6)	1	(10, 4, 2)	1	(10, 9, 7)	2	(11, 7, 4)	4	(11, 11, 1)	1
(8, 8, 0)	1	(10, 4, 3)	1	(10, 9, 8)	1	(11, 7, 5)	3	(11, 11, 2)	2
(8, 8, 2)	1	(10, 4, 4)	2	(10, 9, 9)	1	(11, 7, 6)	3	(11, 11, 3)	3
(8, 8, 4)	1	(10, 5, 1)	1	(10, 10, 0)	2	(11, 7, 7)	2	(11, 11, 4)	2
(8, 8, 6)	1	(10, 5, 3)	1	(10, 10, 2)	2	(11, 8, 1)	3	(11, 11, 5)	3
(8, 8, 8)	1	(10, 6, 0)	2	(10, 10, 3)	2	(11, 8, 2)	2	(11, 11, 6)	3
(9, 3, 0)	1	(10, 6, 2)	2	(10, 10, 4)	4	(11, 8, 3)	5	(11, 11, 7)	3
(9, 4, 3)	1	(10, 6, 3)	1	(10, 10, 5)	2	(11, 8, 4)	4	(11, 11, 8)	2
(9, 4, 4)	1	(10, 6, 4)	3	(10, 10, 6)	4	(11, 8, 5)	5	(11, 11, 9)	1
(9, 5, 0)	1	(10, 6, 5)	1	(10, 10, 7)	2	(11, 8, 6)	4	(11, 11, 10)	1
(9, 5, 2)	1	(10, 6, 6)	2	(10, 10, 8)	2	(11, 8, 7)	3	(11, 11, 11)	1
(9, 6, 1)	1	(10, 7, 1)	2	(10, 10, 9)	2	(11, 8, 8)	1	(12, 0, 0)	2

Table 2. The nonzero $d(\lambda) = \dim V_\lambda^\Gamma$ for $\lambda = (n_1, n_2, n_3)$ with $n_1 \leq 11$.

$G = \mathrm{SO}_8(\mathbb{R})$, $\Gamma = \mathrm{W}^+(\mathrm{E}_8)$.

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0, 0, 0, 0)	1	(10, 9, 1, 0)	1	(11, 8, 5, 2)	1	(11, 11, 3, 3)	2
(4, 4, 4, 4)	1	(10, 9, 4, 3)	1	(11, 8, 6, 1)	1	(11, 11, 4, 4)	1
(6, 6, 0, 0)	1	(10, 9, 5, 0)	1	(11, 8, 6, 3)	1	(11, 11, 5, 1)	1
(6, 6, 6, 6)	1	(10, 9, 6, 1)	1	(11, 8, 7, 2)	1	(11, 11, 5, 5)	2
(7, 7, 3, 3)	1	(10, 9, 7, 0)	1	(11, 8, 7, 4)	1	(11, 11, 6, 2)	1
(7, 7, 7, 7)	1	(10, 9, 9, 2)	1	(11, 8, 8, 3)	1	(11, 11, 6, 6)	2
(8, 0, 0, 0)	1	(10, 10, 0, 0)	1	(11, 9, 2, 0)	1	(11, 11, 7, 1)	2
(8, 4, 4, 0)	1	(10, 10, 2, 2)	1	(11, 9, 3, 1)	1	(11, 11, 7, 7)	2
(8, 6, 6, 0)	1	(10, 10, 3, 3)	1	(11, 9, 4, 2)	1	(11, 11, 8, 0)	2
(8, 7, 7, 0)	1	(10, 10, 4, 0)	1	(11, 9, 5, 1)	1	(11, 11, 8, 4)	1
(8, 8, 0, 0)	1	(10, 10, 4, 4)	2	(11, 9, 5, 3)	1	(11, 11, 8, 8)	1
(8, 8, 2, 2)	1	(10, 10, 5, 5)	1	(11, 9, 6, 0)	2	(11, 11, 9, 3)	1
(8, 8, 4, 4)	1	(10, 10, 6, 0)	1	(11, 9, 6, 4)	1	(11, 11, 9, 9)	1
(8, 8, 6, 6)	1	(10, 10, 6, 2)	1	(11, 9, 7, 1)	1	(11, 11, 10, 2)	1
(8, 8, 8, 0)	1	(10, 10, 6, 6)	2	(11, 9, 7, 3)	1	(11, 11, 10, 10)	1
(8, 8, 8, 8)	1	(10, 10, 7, 1)	1	(11, 9, 7, 5)	1	(11, 11, 11, 3)	1
(9, 6, 3, 0)	1	(10, 10, 7, 7)	1	(11, 9, 8, 2)	1	(11, 11, 11, 11)	1
(9, 7, 4, 2)	1	(10, 10, 8, 0)	1	(11, 9, 9, 3)	1	(12, 0, 0, 0)	1
(9, 8, 6, 1)	1	(10, 10, 8, 4)	1	(11, 10, 1, 0)	1	(12, 4, 0, 0)	1
(9, 9, 3, 3)	1	(10, 10, 8, 8)	1	(11, 10, 3, 2)	1	(12, 4, 4, 0)	1
(9, 9, 4, 4)	1	(10, 10, 9, 3)	1	(11, 10, 4, 1)	1	(12, 4, 4, 4)	1
(9, 9, 6, 6)	1	(10, 10, 9, 9)	1	(11, 10, 4, 3)	1	(12, 5, 3, 2)	1
(9, 9, 9, 9)	1	(10, 10, 10, 2)	2	(11, 10, 5, 0)	1	(12, 6, 0, 0)	1
(10, 4, 0, 0)	1	(10, 10, 10, 10)	2	(11, 10, 5, 2)	1	(12, 6, 2, 0)	1
(10, 4, 4, 2)	1	(11, 4, 4, 3)	1	(11, 10, 5, 4)	1	(12, 6, 4, 0)	1
(10, 6, 0, 0)	1	(11, 5, 2, 0)	1	(11, 10, 6, 1)	2	(12, 6, 4, 2)	1
(10, 6, 4, 0)	1	(11, 6, 3, 0)	1	(11, 10, 6, 5)	1	(12, 6, 6, 0)	2
(10, 6, 6, 2)	1	(11, 6, 4, 3)	1	(11, 10, 7, 0)	3	(12, 6, 6, 4)	1
(10, 7, 4, 1)	1	(11, 6, 6, 3)	1	(11, 10, 7, 2)	1	(12, 7, 3, 0)	1
(10, 7, 6, 3)	1	(11, 7, 3, 1)	1	(11, 10, 7, 6)	1	(12, 7, 3, 2)	1
(10, 7, 7, 2)	1	(11, 7, 4, 0)	1	(11, 10, 8, 1)	1	(12, 7, 4, 1)	1
(10, 8, 2, 0)	1	(11, 7, 5, 1)	1	(11, 10, 8, 3)	1	(12, 7, 4, 3)	1
(10, 8, 4, 0)	1	(11, 7, 6, 2)	1	(11, 10, 9, 2)	1	(12, 7, 5, 2)	1
(10, 8, 4, 2)	1	(11, 7, 7, 3)	1	(11, 10, 9, 4)	1	(12, 7, 6, 1)	1
(10, 8, 6, 0)	1	(11, 8, 3, 0)	1	(11, 10, 10, 3)	2	(12, 7, 6, 3)	1
(10, 8, 6, 4)	1	(11, 8, 4, 1)	1	(11, 11, 1, 1)	1	(12, 7, 7, 0)	1
(10, 8, 8, 2)	1	(11, 8, 5, 0)	1	(11, 11, 2, 2)	1	(12, 7, 7, 4)	1

TABLE 3. The nonzero $d(\lambda) = \dim V_\lambda^\Gamma$ for $\lambda = (n_1, n_2, n_3, n_4)$ with $n_1 \leq 11$.

$G = \mathrm{SO}_9(\mathbb{R})$, $\Gamma = \mathrm{W}(\mathrm{E}_8)$.

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0, 0, 0, 0)	1	(8, 6, 6, 2)	1	(9, 3, 0, 0)	1	(9, 7, 7, 3)	2	(9, 8, 8, 8)	1
(2, 0, 0, 0)	1	(8, 6, 6, 4)	1	(9, 4, 4, 1)	1	(9, 7, 7, 4)	2	(9, 9, 3, 1)	1
(4, 0, 0, 0)	1	(8, 6, 6, 6)	2	(9, 4, 4, 3)	1	(9, 7, 7, 6)	1	(9, 9, 3, 3)	2
(4, 4, 4, 4)	1	(8, 7, 3, 3)	1	(9, 4, 4, 4)	1	(9, 7, 7, 7)	1	(9, 9, 4, 0)	1
(5, 4, 4, 4)	1	(8, 7, 4, 1)	1	(9, 5, 0, 0)	1	(9, 8, 1, 0)	1	(9, 9, 4, 2)	2
(6, 0, 0, 0)	1	(8, 7, 4, 3)	2	(9, 5, 4, 0)	1	(9, 8, 2, 2)	1	(9, 9, 4, 3)	2
(6, 4, 4, 4)	1	(8, 7, 5, 3)	1	(9, 5, 4, 2)	1	(9, 8, 3, 0)	2	(9, 9, 4, 4)	2
(6, 6, 0, 0)	1	(8, 7, 6, 1)	1	(9, 5, 4, 4)	1	(9, 8, 3, 2)	2	(9, 9, 5, 1)	1
(6, 6, 2, 0)	1	(8, 7, 6, 3)	2	(9, 6, 1, 0)	1	(9, 8, 4, 1)	2	(9, 9, 5, 2)	1
(6, 6, 4, 0)	1	(8, 7, 6, 5)	1	(9, 6, 3, 0)	2	(9, 8, 4, 2)	2	(9, 9, 5, 3)	3
(6, 6, 6, 0)	1	(8, 7, 7, 1)	1	(9, 6, 3, 2)	1	(9, 8, 4, 3)	3	(9, 9, 5, 4)	2
(6, 6, 6, 6)	1	(8, 7, 7, 3)	2	(9, 6, 4, 1)	2	(9, 8, 4, 4)	2	(9, 9, 6, 0)	1
(7, 4, 4, 4)	1	(8, 7, 7, 5)	1	(9, 6, 4, 3)	2	(9, 8, 5, 0)	2	(9, 9, 6, 1)	1
(7, 6, 1, 0)	1	(8, 7, 7, 7)	2	(9, 6, 5, 0)	2	(9, 8, 5, 2)	3	(9, 9, 6, 2)	3
(7, 6, 3, 0)	1	(8, 8, 0, 0)	2	(9, 6, 5, 2)	1	(9, 8, 5, 3)	1	(9, 9, 6, 3)	3
(7, 6, 5, 0)	1	(8, 8, 2, 0)	1	(9, 6, 6, 1)	2	(9, 8, 5, 4)	2	(9, 9, 6, 4)	3
(7, 6, 6, 6)	1	(8, 8, 2, 2)	1	(9, 6, 6, 3)	2	(9, 8, 6, 1)	3	(9, 9, 6, 5)	1
(7, 7, 3, 3)	1	(8, 8, 3, 2)	1	(9, 6, 6, 5)	1	(9, 8, 6, 2)	3	(9, 9, 6, 6)	2
(7, 7, 4, 3)	1	(8, 8, 4, 0)	2	(9, 6, 6, 6)	1	(9, 8, 6, 3)	4	(9, 9, 7, 1)	1
(7, 7, 5, 3)	1	(8, 8, 4, 2)	2	(9, 7, 0, 0)	1	(9, 8, 6, 4)	3	(9, 9, 7, 2)	2
(7, 7, 6, 3)	1	(8, 8, 4, 4)	2	(9, 7, 3, 1)	1	(9, 8, 6, 5)	2	(9, 9, 7, 3)	3
(7, 7, 7, 3)	1	(8, 8, 5, 2)	1	(9, 7, 3, 3)	2	(9, 8, 6, 6)	2	(9, 9, 7, 4)	3
(7, 7, 7, 7)	1	(8, 8, 5, 4)	1	(9, 7, 4, 0)	2	(9, 8, 7, 0)	1	(9, 9, 7, 5)	1
(8, 0, 0, 0)	2	(8, 8, 6, 0)	2	(9, 7, 4, 2)	3	(9, 8, 7, 1)	2	(9, 9, 7, 6)	2
(8, 2, 0, 0)	1	(8, 8, 6, 2)	2	(9, 7, 4, 3)	2	(9, 8, 7, 2)	3	(9, 9, 8, 1)	1
(8, 4, 0, 0)	1	(8, 8, 6, 4)	2	(9, 7, 4, 4)	2	(9, 8, 7, 3)	3	(9, 9, 8, 2)	1
(8, 4, 4, 0)	1	(8, 8, 6, 6)	2	(9, 7, 5, 1)	1	(9, 8, 7, 4)	3	(9, 9, 8, 3)	2
(8, 4, 4, 2)	1	(8, 8, 7, 0)	1	(9, 7, 5, 2)	1	(9, 8, 7, 5)	2	(9, 9, 8, 4)	2
(8, 4, 4, 4)	2	(8, 8, 7, 2)	2	(9, 7, 5, 3)	3	(9, 8, 7, 6)	2	(9, 9, 8, 5)	1
(8, 5, 4, 1)	1	(8, 8, 7, 4)	2	(9, 7, 5, 4)	1	(9, 8, 7, 7)	1	(9, 9, 8, 6)	2
(8, 5, 4, 3)	1	(8, 8, 7, 6)	2	(9, 7, 6, 0)	2	(9, 8, 8, 1)	2	(9, 9, 9, 3)	1
(8, 6, 0, 0)	2	(8, 8, 8, 0)	2	(9, 7, 6, 2)	3	(9, 8, 8, 2)	2	(9, 9, 9, 4)	1
(8, 6, 2, 0)	1	(8, 8, 8, 2)	2	(9, 7, 6, 3)	2	(9, 8, 8, 3)	2	(9, 9, 9, 6)	1
(8, 6, 4, 0)	2	(8, 8, 8, 4)	2	(9, 7, 6, 4)	2	(9, 8, 8, 4)	2	(9, 9, 9, 9)	1
(8, 6, 4, 2)	1	(8, 8, 8, 6)	2	(9, 7, 6, 6)	1	(9, 8, 8, 5)	2	(10, 0, 0, 0)	2
(8, 6, 4, 4)	1	(8, 8, 8, 8)	2	(9, 7, 7, 0)	1	(9, 8, 8, 6)	2	(10, 2, 0, 0)	1
(8, 6, 6, 0)	2	(9, 1, 0, 0)	1	(9, 7, 7, 2)	2	(9, 8, 8, 7)	1	(10, 4, 0, 0)	2

TABLE 4. The nonzero $d(\lambda) = \dim V_\lambda^\Gamma$ for $\lambda = (n_1, n_2, n_3, n_4)$ with $n_1 \leq 9$.

$G = \mathrm{G}_2(\mathbb{R})$, $\Gamma = \mathrm{G}_2(\mathbb{Z})$.

(w,v)	$m(w,v)$	(w,v)	$m(w,v)$	(w,v)	$m(w,v)$	(w,v)	$m(w,v)$	(w,v)	$m(w,v)$
(4, 2)	1	(34, 2)	2	(26, 16)	4	(30, 18)	17	(42, 12)	54
(16, 2)	1	(32, 4)	3	(24, 18)	6	(28, 20)	15	(40, 14)	60
(20, 2)	1	(30, 6)	4	(22, 20)	3	(26, 22)	3	(38, 16)	45
(12, 10)	1	(28, 8)	6	(42, 2)	5	(48, 2)	14	(36, 18)	47
(16, 8)	1	(26, 10)	3	(40, 4)	11	(46, 4)	22	(34, 20)	38
(24, 2)	1	(24, 12)	4	(38, 6)	13	(44, 6)	31	(32, 22)	24
(22, 4)	1	(22, 14)	3	(36, 8)	15	(42, 8)	31	(30, 24)	15
(20, 6)	1	(36, 2)	4	(34, 10)	16	(40, 10)	37	(28, 26)	13
(16, 10)	1	(34, 4)	4	(32, 12)	17	(38, 12)	37	(54, 2)	20
(24, 4)	1	(32, 6)	8	(30, 14)	12	(36, 14)	32	(52, 4)	39
(22, 6)	1	(30, 8)	6	(28, 16)	11	(34, 16)	28	(50, 6)	51
(20, 8)	1	(28, 10)	8	(26, 18)	9	(32, 18)	29	(48, 8)	60
(18, 10)	1	(26, 12)	6	(24, 20)	2	(30, 20)	15	(46, 10)	66
(28, 2)	3	(24, 14)	4	(44, 2)	10	(28, 22)	12	(44, 12)	72
(24, 6)	2	(22, 16)	2	(42, 4)	14	(26, 24)	5	(42, 14)	64
(22, 8)	2	(20, 18)	4	(40, 6)	18	(50, 2)	13	(40, 16)	64
(20, 10)	1	(38, 2)	3	(38, 8)	20	(48, 4)	27	(38, 18)	60
(16, 14)	2	(36, 4)	7	(36, 10)	25	(46, 6)	33	(36, 20)	45
(30, 2)	1	(34, 6)	7	(34, 12)	17	(44, 8)	41	(34, 22)	37
(28, 4)	2	(32, 8)	9	(32, 14)	20	(42, 10)	44	(32, 24)	30
(26, 6)	2	(30, 10)	9	(30, 16)	17	(40, 12)	42	(30, 26)	10
(24, 8)	2	(28, 12)	7	(28, 18)	11	(38, 14)	41	(56, 2)	29
(22, 10)	2	(26, 14)	6	(26, 20)	6	(36, 16)	41	(54, 4)	48
(20, 12)	2	(24, 16)	6	(24, 22)	6	(34, 18)	30	(52, 6)	63
(32, 2)	3	(22, 18)	2	(46, 2)	9	(32, 20)	26	(50, 8)	74
(30, 4)	3	(40, 2)	8	(44, 4)	16	(30, 22)	20	(48, 10)	88
(28, 6)	3	(38, 4)	8	(42, 6)	21	(28, 24)	6	(46, 12)	82
(26, 8)	3	(36, 6)	12	(40, 8)	28	(52, 2)	23	(44, 14)	87
(24, 10)	5	(34, 8)	13	(38, 10)	25	(50, 4)	29	(42, 16)	83
(22, 12)	2	(32, 10)	12	(36, 12)	27	(48, 6)	45	(40, 18)	72
(20, 14)	2	(30, 12)	11	(34, 14)	26	(46, 8)	52	(38, 20)	63
(18, 16)	1	(28, 14)	13	(32, 16)	19	(44, 10)	54	(36, 22)	58

TABLE 5. The nonzero $m(w,v) = \dim U^{\Gamma}_{w,v}$ for $v + w \leq 56$.

TABLE 6. The nonzero $S(\underline{w})$ for $\underline{w} = (w_1, w_2)$ and $w_1 \leq 43$, using Tsushima's formula [TSU83].

$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$
(19, 7)	1	(29, 21)	2	(35, 13)	5	(39, 15)	10	(43, 5)	3
(21, 5)	1	(29, 25)	1	(35, 15)	6	(39, 17)	8	(43, 7)	9
(21, 9)	1	(31, 3)	2	(35, 17)	5	(39, 19)	11	(43, 9)	7
(21, 13)	1	(31, 5)	1	(35, 19)	7	(39, 21)	10	(43, 11)	11
(23, 7)	1	(31, 7)	3	(35, 21)	6	(39, 23)	10	(43, 13)	11
(23, 9)	1	(31, 9)	2	(35, 23)	5	(39, 25)	10	(43, 15)	15
(23, 13)	1	(31, 11)	3	(35, 25)	5	(39, 27)	9	(43, 17)	13
(25, 5)	1	(31, 13)	4	(35, 27)	3	(39, 29)	7	(43, 19)	17
(25, 7)	1	(31, 15)	4	(35, 29)	2	(39, 31)	6	(43, 21)	14
(25, 9)	2	(31, 17)	3	(35, 31)	1	(39, 33)	4	(43, 23)	16
(25, 11)	1	(31, 19)	4	(37, 1)	1	(39, 35)	1	(43, 25)	16
(25, 13)	2	(31, 21)	3	(37, 5)	4	(39, 37)	1	(43, 27)	16
(25, 15)	1	(31, 23)	2	(37, 7)	3	(41, 1)	1	(43, 29)	14
(25, 17)	1	(31, 25)	2	(37, 9)	7	(41, 3)	1	(43, 31)	14
(25, 19)	1	(33, 5)	3	(37, 11)	5	(41, 5)	6	(43, 33)	11
(27, 3)	1	(33, 7)	2	(37, 13)	9	(41, 7)	4	(43, 35)	8
(27, 7)	2	(33, 9)	5	(37, 15)	6	(41, 9)	9	(43, 37)	7
(27, 9)	1	(33, 11)	2	(37, 17)	9	(41, 11)	6	(43, 39)	3
(27, 11)	2	(33, 13)	6	(37, 19)	8	(41, 13)	13	(45, 1)	2
(27, 13)	2	(33, 15)	4	(37, 21)	10	(41, 15)	10	(45, 3)	1
(27, 15)	2	(33, 17)	6	(37, 23)	7	(41, 17)	13	(45, 5)	8
(27, 17)	1	(33, 19)	5	(37, 25)	9	(41, 19)	11	(45, 7)	6
(27, 19)	1	(33, 21)	5	(37, 27)	6	(41, 21)	14	(45, 9)	13
(27, 21)	1	(33, 23)	3	(37, 29)	5	(41, 23)	11	(45, 11)	9
(29, 5)	2	(33, 25)	4	(37, 31)	4	(41, 25)	15	(45, 13)	17
(29, 7)	1	(33, 27)	2	(37, 33)	2	(41, 27)	11	(45, 15)	13
(29, 9)	3	(33, 29)	1	(39, 3)	3	(41, 29)	11	(45, 17)	19
(29, 11)	1	(35, 3)	2	(39, 5)	2	(41, 31)	9	(45, 19)	17
(29, 13)	4	(35, 5)	1	(39, 7)	7	(41, 33)	8	(45, 21)	21
(29, 15)	2	(35, 7)	5	(39, 9)	5	(41, 35)	4	(45, 23)	16
(29, 17)	3	(35, 9)	4	(39, 11)	8	(41, 37)	3	(45, 25)	22
(29, 19)	2	(35, 11)	5	(39, 13)	8	(43, 3)	5	(45, 27)	18

TABLE 7. The nonzero $S(\underline{w})$ for $\underline{w} = (w_1, w_2, w_3)$ and $w_1 \leq 29$.

$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$
(23, 13, 5)	1	(27, 17, 11)	1	(29, 11, 5)	1	(29, 21, 19)	1
(23, 15, 3)	1	(27, 17, 13)	1	(29, 13, 3)	1	(29, 23, 1)	1
(23, 15, 7)	1	(27, 19, 3)	2	(29, 13, 5)	1	(29, 23, 3)	2
(23, 17, 5)	1	(27, 19, 5)	2	(29, 13, 7)	3	(29, 23, 5)	5
(23, 17, 9)	1	(27, 19, 7)	3	(29, 13, 9)	1	(29, 23, 7)	5
(23, 19, 3)	1	(27, 19, 9)	3	(29, 15, 1)	1	(29, 23, 9)	6
(23, 19, 11)	1	(27, 19, 11)	3	(29, 15, 5)	3	(29, 23, 11)	7
(25, 13, 3)	1	(27, 19, 13)	2	(29, 15, 7)	2	(29, 23, 13)	5
(25, 13, 7)	1	(27, 19, 15)	1	(29, 15, 9)	3	(29, 23, 15)	5
(25, 15, 5)	1	(27, 21, 1)	1	(29, 15, 13)	1	(29, 23, 17)	3
(25, 15, 9)	1	(27, 21, 5)	4	(29, 17, 3)	3	(29, 23, 19)	1
(25, 17, 3)	2	(27, 21, 7)	2	(29, 17, 5)	1	(29, 25, 3)	3
(25, 17, 7)	2	(27, 21, 9)	4	(29, 17, 7)	6	(29, 25, 5)	3
(25, 17, 11)	1	(27, 21, 11)	2	(29, 17, 9)	3	(29, 25, 7)	7
(25, 19, 1)	1	(27, 21, 13)	3	(29, 17, 11)	3	(29, 25, 9)	4
(25, 19, 5)	2	(27, 21, 15)	1	(29, 17, 13)	1	(29, 25, 11)	7
(25, 19, 9)	2	(27, 21, 17)	1	(29, 19, 1)	1	(29, 25, 13)	4
(25, 19, 13)	1	(27, 23, 3)	1	(29, 19, 3)	1	(29, 25, 15)	5
(25, 21, 3)	2	(27, 23, 5)	3	(29, 19, 5)	6	(29, 25, 17)	3
(25, 21, 7)	2	(27, 23, 7)	1	(29, 19, 7)	3	(29, 25, 19)	2
(25, 21, 11)	2	(27, 23, 9)	2	(29, 19, 9)	7	(29, 25, 21)	1
(25, 21, 15)	1	(27, 23, 11)	2	(29, 19, 11)	4	(29, 27, 1)	1
(27, 9, 5)	1	(27, 23, 13)	1	(29, 19, 13)	5	(29, 27, 5)	1
(27, 13, 5)	2	(27, 23, 15)	1	(29, 19, 15)	1	(29, 27, 7)	2
(27, 13, 7)	1	(27, 23, 17)	1	(29, 19, 17)	1	(29, 27, 9)	3
(27, 13, 9)	1	(27, 25, 5)	2	(29, 21, 3)	5	(29, 27, 11)	1
(27, 15, 3)	1	(27, 25, 7)	1	(29, 21, 5)	1	(29, 27, 13)	2
(27, 15, 5)	1	(27, 25, 9)	1	(29, 21, 7)	10	(29, 27, 15)	1
(27, 15, 7)	2	(27, 25, 11)	1	(29, 21, 9)	4	(29, 27, 17)	1
(27, 15, 9)	1	(27, 25, 13)	1	(29, 21, 11)	8	(29, 27, 19)	1
(27, 17, 5)	4	(27, 25, 15)	1	(29, 21, 13)	4	(31, 9, 5)	1
(27, 17, 7)	1	(27, 25, 17)	1	(29, 21, 15)	5	(31, 11, 3)	1
(27, 17, 9)	3	(29, 9, 7)	1	(29, 21, 17)	1	(31, 11, 7)	1

TABLE 8. The nonzero S($\underline{w}$) for $\underline{w} = (w_1, w_2, w_3, w_4)$ and $w_1 \le 27$.

$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$	$\underline{w}$	$S(\underline{w})$
(25, 17, 9, 5)	1	(27, 17, 13, 7)	2	(27, 21, 19, 7)	1	(27, 23, 21, 9)	1
(25, 17, 13, 5)	1	(27, 19, 9, 5)	1	(27, 21, 19, 9)	1	(27, 25, 9, 3)	2
(25, 19, 9, 3)	1	(27, 19, 11, 3)	2	(27, 21, 19, 11)	1	(27, 25, 11, 1)	1
(25, 19, 11, 5)	1	(27, 19, 11, 5)	1	(27, 23, 7, 3)	2	(27, 25, 11, 3)	1
(25, 19, 13, 3)	1	(27, 19, 13, 1)	1	(27, 23, 9, 1)	1	(27, 25, 11, 5)	2
(25, 19, 13, 5)	1	(27, 19, 13, 3)	1	(27, 23, 9, 5)	2	(27, 25, 13, 3)	5
(25, 19, 13, 7)	1	(27, 19, 13, 5)	4	(27, 23, 11, 3)	5	(27, 25, 13, 5)	1
(25, 19, 13, 9)	1	(27, 19, 13, 7)	1	(27, 23, 11, 5)	1	(27, 25, 13, 7)	4
(25, 19, 15, 5)	1	(27, 19, 13, 9)	3	(27, 23, 11, 7)	4	(27, 25, 13, 9)	1
(25, 21, 11, 7)	1	(27, 19, 15, 3)	2	(27, 23, 13, 1)	4	(27, 25, 15, 1)	3
(25, 21, 13, 5)	1	(27, 19, 15, 5)	1	(27, 23, 13, 3)	1	(27, 25, 15, 3)	2
(25, 21, 13, 7)	1	(27, 19, 15, 7)	1	(27, 23, 13, 5)	6	(27, 25, 15, 5)	5
(25, 21, 15, 3)	1	(27, 19, 15, 9)	1	(27, 23, 13, 7)	3	(27, 25, 15, 7)	3
(25, 21, 15, 5)	1	(27, 19, 17, 5)	1	(27, 23, 13, 9)	6	(27, 25, 15, 9)	5
(25, 21, 15, 7)	2	(27, 19, 17, 9)	1	(27, 23, 15, 3)	7	(27, 25, 15, 11)	1
(25, 21, 15, 9)	1	(27, 21, 9, 3)	2	(27, 23, 15, 5)	3	(27, 25, 17, 3)	7
(25, 21, 17, 5)	1	(27, 21, 9, 7)	1	(27, 23, 15, 7)	7	(27, 25, 17, 5)	2
(25, 21, 17, 7)	1	(27, 21, 11, 3)	1	(27, 23, 15, 9)	4	(27, 25, 17, 7)	7
(25, 21, 17, 9)	1	(27, 21, 11, 5)	2	(27, 23, 15, 11)	5	(27, 25, 17, 9)	4
(25, 23, 9, 3)	1	(27, 21, 11, 7)	2	(27, 23, 15, 13)	1	(27, 25, 17, 11)	5
(25, 23, 11, 1)	1	(27, 21, 13, 3)	5	(27, 23, 17, 1)	5	(27, 25, 17, 13)	1
(25, 23, 11, 5)	2	(27, 21, 13, 5)	2	(27, 23, 17, 3)	2	(27, 25, 19, 1)	3
(25, 23, 13, 3)	1	(27, 21, 13, 7)	6	(27, 23, 17, 5)	6	(27, 25, 19, 3)	2
(25, 23, 13, 7)	1	(27, 21, 13, 9)	2	(27, 23, 17, 7)	5	(27, 25, 19, 5)	5
(25, 23, 15, 1)	1	(27, 21, 15, 1)	1	(27, 23, 17, 9)	7	(27, 25, 19, 7)	3
(25, 23, 15, 5)	3	(27, 21, 15, 3)	2	(27, 23, 17, 11)	3	(27, 25, 19, 9)	6
(25, 23, 15, 9)	1	(27, 21, 15, 5)	4	(27, 23, 17, 13)	4	(27, 25, 19, 11)	3
(25, 23, 15, 11)	1	(27, 21, 15, 7)	4	(27, 23, 19, 3)	5	(27, 25, 19, 13)	3
(25, 23, 17, 3)	1	(27, 21, 15, 9)	4	(27, 23, 19, 5)	1	(27, 25, 21, 3)	4
(25, 23, 17, 5)	1	(27, 21, 15, 11)	2	(27, 23, 19, 7)	6	(27, 25, 21, 7)	4
(25, 23, 17, 7)	1	(27, 21, 17, 3)	5	(27, 23, 19, 9)	2	(27, 25, 21, 9)	2
(25, 23, 17, 11)	1	(27, 21, 17, 7)	6	(27, 23, 19, 11)	3	(27, 25, 21, 11)	3
(25, 23, 19, 5)	1	(27, 21, 17, 9)	2	(27, 23, 19, 13)	1	(27, 25, 21, 13)	1
(27, 17, 9, 3)	1	(27, 21, 17, 11)	3	(27, 23, 19, 15)	1	(27, 25, 21, 15)	1
(27, 17, 9, 7)	1	(27, 21, 19, 3)	1	(27, 23, 21, 1)	1	(27, 25, 23, 3)	1
(27, 17, 13, 3)	2	(27, 21, 19, 5)	1	(27, 23, 21, 5)	1	(27, 25, 23, 9)	1
(27, 25, 23, 11)	1						

Table 9. The nonzero $O(\underline{w})$ for $\underline{w} = (w_1, w_2, w_3, w_4)$ and $0 < w_4 < w_1 \leq 30$.

$\underline{w}$	$O(\underline{w})$	$\underline{w}$	$O(\underline{w})$	$\underline{w}$	$O(\underline{w})$	$\underline{w}$	$O(\underline{w})$
(24, 18, 10, 4)	1	(28, 24, 14, 2)	2	(30, 22, 14, 2)	2	(30, 26, 16, 12)	1
(24, 20, 14, 2)	1	(28, 24, 14, 10)	1	(30, 22, 14, 6)	3	(30, 26, 18, 2)	3
(26, 18, 10, 2)	1	(28, 24, 16, 4)	1	(30, 22, 16, 4)	2	(30, 26, 18, 6)	2
(26, 18, 14, 6)	1	(28, 24, 16, 12)	1	(30, 22, 16, 8)	1	(30, 26, 18, 10)	1
(26, 20, 10, 4)	1	(28, 24, 18, 2)	1	(30, 22, 18, 2)	1	(30, 26, 18, 14)	1
(26, 20, 14, 8)	1	(28, 24, 18, 6)	1	(30, 22, 18, 6)	1	(30, 26, 20, 4)	3
(26, 22, 10, 6)	1	(28, 24, 20, 4)	1	(30, 22, 18, 10)	1	(30, 26, 20, 8)	1
(26, 22, 14, 2)	1	(28, 24, 20, 8)	1	(30, 24, 8, 2)	1	(30, 26, 22, 2)	1
(26, 24, 14, 4)	1	(28, 26, 12, 2)	1	(30, 24, 10, 4)	3	(30, 26, 22, 6)	2
(26, 24, 16, 2)	1	(28, 26, 14, 4)	1	(30, 24, 12, 2)	2	(30, 26, 22, 10)	1
(26, 24, 18, 8)	1	(28, 26, 16, 2)	2	(30, 24, 12, 6)	2	(30, 28, 10, 4)	1
(26, 24, 20, 6)	1	(28, 26, 18, 8)	1	(30, 24, 14, 4)	2	(30, 28, 10, 8)	1
(28, 16, 10, 6)	1	(28, 26, 20, 6)	1	(30, 24, 14, 8)	3	(30, 28, 12, 2)	1
(28, 18, 8, 2)	1	(28, 26, 22, 4)	1	(30, 24, 16, 2)	3	(30, 28, 14, 4)	3
(28, 18, 12, 2)	1	(30, 14, 8, 4)	1	(30, 24, 16, 6)	2	(30, 28, 14, 12)	1
(28, 18, 14, 4)	1	(30, 16, 10, 4)	1	(30, 24, 16, 10)	2	(30, 28, 16, 2)	2
(28, 20, 10, 2)	1	(30, 18, 8, 4)	1	(30, 24, 18, 4)	4	(30, 28, 16, 6)	1
(28, 20, 12, 4)	1	(30, 18, 10, 2)	1	(30, 24, 18, 8)	1	(30, 28, 18, 4)	1
(28, 20, 14, 2)	1	(30, 18, 10, 6)	1	(30, 24, 18, 12)	2	(30, 28, 18, 8)	2
(28, 20, 14, 6)	1	(30, 18, 12, 4)	1	(30, 24, 20, 2)	2	(30, 28, 20, 2)	1
(28, 20, 16, 4)	1	(30, 18, 14, 2)	1	(30, 24, 20, 6)	2	(30, 28, 20, 6)	3
(28, 20, 16, 8)	1	(30, 18, 14, 6)	1	(30, 24, 20, 10)	1	(30, 28, 20, 10)	1
(28, 22, 8, 2)	1	(30, 20, 6, 4)	1	(30, 24, 20, 14)	1	(30, 28, 22, 4)	4
(28, 22, 10, 4)	1	(30, 20, 10, 4)	1	(30, 24, 22, 8)	1	(30, 28, 22, 8)	1
(28, 22, 12, 2)	1	(30, 20, 10, 8)	1	(30, 24, 22, 16)	1	(30, 28, 22, 12)	1
(28, 22, 12, 6)	1	(30, 20, 12, 2)	1	(30, 26, 6, 2)	1	(32, 16, 8, 4)	1
(28, 22, 14, 8)	1	(30, 20, 14, 4)	3	(30, 26, 8, 4)	1	(32, 16, 10, 2)	1
(28, 22, 16, 2)	1	(30, 20, 14, 8)	1	(30, 26, 10, 2)	1	(32, 16, 10, 6)	1
(28, 22, 16, 6)	1	(30, 20, 14, 12)	1	(30, 26, 10, 6)	2	(32, 18, 8, 2)	1
(28, 22, 16, 10)	1	(30, 20, 16, 2)	1	(30, 26, 12, 4)	2	(32, 18, 8, 6)	1
(28, 22, 18, 4)	1	(30, 20, 16, 6)	1	(30, 26, 12, 8)	1	(32, 18, 10, 4)	2
(28, 24, 8, 4)	1	(30, 20, 18, 8)	1	(30, 26, 14, 2)	3	(32, 18, 10, 8)	1
(28, 24, 10, 2)	1	(30, 22, 8, 4)	1	(30, 26, 14, 6)	1	(32, 18, 12, 2)	1
(28, 24, 10, 6)	1	(30, 22, 10, 2)	2	(30, 26, 14, 10)	2	(32, 18, 12, 6)	1
(28, 24, 12, 4)	1	(30, 22, 10, 6)	1	(30, 26, 16, 4)	2	(32, 18, 14, 4)	1
(28, 24, 12, 8)	1	(30, 22, 12, 4)	1	(30, 26, 16, 8)	1	(32, 18, 14, 8)	1

TABLE 10. The nonzero $\mathrm{O}(\underline{w}) = 2 \cdot \mathrm{O}(w_1, w_2, w_3, 0) + \mathrm{O}^*(w_1, w_2, w_3)$ for $w_1 \leq 34$.

$\underline{w}$	$O(\underline{w})$	$\underline{w}$	$O(\underline{w})$	$\underline{w}$	$O(\underline{w})$	$\underline{w}$	$O(\underline{w})$
(24, 16, 8, 0)	1	(30, 28, 10, 0)	2	(32, 30, 10, 0)	2	(34, 28, 26, 0)	2
(26, 16, 10, 0)	1	(30, 28, 14, 0)	3	(32, 30, 14, 0)	4	(34, 30, 4, 0)	2
(26, 20, 6, 0)	1	(30, 28, 18, 0)	5	(32, 30, 18, 0)	6	(34, 30, 8, 0)	2
(26, 20, 10, 0)	1	(30, 28, 26, 0)	1	(32, 30, 26, 0)	6	(34, 30, 12, 0)	7
(26, 20, 14, 0)	1	(32, 12, 8, 0)	1	(34, 12, 6, 0)	1	(34, 30, 16, 0)	14
(26, 24, 10, 0)	1	(32, 14, 10, 0)	1	(34, 14, 8, 0)	1	(34, 30, 20, 0)	6
(26, 24, 14, 0)	1	(32, 16, 4, 0)	1	(34, 16, 6, 0)	1	(34, 30, 24, 0)	7
(26, 24, 18, 0)	1	(32, 16, 8, 0)	1	(34, 16, 10, 0)	3	(34, 32, 2, 0)	1
(28, 14, 6, 0)	1	(32, 16, 12, 0)	1	(34, 16, 14, 0)	1	(34, 32, 6, 0)	2
(28, 16, 8, 0)	1	(32, 18, 6, 0)	1	(34, 18, 4, 0)	1	(34, 32, 10, 0)	6
(28, 18, 10, 0)	1	(32, 18, 10, 0)	1	(34, 18, 8, 0)	1	(34, 32, 14, 0)	8
(28, 20, 8, 0)	1	(32, 18, 14, 0)	3	(34, 18, 12, 0)	3	(34, 32, 18, 0)	13
(28, 20, 12, 0)	1	(32, 20, 4, 0)	1	(34, 20, 6, 0)	3	(34, 32, 22, 0)	3
(28, 22, 14, 0)	2	(32, 20, 8, 0)	2	(34, 20, 10, 0)	3	(34, 32, 26, 0)	14
(28, 24, 4, 0)	1	(32, 20, 12, 0)	2	(34, 20, 14, 0)	8	(36, 12, 8, 0)	1
(28, 24, 12, 0)	1	(32, 20, 16, 0)	3	(34, 20, 18, 0)	2	(36, 14, 6, 0)	1
(28, 24, 16, 0)	3	(32, 22, 6, 0)	1	(34, 22, 4, 0)	1	(36, 14, 10, 0)	1
(28, 26, 18, 0)	2	(32, 22, 10, 0)	4	(34, 22, 8, 0)	3	(36, 16, 4, 0)	1
(30, 16, 6, 0)	1	(32, 22, 14, 0)	1	(34, 22, 12, 0)	3	(36, 16, 8, 0)	3
(30, 16, 10, 0)	1	(32, 22, 18, 0)	3	(34, 22, 16, 0)	5	(36, 16, 12, 0)	2
(30, 16, 14, 0)	1	(32, 24, 4, 0)	1	(34, 24, 6, 0)	3	(36, 18, 6, 0)	2
(30, 18, 8, 0)	1	(32, 24, 8, 0)	5	(34, 24, 10, 0)	11	(36, 18, 10, 0)	5
(30, 20, 6, 0)	1	(32, 24, 12, 0)	5	(34, 24, 14, 0)	7	(36, 18, 14, 0)	4
(30, 20, 10, 0)	4	(32, 24, 16, 0)	4	(34, 24, 18, 0)	12	(36, 20, 4, 0)	2
(30, 20, 14, 0)	1	(32, 24, 20, 0)	6	(34, 24, 22, 0)	2	(36, 20, 8, 0)	4
(30, 20, 18, 0)	1	(32, 26, 6, 0)	2	(34, 26, 4, 0)	1	(36, 20, 12, 0)	6
(30, 22, 8, 0)	1	(32, 26, 10, 0)	4	(34, 26, 8, 0)	6	(36, 20, 16, 0)	6
(30, 22, 12, 0)	1	(32, 26, 14, 0)	8	(34, 26, 12, 0)	9	(36, 22, 6, 0)	3
(30, 24, 6, 0)	2	(32, 26, 18, 0)	3	(34, 26, 16, 0)	7	(36, 22, 10, 0)	6
(30, 24, 10, 0)	2	(32, 26, 22, 0)	5	(34, 26, 20, 0)	12	(36, 22, 14, 0)	10
(30, 24, 14, 0)	5	(32, 28, 4, 0)	2	(34, 26, 24, 0)	2	(36, 22, 18, 0)	6
(30, 24, 18, 0)	2	(32, 28, 8, 0)	2	(34, 28, 6, 0)	6	(36, 24, 4, 0)	3
(30, 26, 8, 0)	1	(32, 28, 12, 0)	5	(34, 28, 10, 0)	8	(36, 24, 8, 0)	8
(30, 26, 12, 0)	1	(32, 28, 16, 0)	9	(34, 28, 14, 0)	16	(36, 24, 12, 0)	13
(30, 26, 16, 0)	3	(32, 28, 20, 0)	4	(34, 28, 18, 0)	11	(36, 24, 16, 0)	16
(30, 28, 2, 0)	1	(32, 28, 24, 0)	3	(34, 28, 22, 0)	14	(36, 24, 20, 0)	12

Table 11. The nonzero $G_2(\underline{w})$ for $\underline{w} = (w, v)$ and $w + v \leq 58$.

(w,v)	$G_2(\underline{w})$	(w,v)	$G_2(\underline{w})$	(w,v)	$G_2(\underline{w})$	(w,v)	$G_2(\underline{w})$	(w,v)	$G_2(\underline{w})$
(16, 8)	1	(30, 8)	4	(44, 2)	7	(28, 22)	12	(44, 12)	72
(20, 6)	1	(28, 10)	8	(42, 4)	13	(26, 24)	4	(42, 14)	61
(16, 10)	1	(26, 12)	6	(40, 6)	18	(50, 2)	11	(40, 16)	64
(24, 4)	1	(24, 14)	4	(38, 8)	18	(48, 4)	27	(38, 18)	58
(20, 8)	1	(20, 18)	3	(36, 10)	25	(46, 6)	29	(36, 20)	45
(18, 10)	1	(38, 2)	2	(34, 12)	15	(44, 8)	41	(34, 22)	34
(28, 2)	1	(36, 4)	7	(32, 14)	20	(42, 10)	42	(32, 24)	30
(24, 6)	2	(34, 6)	5	(30, 16)	15	(40, 12)	42	(30, 26)	7
(22, 8)	1	(32, 8)	9	(28, 18)	11	(38, 14)	39	(56, 2)	25
(20, 10)	1	(30, 10)	8	(26, 20)	6	(36, 16)	41	(54, 4)	44
(16, 14)	1	(28, 12)	7	(24, 22)	4	(34, 18)	27	(52, 6)	63
(28, 4)	2	(26, 14)	6	(46, 2)	7	(32, 20)	26	(50, 8)	72
(26, 6)	2	(24, 16)	6	(44, 4)	16	(30, 22)	18	(48, 10)	88
(24, 8)	2	(40, 2)	5	(42, 6)	19	(28, 24)	6	(46, 12)	76
(22, 10)	1	(38, 4)	6	(40, 8)	28	(52, 2)	19	(44, 14)	87
(20, 12)	2	(36, 6)	12	(38, 10)	23	(50, 4)	27	(42, 16)	81
(32, 2)	1	(34, 8)	12	(36, 12)	27	(48, 6)	45	(40, 18)	72
(30, 4)	2	(32, 10)	12	(34, 14)	24	(46, 8)	48	(38, 20)	60
(28, 6)	3	(30, 12)	9	(32, 16)	19	(44, 10)	54	(36, 22)	58
(26, 8)	3	(28, 14)	13	(30, 18)	15	(42, 12)	52	(34, 24)	29
(24, 10)	5	(26, 16)	4	(28, 20)	15	(40, 14)	60	(32, 26)	26
(20, 14)	2	(24, 18)	6	(26, 22)	3	(38, 16)	42	(30, 28)	6
(34, 2)	1	(42, 2)	3	(48, 2)	11	(36, 18)	47	(58, 2)	25
(32, 4)	3	(40, 4)	11	(46, 4)	18	(34, 20)	36	(56, 4)	54
(30, 6)	3	(38, 6)	12	(44, 6)	31	(32, 22)	24	(54, 6)	69
(28, 8)	6	(36, 8)	15	(42, 8)	29	(30, 24)	12	(52, 8)	93
(26, 10)	3	(34, 10)	14	(40, 10)	37	(28, 26)	11	(50, 10)	92
(24, 12)	4	(32, 12)	17	(38, 12)	35	(54, 2)	16	(48, 12)	104
(22, 14)	2	(30, 14)	10	(36, 14)	32	(52, 4)	39	(46, 14)	102
(36, 2)	2	(28, 16)	11	(34, 16)	26	(50, 6)	49	(44, 16)	96
(34, 4)	3	(26, 18)	9	(32, 18)	29	(48, 8)	60	(42, 18)	89
(32, 6)	8	(24, 20)	2	(30, 20)	12	(46, 10)	62	(40, 20)	88

TABLE 12. The nonempty $\Pi_{w_1,w_2,w_3}(\mathrm{SO}_7)$ for $w_1 \leq 23$

(w_1, w_2, w_3)	$\Pi_{w_1,w_2,w_3}(\mathrm{SO}_7)$	(w_1, w_2, w_3)	$\Pi_{w_1,w_2,w_3}(\mathrm{SO}_7)$
(5,3,1)	$[6]$	(21,19,17)	$\Delta_{19}[3]$
(13,11,9)	$\Delta_{11}[3]$	(23,9,1)	$\Delta_{23,9} \oplus [2]$
(17,3,1)	$\Delta_{17} \oplus [4]$	(23,11,7)	$\Delta_{23,7} \oplus \Delta_{11}$
(17,11,1)	$\Delta_{17} \oplus \Delta_{11} \oplus [2]$	(23,11,9)	$\Delta_{23,9} \oplus \Delta_{11}$
(17,15,1)	$\Delta_{17} \oplus \Delta_{15} \oplus [2]$	(23,13,1)	$\Delta_{23,13} \oplus [2]$
(17,15,13)	$\Delta_{15}[3]$	(23,13,5)	$\Delta_{23,13,5}$
(19,11,7)	$\Delta_{19,7} \oplus \Delta_{11}$	(23,15,3)	$\Delta_{23,15,3}$
(19,15,7)	$\Delta_{19,7} \oplus \Delta_{15}$	(23,15,7)	$\Delta_{23,7} \oplus \Delta_{15}$, $\Delta_{23,15,7}$
(19,17,7)	$\Delta_{19,7} \oplus \Delta_{17}$	(23,15,9)	$\Delta_{23,9} \oplus \Delta_{15}$
(19,17,15)	$\Delta_{17}[3]$	(23,15,13)	$\Delta_{23,13} \oplus \Delta_{15}$
(21,3,1)	$\Delta_{21} \oplus [4]$	(23,17,5)	$\Delta_{23,17,5}$
(21,11,1)	$\Delta_{21} \oplus \Delta_{11} \oplus [2]$	(23,17,7)	$\Delta_{23,7} \oplus \Delta_{17}$
(21,11,5)	$\Delta_{21,5} \oplus \Delta_{11}$	(23,17,9)	$\Delta_{23,9} \oplus \Delta_{17}$, $\Delta_{23,17,9}$
(21,11,9)	$\Delta_{21,9} \oplus \Delta_{11}$	(23,17,13)	$\Delta_{23,13} \oplus \Delta_{17}$
(21,15,1)	$\Delta_{21} \oplus \Delta_{15} \oplus [2]$	(23,19,3)	$\Delta_{23,19,3}$
(21,15,5)	$\Delta_{21,5} \oplus \Delta_{15}$	(23,19,7)	$\Delta_{23,7} \oplus \Delta_{19}$
(21,15,9)	$\Delta_{21,9} \oplus \Delta_{15}$	(23,19,9)	$\Delta_{23,9} \oplus \Delta_{19}$
(21,15,13)	$\Delta_{21,13} \oplus \Delta_{15}$	(23,19,11)	$\Delta_{23,19,11}$
(21,17,5)	$\Delta_{21,5} \oplus \Delta_{17}$	(23,19,13)	$\Delta_{23,13} \oplus \Delta_{19}$
(21,17,9)	$\Delta_{21,9} \oplus \Delta_{17}$	(23,21,1)	$\mathrm{Sym}^2\Delta_{11}[2]$
(21,17,13)	$\Delta_{21,13} \oplus \Delta_{17}$	(23,21,7)	$\Delta_{23,7} \oplus \Delta_{21}$
(21,19,1)	$\Delta_{21} \oplus \Delta_{19} \oplus [2]$	(23,21,9)	$\Delta_{23,9} \oplus \Delta_{21}$
(21,19,5)	$\Delta_{21,5} \oplus \Delta_{19}$	(23,21,13)	$\Delta_{23,13} \oplus \Delta_{21}$
(21,19,9)	$\Delta_{21,9} \oplus \Delta_{19}$	(23,21,19)	$\Delta_{21}[3]$
(21,19,13)	$\Delta_{21,13} \oplus \Delta_{19}$		

TABLE 13. The nonempty $\Pi_{25,w_2,w_3}(\mathrm{SO}_7)$

(w_1,w_2,w_3)	$\Pi_{w_1,w_2,w_3}(\mathrm{SO}_7)$	(w_1,w_2,w_3)	$\Pi_{w_1,w_2,w_3}(\mathrm{SO}_7)$
(25,3,1)	$\Delta_{25} \oplus [4]$	(25,19,9)	$\Delta^2_{25,9} \oplus \Delta_{19},\ \Delta^2_{25,19,9}$
(25,7,1)	$\Delta_{25,7} \oplus [2]$	(25,19,11)	$\Delta_{25,11} \oplus \Delta_{19}$
(25,11,1)	$\Delta_{25,11} \oplus [2],\ \Delta_{25} \oplus \Delta_{11} \oplus [2]$	(25,19,13)	$\Delta^2_{25,13} \oplus \Delta_{19},\ \Delta_{25,19,13}$
(25,11,5)	$\Delta_{25,5} \oplus \Delta_{11}$	(25,19,15)	$\Delta_{25,15} \oplus \Delta_{19}$
(25,11,7)	$\Delta_{25,7} \oplus \Delta_{11}$	(25,19,17)	$\Delta_{25,17} \oplus \Delta_{19}$
(25,11,9)	$\Delta^2_{25,9} \oplus \Delta_{11}$	(25,21,3)	$\Delta^2_{25,21,3}$
(25,13,3)	$\Delta_{25,13,3}$	(25,21,5)	$\Delta_{25,5} \oplus \Delta_{21}$
(25,13,7)	$\Delta_{25,13,7}$	(25,21,7)	$\Delta_{25,7} \oplus \Delta_{21},\ \Delta^2_{25,21,7}$
(25,15,1)	$\Delta_{25,15} \oplus [2],\ \Delta_{25} \oplus \Delta_{15} \oplus [2]$	(25,21,9)	$\Delta^2_{25,9} \oplus \Delta_{21}$
(25,15,5)	$\Delta_{25,5} \oplus \Delta_{15},\ \Delta_{25,15,5}$	(25,21,11)	$\Delta_{25,11} \oplus \Delta_{21},\ \Delta^2_{25,21,11}$
(25,15,7)	$\Delta_{25,7} \oplus \Delta_{15}$	(25,21,13)	$\Delta^2_{25,13} \oplus \Delta_{21}$
(25,15,9)	$\Delta^2_{25,9} \oplus \Delta_{15},\ \Delta_{25,15,9}$	(25,21,15)	$\Delta_{25,15} \oplus \Delta_{21},\ \Delta_{25,21,15}$
(25,15,11)	$\Delta_{25,11} \oplus \Delta_{15}$	(25,21,17)	$\Delta_{25,17} \oplus \Delta_{21}$
(25,15,13)	$\Delta^2_{25,13} \oplus \Delta_{15}$	(25,21,19)	$\Delta_{25,19} \oplus \Delta_{21}$
(25,17,3)	$\Delta^2_{25,17,3}$	(25,23,1)	$\Delta_{25} \oplus \Delta^2_{23} \oplus [2]$
(25,17,5)	$\Delta_{25,5} \oplus \Delta_{17}$	(25,23,5)	$\Delta_{25,5} \oplus \Delta^2_{23}$
(25,17,7)	$\Delta_{25,7} \oplus \Delta_{17},\ \Delta^2_{25,17,7}$	(25,23,7)	$\Delta_{25,7} \oplus \Delta^2_{23}$
(25,17,9)	$\Delta^2_{25,9} \oplus \Delta_{17}$	(25,23,9)	$\Delta^2_{25,9} \oplus \Delta^2_{23}$
(25,17,11)	$\Delta_{25,11} \oplus \Delta_{17},\ \Delta_{25,17,11}$	(25,23,11)	$\Delta_{25,11} \oplus \Delta^2_{23}$
(25,17,13)	$\Delta^2_{25,13} \oplus \Delta_{17}$	(25,23,13)	$\Delta^2_{25,13} \oplus \Delta^2_{23}$
(25,17,15)	$\Delta_{25,15} \oplus \Delta_{17}$	(25,23,15)	$\Delta_{25,15} \oplus \Delta^2_{23}$
(25,19,1)	$\Delta_{25,19} \oplus [2],\ \Delta_{25} \oplus \Delta_{19} \oplus [2],\ \Delta_{25,19,1}$	(25,23,17)	$\Delta_{25,17} \oplus \Delta^2_{23}$
(25,19,5)	$\Delta_{25,5} \oplus \Delta_{19},\ \Delta^2_{25,19,5}$	(25,23,19)	$\Delta_{25,19} \oplus \Delta^2_{23}$
(25,19,7)	$\Delta_{25,7} \oplus \Delta_{19}$	(25,23,21)	$\Delta^2_{23}[3]$

Table 14. The nonempty $\Pi_{w_1,w_2,w_3,w_4}(\mathrm{SO}_9)$ for $w_1 \leq 23$

(w_1, w_2, w_3, w_4)	$\Pi_{w_1,w_2,w_3,w_4}(\mathrm{SO}_9)$	(w_1, w_2, w_3, w_4)	$\Pi_{w_1,w_2,w_3,w_4}(\mathrm{SO}_9)$
(7, 5, 3, 1)	$[8]$	(23, 17, 15, 5)	$\Delta_{23,17,5} \oplus \Delta_{15}$
(11, 5, 3, 1)	$\Delta_{11} \oplus [6]$	(23, 17, 15, 9)	$\Delta_{23,17,9} \oplus \Delta_{15}$
(15, 5, 3, 1)	$\Delta_{15} \oplus [6]$	(23, 17, 15, 13)	$\Delta_{23}^2 \oplus \Delta_{15}[3]$
(15, 13, 11, 9)	$\Delta_{15} \oplus \Delta_{11}[3]$	(23, 19, 9, 7)	$\Delta_{23,9} \oplus \Delta_{19,7}$
(17, 13, 11, 9)	$\Delta_{17} \oplus \Delta_{11}[3]$	(23, 19, 11, 3)	$\Delta_{23,19,3} \oplus \Delta_{11}$
(19, 5, 3, 1)	$\Delta_{19} \oplus [6]$	(23, 19, 11, 7)	$\Delta_{23}^2 \oplus \Delta_{19,7} \oplus \Delta_{11}$
(19, 13, 11, 9)	$\Delta_{19} \oplus \Delta_{11}[3]$	(23, 19, 13, 7)	$\Delta_{23,13} \oplus \Delta_{19,7}$
(19, 17, 3, 1)	$\Delta_{19} \oplus \Delta_{17} \oplus [4]$	(23, 19, 15, 3)	$\Delta_{23,19,3} \oplus \Delta_{15}$
(19, 17, 7, 1)	$\Delta_{19,7} \oplus \Delta_{17} \oplus [2]$	(23, 19, 15, 7)	$\Delta_{23}^2 \oplus \Delta_{19,7} \oplus \Delta_{15}$
(19, 17, 11, 1)	$\Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11} \oplus [2]$	(23, 19, 15, 11)	$\Delta_{23,19,11} \oplus \Delta_{15}$
(19, 17, 15, 1)	$\Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [2]$	(23, 19, 17, 3)	$\Delta_{23,19,3} \oplus \Delta_{17}$
(19, 17, 15, 13)	$\Delta_{19} \oplus \Delta_{15}[3]$	(23, 19, 17, 7)	$\Delta_{23}^2 \oplus \Delta_{19,7} \oplus \Delta_{17}$
(21, 13, 11, 9)	$\Delta_{21} \oplus \Delta_{11}[3]$	(23, 19, 17, 11)	$\Delta_{23,19,11} \oplus \Delta_{17}$
(21, 17, 5, 1)	$\Delta_{21,5} \oplus \Delta_{17} \oplus [2]$	(23, 19, 17, 15)	$\Delta_{23}^2 \oplus \Delta_{17}[3]$
(21, 17, 9, 1)	$\Delta_{21,9} \oplus \Delta_{17} \oplus [2]$	(23, 21, 3, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus [4]$
(21, 17, 13, 1)	$\Delta_{21,13} \oplus \Delta_{17} \oplus [2]$	(23, 21, 7, 1)	$\Delta_{23,7} \oplus \Delta_{21} \oplus [2]$
(21, 17, 15, 13)	$\Delta_{21} \oplus \Delta_{15}[3]$	(23, 21, 7, 5)	$\Delta_{23,7} \oplus \Delta_{21,5}$
(21, 19, 9, 7)	$\Delta_{21,9} \oplus \Delta_{19,7}$	(23, 21, 9, 5)	$\Delta_{23,9} \oplus \Delta_{21,5}$
(21, 19, 11, 7)	$\Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{11}$	(23, 21, 11, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{11} \oplus [2]$
(21, 19, 13, 7)	$\Delta_{21,13} \oplus \Delta_{19,7}$	(23, 21, 11, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{11}$
(21, 19, 15, 7)	$\Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{15}$	(23, 21, 11, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{11}$
(21, 19, 17, 7)	$\Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{17}$	(23, 21, 13, 5)	$\Delta_{23,13} \oplus \Delta_{21,5}$
(21, 19, 17, 15)	$\Delta_{21} \oplus \Delta_{17}[3]$	(23, 21, 13, 9)	$\Delta_{23,13} \oplus \Delta_{21,9}$
(23, 5, 3, 1)	$\Delta_{23}^2 \oplus [6]$	(23, 21, 15, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{15} \oplus [2]$
(23, 9, 3, 1)	$\Delta_{23,9} \oplus [4]$	(23, 21, 15, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{15}$
(23, 13, 3, 1)	$\Delta_{23,13} \oplus [4]$	(23, 21, 15, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{15}$
(23, 13, 11, 1)	$\Delta_{23,13} \oplus \Delta_{11} \oplus [2]$	(23, 21, 15, 13)	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{15}$
(23, 13, 11, 5)	$\Delta_{23,13,5} \oplus \Delta_{11}$	(23, 21, 17, 1)	$\mathrm{Sym}^2\Delta_{11}[2] \oplus \Delta_{17}$
(23, 13, 11, 9)	$\Delta_{23}^2 \oplus \Delta_{11}[3]$	(23, 21, 17, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{17}$
(23, 15, 11, 3)	$\Delta_{23,15,3} \oplus \Delta_{11}$	(23, 21, 17, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{17}$
(23, 15, 11, 7)	$\Delta_{23,15,7} \oplus \Delta_{11}$	(23, 21, 17, 13)	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{17}$
(23, 17, 3, 1)	$\Delta_{23}^2 \oplus \Delta_{17} \oplus [4]$	(23, 21, 19, 1)	$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus [2]$
(23, 17, 7, 1)	$\Delta_{23,7} \oplus \Delta_{17} \oplus [2]$	(23, 21, 19, 5)	$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{19}$
(23, 17, 11, 1)	$\Delta_{23}^2 \oplus \Delta_{17} \oplus \Delta_{11} \oplus [2]$	(23, 21, 19, 9)	$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{19}$
(23, 17, 11, 5)	$\Delta_{23,17,5} \oplus \Delta_{11}$	(23, 21, 19, 13)	$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{19}$
(23, 17, 11, 9)	$\Delta_{23,17,9} \oplus \Delta_{11}$	(23, 21, 19, 17)	$\Delta_{23}^2 \oplus \Delta_{19}[3]$
(23, 17, 15, 1)	$\Delta_{23}^2 \oplus \Delta_{17} \oplus \Delta_{15} \oplus [2]$	(25, 7, 3, 1)	$\Delta_{25,7} \oplus [4]$

Appendix D. The 121 level 1 automorphic representations of SO_{25} with trivial coefficients

$[24]$

$\Delta_{15}[9] \oplus [6]$

$\Delta_{17}[7] \oplus [10]$

$\Delta_{19}[5] \oplus [14]$

$\Delta_{21}[3] \oplus [18]$

$\Delta_{23}^2 \oplus [22]$

$\Delta_{23}^2 \oplus \Delta_{11}[11]$

$\mathrm{Sym}^2\Delta_{11}[2] \oplus \Delta_{11}[9]$

$\Delta_{19}[5] \oplus \Delta_{11}[3] \oplus [8]$

$\Delta_{21}[3] \oplus \Delta_{11}[7] \oplus [4]$

$\Delta_{21}[3] \oplus \Delta_{15}[3] \oplus [12]$

$\Delta_{21}[3] \oplus \Delta_{17} \oplus [16]$

$\Delta_{23}^2 \oplus \Delta_{15}[7] \oplus [8]$

$\Delta_{23}^2 \oplus \Delta_{17}[5] \oplus [12]$

$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus [16]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus [20]$

$\Delta_{21,9}[3] \oplus \Delta_{15}[3] \oplus [6]$

$\Delta_{21,13}[3] \oplus \Delta_{17} \oplus [10]$

$\Delta_{23,7} \oplus \Delta_{15}[7] \oplus [6]$

$\Delta_{21}[3] \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [10]$

$\Delta_{21}[3] \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [6]$

$\Delta_{21}[3] \oplus \Delta_{17} \oplus \Delta_{15} \oplus [14]$

$\Delta_{23}^2 \oplus \Delta_{17}[5] \oplus \Delta_{11} \oplus [10]$

$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus \Delta_{11}[5] \oplus [6]$

$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus [14]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{11}[9] \oplus [2]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{15}[5] \oplus [10]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus [14]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus [18]$

$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{15}[5] \oplus [8]$

$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{17}[3] \oplus [12]$

$\Delta_{23,7} \oplus \Delta_{21,9} \oplus \Delta_{15}[5] \oplus [6]$

$\Delta_{23,9} \oplus \Delta_{17}[5] \oplus \Delta_{11} \oplus [8]$

$\Delta_{23,9} \oplus \Delta_{21} \oplus \Delta_{15}[5] \oplus [8]$

$\Delta_{23,13} \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus [12]$

$\Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus [12]$

$\Delta_{23,19,3} \oplus \Delta_{21} \oplus \Delta_{11}[7] \oplus [2]$

$\Delta_{23,19,11} \oplus \Delta_{21} \oplus \Delta_{15}[3] \oplus [10]$

$\Delta_{23,19,11} \oplus \Delta_{21,9} \oplus \Delta_{15}[3] \oplus [8]$

$\Delta_{23,15,7} \oplus \Delta_{19}[3] \oplus \Delta_{11}[3] \oplus [6]$

$\Delta_{21}[3] \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [8]$

$\Delta_{23}^2 \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [8]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus \Delta_{11}[3] \oplus [8]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{11}[7] \oplus [4]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus [12]$

$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus [16]$

$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [10]$

$$\Delta_{21,5}[3] \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [2]$$
$$\Delta_{23,7} \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [6]$$
$$\Delta_{23,7} \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus \Delta_{11}[3] \oplus [6]$$
$$\Delta_{23,7} \oplus \Delta_{21,5} \oplus \Delta_{17}[3] \oplus \Delta_{11}[3] \oplus [4]$$
$$\Delta_{23,9} \oplus \Delta_{21,13} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [8]$$
$$\Delta_{23,13} \oplus \Delta_{19}[3] \oplus \Delta_{15} \oplus \Delta_{11} \oplus [10]$$
$$\Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [10]$$
$$\Delta_{23,13} \oplus \Delta_{19,7}[3] \oplus \Delta_{15} \oplus \Delta_{11} \oplus [4]$$
$$\Delta_{23,13} \oplus \Delta_{21,9} \oplus \Delta_{17}[3] \oplus \Delta_{11} \oplus [8]$$
$$\Delta_{23,17,5} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{11}[5] \oplus [4]$$
$$\Delta_{23,19,3} \oplus \Delta_{21,5} \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [2]$$
$$\Delta_{23,19,11} \oplus \Delta_{21,13} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [10]$$
$$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [10]$$
$$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [6]$$
$$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [14]$$
$$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[5] \oplus [4]$$
$$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [8]$$
$$\Delta_{23}^2 \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [6]$$
$$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [12]$$
$$\Delta_{23,7} \oplus \Delta_{21,9} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [6]$$
$$\Delta_{23,9} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [8]$$
$$\Delta_{23,9} \oplus \Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [6]$$
$$\Delta_{23,9} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{15}[3] \oplus \Delta_{11} \oplus [4]$$
$$\Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus [12]$$
$$\Delta_{23,17,5} \oplus \Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [4]$$
$$\Delta_{23,17,9} \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [8]$$

$$\Delta_{23,17,9} \oplus \Delta_{21,13} \oplus \Delta_{19,7} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [6]$$
$$\Delta_{23,15,3} \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [2]$$
$$\Delta_{23,15,7} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [6]$$
$$\Delta_{23,15,7} \oplus \Delta_{21,5} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{11}[3] \oplus [4]$$
$$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [8]$$
$$\Delta_{23}^2 \oplus \Delta_{21} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [6]$$
$$\Delta_{23}^2 \oplus \Delta_{21,5} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [4]$$
$$\Delta_{23}^2 \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [10]$$
$$\Delta_{23,7} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [6]$$
$$\Delta_{23,7} \oplus \Delta_{21,5} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11}[3] \oplus [4]$$
$$\Delta_{23,9} \oplus \Delta_{21,13} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [8]$$
$$\Delta_{23,9} \oplus \Delta_{21,13} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [6]$$
$$\Delta_{23,13} \oplus \Delta_{21} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [10]$$
$$\Delta_{23,13} \oplus \Delta_{21,9} \oplus \Delta_{19} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [8]$$
$$\Delta_{23,13} \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [6]$$
$$\Delta_{23,13,5} \oplus \Delta_{21,9} \oplus \Delta_{19,7} \oplus \Delta_{17} \oplus \Delta_{15} \oplus \Delta_{11} \oplus [4]$$

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