Automorphic forms
and
even unimodular lattices

Kneser neighbors of Niemeier lattices

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Preface

Automorphic forms are functions defined on *adèle groups*, derived from harmonic analysis, whose theory represents a far-reaching generalization of the one of modular forms. The famous *functoriality conjecture* by Langlands predicts unexpected connections between automorphic forms associated to quite different groups. Recent advances confirm a part of these general conjectures, as well as their refinements by Arthur, for classical groups. The technicality of the proofs is formidable but in contrast the statements are fascinating by their extreme beauty, their wide range of applications and to some extent their simplicity. Our aim in this memoir is to reconsider some problems of classical origin, of number theory and of the theory of quadratic forms, from the angle of these recent results.

A special case, in which the Langlands conjectures keep however their whole flavour, while freed from numerous difficulties existing in general, is the case where one restricts oneself to the study of automorphic forms *unramified at each prime*. An alternative terminology is *level* 1 automorphic forms. When one deals with classical modular forms or Siegel's modular forms, historical examples if there were any of automorphic forms, this assumption means that one considers only forms which are modular for the groups $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{Sp}_{2g}(\mathbb{Z})$, and not for general congruence subgroups.

What is the interest of the case of level 1 automorphic forms is not only the simplifications it provides, it is also very appealing to the arithmetician by the mix of rarity and elegance of the examples (again think of the theory of modular forms for $\mathrm{SL}_2(\mathbb{Z})$). Furthermore it is linked, sometimes very directly, sometimes much less so, sometimes only conjecturally, to the objects of algebraic geometry (varieties, stacks) which are both proper and smooth over the ring $\mathbb{Z}$ of integers and even to *motives* over the rationals with everywhere good reduction, objects as fascinating as mysterious.

In this work we aim to study the conjectures by Arthur and Langlands in this context of level 1 automorphic forms, to give precise formulations of the statements arising from the work of Arthur in this framework, and to illustrate the latter by examples which are more specific but particularly spicy. We will confront also Arthur's results to the ones derived from certain more classical constructions, *theta series*, which put within reach numerous examples. Some of these constructions appear to be even richer, as we discovered, when they are combined with the *triality principle*. Let us emphasize that we want to work if possible with groups of large rank, as they best reveal the riches of the general phenomena, and to move away from the classical examples given by “small" groups such as GL_2 which have already been the subject of an extensive literature.

Our illustrations will mainly concern the theory of quadratic forms over $\mathbb{Z}$ which are non-degenerate and positive definite, in other words the theory of *even (integral)* Euclidean lattices whose determinant is 1 or 2. This assumption about the determinant exactly means that the associated projective quadric is smooth over $\mathbb{Z}$, in which case the associated special orthogonal group is smooth (and even reductive) over $\mathbb{Z}$. In the dimensions (less than or equal to 25) for which these objects are classified, the concrete problem we are going to address is the determination, for each prime number p, of the number of *p-neighborhoods* in Kneser's sense, between the classes of such objects. We will call it the *p-neighbor problem.*

The p-neighbor problem can be quite elementarily approached: this is the point of view that we choosed to follow in the introduction (Chapter I) and also in the organization of the memoir where it will serve as a connecting thread. This will also make it possible to begin by exposing the rich and fascinating history of the subject, and to bring out some simple but striking statements which are consequences of our results (dimension 16 case, determination of the *p-neighborhood graphs* in dimension 24, affirmation of the conjecture by Nebe-Venkov about the linear combinations of higher genus theta series of Niemeier lattices. . .). However, we think it helpful to explain upstream our original motivation which was to test Arthur's results in a context both concrete and of high dimension, a motivation which will only be lying in the background in the introduction.

In the rest of this preface, we will explain the place of the p-neighbor problem in the general landscape of Langlands' conjectures, or even motives, as well as the line of thoughts that led us to this problem. We hope that this enlightenment (or darkening depending upon the viewpoint!) will arouse the interest of the readers who are maybe less sensitive to the appeal of the theory of Euclidean lattices. In any case this passage will be inevitable in order to understand the ideas of the solution we propose of the p-neighbor problem, which uses in a crucial way the aforementioned recent developments. This apparent disproportion between the sophistication of methods and the elementary aspect of the p-neighbor problem is one of the charms of it.

The plan will be as follows. First, we will come back in a more precise way on the notion of level 1 automorphic form (studied in Chapter IV of the memoir). After having discussed a few examples, we will present briefly Langlands' conjectures, emphasizing a statement that we call the *Arthur-Langlands conjecture* (Chapters VI & VIII). We will explain how Langlands and Arthur motivate this conjecture by means of a certain hypothetical group, the *Langlands group of* $\mathbb{Z}$, that we denote by $\mathrm{L}_{\mathbb{Z}}$. When one specializes the statements to *algebraic* automorphic forms, one may to a large extent replace $\mathrm{L}_{\mathbb{Z}}$ with the absolute Galois group of $\mathbb{Q}$. Then we will be in a position to provide the enlightenments we promised above, and also to glimpse some of the problems still to be solved once Arthur's results have been "put in the engine".

Automorphic forms of level 1

Let us fix an algebraic group (scheme) G defined and *reductive* over the ring $\mathbb{Z}$ of integers. This means that G is connected, smooth over $\mathbb{Z}$, and that its reduction modulo p is reductive over $\mathbb{Z}/p\mathbb{Z}$ for each prime p. The most important examples are GL_n and the famous Chevalley groups, or the groups which are isogenous to them such as PGL_n, but other examples will play a role in the sequel.

The adele group $G(\mathbb{A})$ is a locally compact topological group in a natural way, *restricted* product of the real Lie group $G(\mathbb{R})$ and of the p-adic Lie groups $G(\mathbb{Q}_p)$ over all primes p; the subgroup $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$. We denote by Z the neutral component of the center of $G(\mathbb{R})$ (so Z equals 1 if G is semisimple). The homogeneous space $G(\mathbb{Q})\backslash G(\mathbb{A})/Z$ is equipped with a finite $G(\mathbb{A})$-invariant Borel measure. A central question is to describe the Hilbert space $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$ of *square integrable automorphic forms of* G, seen as a unitary representation of $G(\mathbb{A})$ for the right translations. According to our objectives, we shall limit ourselves to considering the subspace

$$\mathcal{A}^2(G) = \mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z \cdot G(\widehat{\mathbb{Z}}))$$

of automorphic forms *of level* 1, which is nothing else than the subspace of $G(\widehat{\mathbb{Z}})$-invariants of $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$. This is a Hilbert space equipped with a natural unitary representation of $G(\mathbb{R})$, and for each prime p, with an action of the convolution ring

$$\mathrm{H}_p(G) = \mathbb{Z}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)],$$

whose elements are the *Hecke operators* at p. The aim is to describe $\mathcal{A}^2(G)$ equipped with these commuting actions of $G(\mathbb{R})$ and of the commutative ring $\mathrm{H}(G) := \bigotimes_p \mathrm{H}_p(G)$.

Denote by $\Pi(G)$ the set of isomorphism classes of objects of the form $\pi_\infty \otimes \pi_f$, with π_∞ an irreducible unitary representation of $G(\mathbb{R})$ and π_f a one-dimensional complex representation of the ring $\mathrm{H}(G)$. Such a π_f may equally be viewed as a collection of ring homomorphisms[1] $\pi_p : \mathrm{H}_p(G) \to \mathbb{C}$; we also talk about *systems of eigenvalues of Hecke operators*. Denote moreover by $\mathrm{m}(\pi)$ the *multiplicity* of π as a subrepresentation of $\mathcal{A}^2(G)$; it is finite according to Harish-Chandra. A *discrete automorphic representation* of G is an element π of $\Pi(G)$ with $\mathrm{m}(\pi) \neq 0$. Let us denote at last by $\Pi_{\mathrm{disc}}(G) \subset \Pi(G)$ the subset of those representations. For general reasons, we may write

$$(1) \quad \mathcal{A}^2(G) = \mathcal{A}^2_{\mathrm{disc}}(G) \overset{\perp}{\oplus} \mathcal{A}^2_{\mathrm{cont}}(G) \quad \text{with} \quad \mathcal{A}^2_{\mathrm{disc}}(G) \simeq \overset{\perp}{\bigoplus_{\pi\in\Pi_{\mathrm{disc}}(G)}} \mathrm{m}(\pi)\ \pi.$$

The space $\mathcal{A}^2_{\mathrm{disc}}(G)$ contains the subspace $\mathcal{A}^2_{\mathrm{cusp}}(G)$ of *cuspforms*, whose definition is a natural generalization of the one of cuspidal modular form. We denote by $\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G)$ the subset of elements appearing in $\mathcal{A}^2_{\mathrm{cusp}}(G)$. The description of the subsets $\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G)$ of $\Pi(G)$ is the heart of the problem. Indeed, we know since Langlands how to describe the *continuous* part $\mathcal{A}^2_{\mathrm{cont}}(G)$ in terms of the $\mathcal{A}^2_{\mathrm{cusp}}(L)$ where L runs through the Levi subgroups of all the proper parabolic subgroups of G defined over $\mathbb{Z}$. We will not be interested in $\mathcal{A}^2_{\mathrm{cont}}(G)$ in this memoir.

Two examples

The representations π in $\Pi_{\mathrm{disc}}(G)$ have very different concrete manifestations depending on the nature of their *Archimedean component* π_∞. If U is an arbitrary irreducible unitary representation of $G(\mathbb{R})$, and if we set $\mathcal{A}_U(G) := \mathrm{Hom}_{G(\mathbb{R})}(U, \mathcal{A}^2(G))$, then we have

$$\mathcal{A}_U(G) = \mathrm{Hom}_{G(\mathbb{R})}(U, \mathcal{A}^2_{\mathrm{disc}}(G)) \simeq \bigoplus_{\{\pi\,\in\,\Pi_{\mathrm{disc}}(G)\,|\,\pi_\infty\simeq U\}} \mathrm{m}(\pi)\ \pi_f$$

This is an $\mathrm{H}(G)$-module in an obvious way, and a finite dimensional complex vector space according to Harish-Chandra. It is equivalent to describe the whole of $\Pi_{\mathrm{disc}}(G)$ or the $\mathrm{H}(G)$-modules $\mathcal{A}_U(G)$ when U runs through the unitary dual of $G(\mathbb{R})$.

In order to illustrate these notions, it is instructive to specify them in the special case of the group $G = \mathrm{PGL}_2$.[2] If U is a *discrete series* representation,

[1] We do not follow here the tradition according to which π_p rather denotes the isomorphism class of the (irreducible) $\mathbb{C}[G(\mathbb{Q}_p)]$-submodule of $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$ generated by an arbitrary nonzero element of π. The difference is however artificial, as it is a well-kwown consequence of the commutativity of $\mathrm{H}_p(G)$ that the two definitions exactly contain the same information.

[2] Following our definitions, we have a canonical isomorphism $\mathcal{A}^2(\mathrm{PGL}_n) \xrightarrow{\sim} \mathcal{A}^2(\mathrm{GL}_n)$.

say with lowest weight the (even) integer $k > 0$, then $\mathcal{A}_U(G)$ naturally identifies with the space of cuspforms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ equipped with the action of the standard Hecke operators on the latter. If $U := U_s$ is a *principal or complementary* series, parameterized in the usual way by an element $s \in i\mathbb{R} \cup [0,1[$, then $\mathcal{A}_{U_s}(G)$ identifies with the Hecke-module of cuspidal *Maass forms* whose eigenvalue is $\frac{1-s^2}{4}$ for the action of the Laplace operator on the Poincaré upper half-plane. Contrary to the previous case, these spaces are very mysterious: Selberg has proved[3] $\mathcal{A}_{U_s}(G) = 0$ for $s > 0$, but we do not know any single exact value of s such that $\mathcal{A}_{U_s}(G)$ is nonzero, or whether the latter space can be of dimension > 1. Finally, the unique unitary representation left of $\mathrm{PGL}_2(\mathbb{R})$ is, according to Bargmann, the trivial representation 1, and we obviously have $\dim \mathcal{A}_1(G) = 1$ (consider the constant functions).

Let us now discuss the example which will be of a great importance in the sequel. Let $n \geq 1$ be an integer and $\mathbb{R}^n$ the standard Euclidean space of dimension n. It turns out that the special orthogonal (compact) group of $\mathbb{R}^n$ is of the form $G(\mathbb{R})$ with G reductive over $\mathbb{Z}$ if, and only if, the integer n is congruent to -1, 0 or $+1$ modulo 8. Let us describe such a G under the assumption $n \equiv 0 \bmod 8$. It is well-known that in this case $\mathbb{R}^n$ possesses *even unimodular lattices*. Such a lattice L is naturally equipped with an integral quadratic form, positive definite and non-degenerate over $\mathbb{Z}$. The associate orthogonal group (scheme) O_L is smooth over $\mathbb{Z}$, and its neutral component SO_L is semisimple over $\mathbb{Z}$, with real points $\mathrm{SO}(\mathbb{R}^n)$.

We will denote by $\mathcal{L}_n$ the set of even unimodular lattices in $\mathbb{R}^n$. Any two elements of $\mathcal{L}_n$ are in the same *genus*, that is are isometric over $\mathbb{Z}_p$ for each prime p (hence over the rationals as well, according to Hasse and Minkowski). This implies first that the space $\mathcal{A}^2(\mathrm{SO}_L)$ depends in an inessential way on the choice of the lattice L. In order to fix ideas, we will focus in this memoir on the group $\mathrm{SO}_n := \mathrm{SO}_{\mathrm{E}_n}$, where E_n denotes the *standard* even unimodular lattice generated by $\frac{1}{2}(1,\ldots,1)$ and the n-tuples of integers $(x_1,\ldots,x_n)$ with $\sum_i x_i$ is even. Another consequence is that we have a natural identification

$$\mathcal{L}_n \xrightarrow{\sim} \mathrm{SO}_n(\mathbb{Q})\backslash\mathrm{SO}_n(\mathbb{A})/\mathrm{SO}_n(\widehat{\mathbb{Z}})$$

compatible with the obvious actions of $\mathrm{SO}_n(\mathbb{R})$ on both sides. If 1 denotes the trivial representation of $G(\mathbb{R})$, and if $\widetilde{\mathrm{X}}_n = \mathrm{SO}(\mathbb{R}^n)\backslash\mathcal{L}_n$ denotes the finite set of proper isometry classes of elements in $\mathcal{L}_n$, we have thus natural

[3] This is an Archimedean analogue of Ramanujan's conjecture, still open for general congruence subgroups.

isomorphisms

$$\mathcal{A}_1(\mathrm{SO}_n) \simeq \{f : \widetilde{\mathrm{X}}_n \to \mathbb{C}\} \simeq \bigoplus_{\{\pi \,\in\, \Pi_{\mathrm{disc}}(\mathrm{SO}_n)\,|\,\pi_\infty \simeq 1\}} \mathrm{m}(\pi)\ \pi_f .$$

The vector space $\mathbb{C}[\widetilde{\mathrm{X}}_n]$, dual of $\mathcal{A}_1(\mathrm{SO}_n)$, is therefore an $\mathrm{H}(\mathrm{SO}_n)$-module in a natural way. For instance, it is an exercise to see that the endomorphism of $\mathbb{C}[\widetilde{\mathrm{X}}_n]$, mapping the class of a lattice to the sum of the classes of its p-neighbors, is induced by an element of $\mathrm{H}_p(\mathrm{SO}_n)$ that we will denote by T_p. The determination of this endomorphism is exactly the problem considered at the beginning of the introduction.[4] Let us add that the spaces $\mathcal{A}_U(\mathrm{SO}_n)$, with U arbitrary (but necessarily finite dimensional), have similar interpretations as spaces of $\mathrm{SO}_n(\mathbb{R})$-equivariant functions $\mathcal{L}_n \to U^*$; many such spaces will play a role in this memoir.

Langlands' functoriality principle

Let us describe in a rather brief way Langlands' general conjectures in the case of level 1 automorphic forms. A starting point is the notion, introduced by Langlands, of *dual* group. If G is reductive over $\mathbb{Z}$, its dual in the sense of Langlands is simply "the" complex linear algebraic reductive group, denoted $\widehat{G}$, whose *based root datum* is *dual* (or *inverse*) to that of $G_{\mathbb{C}}$:

$G_{\mathbb{C}}$	GL_n	PGL_n	Sp_{2g}	PGSp_{2g}	SO_{2n+1}	SO_{2n}	PGSO_{2n}
$\widehat{G}$	GL_n	SL_n	SO_{2g+1}	Spin_{2g+1}	Sp_{2n}	SO_{2n}	Spin_{2n}

This group allows first Langlands to *parameterize* the elements of $\Pi(G)$. He observes that the *Satake isomorphism* provides a canonical bijection, for each prime p, between the set of ring homomorphisms $\mathrm{H}_p(G) \to \mathbb{C}$ and the set of semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$. In a similar way, he interprets the *infinitesimal character* (in the sense of Harish-Chandra) of a unitary representation of $G(\mathbb{R})$ as a semisimple conjugacy class in the Lie algebra of $\widehat{G}$. In the end, to each element π of $\Pi(G)$ is associated a collection of conjugacy classes

$$\mathrm{c}(\pi) = (\mathrm{c}_\infty(\pi), \mathrm{c}_2(\pi), \mathrm{c}_3(\pi), \cdots)$$

which uniquely determines π_p for each prime p premier, as well as the infinitesimal character of π_∞, which only leaves finitely many possibilities for

[4] Actually, we will mostly consider the analogous problem, only slightly simpler, in which SO_n is replaced by $\mathrm{O}_n := \mathrm{O}_{\mathrm{E}_n}$, whose only flaw is that it does not quite fit the conventions adopted here, as O_n is not connected, but this slight difference is inessential.

π_∞. These parameterizations, recalled in chapter VI, have some very concrete aspects. For instance, we will see that for π in $\Pi(\mathrm{SO}_n)$ we have the relation:

$$(2) \qquad \pi_p(\mathrm{T}_p) \;=\; p^{\frac{n}{2}-1}\,\mathrm{trace}\,\mathrm{c}_p(\pi).$$

Let G and G' be two reductive groups over $\mathbb{Z}$, and a morphism of algebraic groups $r : \widehat{G} \to \widehat{G'}$. Langlands' *functoriality principle* predicts, for each *constituent* π of $\mathcal{A}^2(G)$, the existence of a constituent π' of $\mathcal{A}^2(G')$ which *corresponds* to π, in the sense that we have an equality of conjugacy classes $r(\mathrm{c}_v(\pi)) = \mathrm{c}_v(\pi')$ for each v in the set $\mathrm{V} := \{\infty, 2, 3, 5, \dots\}$ of all the places of $\mathbb{Q}$. It is only a principle, rather than a conjecture, as it is not quite accurate as stated, even after a reasonable sense being given to the term "constituent". In what follows, we propose ourselves to make the statement of the functoriality principle precise in the important case $G' = \mathrm{GL}_n$, in which r is nothing else than an n-dimensional representation of the algebraic group $\widehat{G}$. We will later refer to this statement as the *Arthur-Langlands conjecture.*

The Langlands group of $\mathbb{Z}$

Langlands has observed that the formulation of his conjectures is enlightened if one assumes the existence of a certain group, that we will denote here by[5] $\mathrm{L}_\mathbb{Z}$, whose representations in $\widehat{G}$ parameterize the automorphic representations of G in an appropriate sense. We may think of this group as being an extension of the absolute Galois group of $\mathbb{Z}$ (... trivial according to Minkovski!). For our needs in this preface, we will only assume that $\mathrm{L}_\mathbb{Z}$ is a compact Hausdorff topological group (hence an inverse limit of compact Lie groups) satisfying the axioms denoted (L1), (L2) and (L3) that we are going to introduce below.

(L1) $\mathrm{L}_\mathbb{Z}$ *is equipped, for each prime p, with a conjugacy class* $\mathrm{Frob}_p \subset \mathrm{L}_\mathbb{Z}$, *and its complex pro-Lie-algebra, with a semisimple conjugacy class* Frob_∞.

Let G be reductive over $\mathbb{Z}$. Following Arthur and Langlands, we denote by $\Psi(G)$ the set of $\widehat{G}(\mathbb{C})$-conjugacy classes of continuous group homomorphisms

$$(3) \qquad \psi : \mathrm{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}(\mathbb{C})$$

[5] To be completely honest, Langlands rather considers a group which applies to all automorphic forms, rather that to level 1 forms only, of which our $\mathrm{L}_\mathbb{Z}$ would merely be a quotient [Lan79, §2]. Moreover, following Arthur in [Art89, §8], we adopt Kottwitz's point of view [Kot84, §12] on the Langlands group, which amounts to see it as a topological group rather than a pro-algebraic one as Langlands does.

which are polynomial on the $\mathrm{SL}_2(\mathbb{C})$-factor. Such a ψ is called *discrete* if the centralizer C_ψ of $\mathrm{Im}\,\psi$ in $\widehat{G}(\mathbb{C})$ is finite modulo the center $\mathrm{Z}(\widehat{G})$ of $\widehat{G}(\mathbb{C})$. For example, if G is GL_n, in which case we also have $\widehat{G} = \mathrm{GL}_n$ and ψ is nothing else than an n-dimensional representation of $\mathrm{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})$, then ψ is discrete if, and only if, it is an irreducible representation. We denote by $\Psi_{\mathrm{disc}}(G) \subset \Psi(G)$ the subset of classes of discrete morphisms.

In parallel with what has been done for $\Pi(G)$, Arthur and Langlands associate to each ψ in $\Psi(G)$ a collection of conjugacy classes $\mathrm{c}(\psi) = (\mathrm{c}_v(\psi))_{v\in\mathrm{V}}$ defined by $\mathrm{c}_\infty(\psi) = \psi(\mathrm{Frob}_\infty, \mathrm{e}_\infty)$ and $\mathrm{c}_p(\psi) = \psi(\mathrm{Frob}_p, \mathrm{e}_p)$, where the e_v are respectively the elements of $\mathfrak{sl}_2(\mathbb{C})$ for $v = \infty$, and of $\mathrm{SL}_2(\mathbb{C})$ for $v = p$, defined by

$$\mathrm{e}_\infty = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathrm{e}_p = \begin{bmatrix} p^{-\frac{1}{2}} & 0 \\ 0 & p^{\frac{1}{2}} \end{bmatrix}.$$

(L2) *For each integer $n \geq 1$, there is a unique bijection*

$$\pi \mapsto \psi_\pi, \qquad \Pi_{\mathrm{disc}}(\mathrm{GL}_n) \overset{\sim}{\to} \Psi_{\mathrm{disc}}(\mathrm{GL}_n),$$

such that we have $\mathrm{c}(\pi) = \mathrm{c}(\psi_\pi)$ for all $\pi \in \Pi_{\mathrm{disc}}(\mathrm{GL}_n)$. Moreover, ψ_π is trivial on $\mathrm{SL}_2(\mathbb{C})$ if, and only if, we have $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$.

This axiom, together with the compactness of $\mathrm{L}_\mathbb{Z}$, contains the *generalized Ramanujan conjecture*. It also shows $|\mathrm{L}_\mathbb{Z}^{\mathrm{ab}}| = \dim \mathcal{A}(\mathrm{GL}_1) = 1$. We will prove in Chapter IX that (L2) implies as well that $\mathrm{L}_\mathbb{Z}$ is connected.

(L3) *For each G reductive over $\mathbb{Z}$, there exists a decomposition*

$$\mathcal{A}_{\mathrm{disc}}(G) \;=\; \overset{\perp}{\bigoplus}_{\psi\in\Psi_{\mathrm{disc}}(G)} \mathcal{A}_\psi(G),$$

stable under $G(\mathbb{R})$ and $\mathrm{H}(G)$, and satisfying the following property: if $\pi \in \Pi(G)$ appears in $\mathcal{A}_\psi(G)$ then we have $\mathrm{c}(\pi) = \mathrm{c}(\psi)$.

It is Arthur's idea that the failure of Ramanujan's conjecture may be entirely explained in general by the presence of $\mathrm{SL}_2(\mathbb{C})$ in the definition of $\Psi(G)$ (Formula (3)).

Arthur and Langlands strengthen the axiom (L3) by adding a converse statement, called *the multiplicity formula*, whose formulation requires however the introduction of more technical ingredients. Let us simply say that if $\psi \in \Psi_{\mathrm{disc}}(G)$ and $\pi \in \Pi(G)$ satisfy $\mathrm{c}(\pi) = \mathrm{c}(\psi)$, this formula expresses the multiplicity of π in the subspace $\mathcal{A}_\psi(G)$ as the scalar product of two "explicit"[6] characters of the finite group $\mathrm{C}_\psi/\mathrm{Z}(\widehat{G})$.

Arthur-Langlands' conjecture

Let us go back to the statement of the Arthur-Langlands conjecture alluded to above. We assume first the existence of a compact group $\mathrm{L}_\mathbb{Z}$ satisfying the axioms (L1), (L2) and (L3). Let G be reductive over $\mathbb{Z}$, π in $\Pi_{\mathrm{disc}}(G)$ and $r : \widehat{G} \to \mathrm{GL}_n$ a representation. Let $\psi \in \Psi_{\mathrm{disc}}(G)$ be such that π appears in $\mathcal{A}_\psi(G)$, such a ψ exists by (L3). The decomposition into irreducibles of the representation $r \circ \psi$ of the direct product $\mathrm{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})$ may be written as $\oplus_i r_i \otimes \mathrm{Sym}^{d_i-1}\mathbb{C}^2$ for some irreducible representation r_i of dimension n_i of $\mathrm{L}_\mathbb{Z}$, and some integers $d_i \geq 1$. According to (L2), we have $r_i \simeq \psi_{\pi_i}$ for a unique π_i in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_{n_i})$. In particular, for each $v \in \mathrm{V}$ we have the identity between conjugacy classes

$$r(\mathrm{c}_v(\pi)) = \bigoplus_i \mathrm{c}_v(\pi_i) \otimes \mathrm{Sym}^{d_i-1}(e_v) \tag{4}$$

(the reader will have no trouble decrypting the meaning of the right-hand side of this equality).

As a consequence, if we have π in $\Pi_{\mathrm{disc}}(G)$ and a representation $r : \widehat{G} \to \mathrm{GL}_n$, one conjectures the existence of a decomposition $n = \sum_i n_i d_i$ and of representations π_i in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_{n_i})$, unique up to a permutation of the indices i, satisfying equality (4): this is the precise form of the Arthur-Langlands conjecture that had been promised. Observe that $\mathrm{L}_\mathbb{Z}$ does not appear at all in this formulation.

In his work that we already mentionned, Arthur has proved the following special cases of this conjecture: $G_\mathbb{Q}$ is either the symplectic group Sp_{2g} of a symplectic space sur $\mathbb{Q}$ of dimension $2g$, or the special orthogonal group of a quadratic space of dimension $2n$ (resp. $2n+1$) over $\mathbb{Q}$ possessing a totally isotropic subspace of dimension n, $\pi \in \Pi_{\mathrm{disc}}(G)$ is arbitrary, and r is the natural representation of $\widehat{G}$, called the *standard representation*, whose respective dimension is $2g+1$, $2n$ and $2n$. For such groups, Arthur also

[6] The definition of these characters is very delicate. One of them is a group homomorphism $\mathrm{C}_\psi/\mathrm{Z}(\widehat{G}) \to \mathbb{C}^\times$ defined by Arthur in [Art89, p. 55] with the help of the *ε-factors* of certain L-functions associated to ψ. The other one depends on the definition of a certain finite subset of irreducible unitary representations of $G(\mathbb{R})$ associated to ψ, denoted $\Pi_\infty(\psi)$, now usually called an *Arthur packet* [Art89, §4]. This character is nonzero if, and only if, π_∞ belongs to $\Pi_\infty(\psi)$. In the important special case $\mathrm{C}_\psi = \mathrm{Z}(\widehat{G})$, the multiplicity of π in $\mathcal{A}_\psi(G)$ is thus nonzero if, and only if, we have $\pi_\infty \in \Pi_\infty(\psi)$.

proves a version of the multiplicity formula to which we alluded during the discussion of the axiom (L3). We shall state more precise forms of Arthur's results in chapter VIII of the memoir. Let us stress however that we shall not say anything about Arthur's proofs: they go far beyond the scope of this work.

Galois representations and motives

The group $\mathrm{L}_{\mathbb{Z}}$ is subject to several other conjectures. A most tempting one is that it satisfies the *Sato-Tate property*: the Frob_p are equidistributed in the set of conjugacy classes of $\mathrm{L}_{\mathbb{Z}}$, equipped with its invariant probability measure.[7] We will rather discuss in this paragraph the conjectural relation between $\mathrm{L}_{\mathbb{Z}}$, Grothendieck motives and Galois representations.

These links will only concern the quotient of $\mathrm{L}_{\mathbb{Z}}$ whose irreducible representations parameterize, in the sense of axiom (L2), the representations π in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ which are *algebraic*. This adjective means here that, if we denote by λ_i the eigenvalues of the conjugacy class $\mathrm{c}_\infty(\pi) \subset \mathrm{M}_n(\mathbb{C})$, we have $\lambda_i - \lambda_j \in \mathbb{Z}$ for all i, j. We then denote by $\mathrm{w}(\pi)$ the maximum of the differences $\lambda_i - \lambda_j$, and call it the *motivic weight* of π.

Denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the subfield of algebraic numbers. Fix a prime ℓ, $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers, and $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ an embedding. Thanks to the works of a number of mathematicians (including Clozel, Deligne, Fontaine, Grothendieck, Langlands, Mazur, Serre, Shimura, Taniyama, Tate, Weil ...) one conjectures the existence of a natural bijection $\pi \mapsto \rho_{\pi,\iota}$ between the set of algebraic π in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, and the set of isomorphism classes of irreducible continuous representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are *unramified* at each prime $p \neq \ell$ and *crystalline at* ℓ in the sense of Fontaine, with smallest *Hodge-Tate weight* 0. One asks that this bijection satisfies in particular the equality [8]

$$\det(t - \rho_{\pi,\iota}(\mathrm{Frob}_p)) = \iota(\det(t - p^{\mathrm{w}(\pi)/2}\mathrm{c}_p(\pi)))$$

for each prime $p \neq \ell$, which determines it uniquely.

This conjecture may be readily seen as an "algebraic" analogue of the axiom (L2). A number of difficult and important special cases of it are known. According to Fontaine and Mazur, one expects that the Galois representations above are exactly those appearing in the ℓ-adic realizations of pure motives over $\mathbb{Q}$ with *everywhere good reduction*.

[7] Given the connectedness of $\mathrm{L}_{\mathbb{Z}}$, it would be easy to see that this property implies indeed the usual Sato-Tate conjecture for modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

[8] This equality makes sense as we also conjecture that we have $\det(t - \mathrm{c}_p(\pi)) \in \overline{\mathbb{Q}}[t]$ if π is algebraic.

Conclusion

Let G be reductive over $\mathbb{Z}$ and r a representation of $\widehat{G}$. As we have seen, the Arthur-Langlands conjecture predicts that for each π in $\Pi_{\mathrm{disc}}(G)$ the collection of conjugacy classes $r(\mathrm{c}(\pi))$ may be expressed in a very precise way in terms of *building blocks* which are some elements π_i of $\Pi_{\mathrm{cusp}}(\mathrm{GL}_{n_i})$, and some integers d_i, with $\dim r = \sum_i n_i d_i$. Here are some questions which naturally arise: assume that a representation π in $\Pi_{\mathrm{disc}}(G)$ is given, for instance such that π_f appears concretely in a specific $\mathcal{A}_U(G)$, can we determine the associated representations π_i and integers d_i? is it easier to determine them rather than π itself ?

A first obstacle that we encounter, if one tries to illustrate these questions, is to have at our disposal examples of groups G and of irreducible unitary representations U of $G(\mathbb{R})$ for which we know how to determine whether $\mathcal{A}_U(G)$ is nonzero, or better its dimension. When U is a discrete series representation, this is an accessible but notoriously difficult problem: when $G = \mathrm{Sp}_{2g}$ it contains for instance the question of determining the dimension of spaces of Siegel modular cuspforms for $\mathrm{Sp}_{2g}(\mathbb{Z})$, a classical problem that has been solved only very recently by Taïbi in genus $3 \leq g \leq 7$. When U is not in the discrete series, it seems hopeless to obtain a formula for $\dim \mathcal{A}_U(G)$, as is shown by the example $\mathrm{G} = \mathrm{PGL}_2$.

The special case $G(\mathbb{R})$ compact, for which all the irreducible representations are in the discrete series, has the peculiar feature that the question of determining $\dim \mathcal{A}_U(G)$ is significantly more elementary. We will give many such examples with $G = \mathrm{SO}_n$. The case $G = \mathrm{SO}_{24}$ is especially interesting from this point of view, as it is one of the groups of highest rank for which $\dim \mathcal{A}_U(G)$ can be computed for at least *one* U (and with $\mathcal{A}_U(G) \neq 0$). Indeed, we have $\dim \mathcal{A}_1(G) = |\widetilde{\mathrm{X}}_{24}|$ and this cardinality is 25 as the Leech lattice is the only one, among the 24 Niemeier lattices, not to admit any improper isometry. We are forced to ask ourselves the following question:

Question 0. Let r be the standard representation of $\widehat{\mathrm{SO}_{24}}$ and π in $\Pi_{\mathrm{disc}}(\mathrm{SO}_{24})$ with $\pi_\infty = 1$; can we determine the collection of representations π_i and the integers d_i corresponding to π and r according to Arthur-Langlands' conjecture ?[9]

This is the question at the origin of this work. Formulas (4) and (2) show

[9]Observe that Arthur's results do not immediately apply here as SO_n is not (quasi-) split over $\mathbb{Q}$. Nevertheless, we will prove that the Arthur-Langlands conjecture is satisfied when π and r are as in the statement of Question 0, by applying Arthur's results to Sp_{2g} and using some theta series arguments.

that a positive answer to this question gives decisive informations about the p-neighbor problem in dimension 24.

Before saying more about Question 0, let us add that the π_i which appear in its statement are not arbitrary: they are algebraic. More generally, if G is reductive over $\mathbb{Z}$ and if π is in $\Pi_{\mathrm{disc}}(G)$ with π_∞ a discrete series representation, then the eigenvalues of $\mathrm{c}_\infty(\pi)$ in the adjoint representation of $\mathrm{Lie}\,\widehat{G}$ are in $\mathbb{Z}$ (Harish-Chandra); it follows that if r is an arbitrary representation of $\widehat{G}$ then the representations π_i associated to π and r by the Arthur-Langlands conjecture are necessarily algebraic. As a consequence, those π are related to motives and Galois representations, which makes them even more interesting. Those links are deep. We will show for instance that Arthur's multiplicity formula suggests that if π in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_{8k})$ is algebraic, isomorphic to its *dual*, and if the eigenvalues of $\mathrm{c}(\pi_\infty)$ are distinct, then there exists π' in $\Pi_{\mathrm{disc}}(\mathrm{SO}_{8k})$ satisfying $r(\mathrm{c}(\pi')) = \mathrm{c}(\pi)$. These unexpected relations between Galois representations and even unimodular lattices clearly show the interest to study $\Pi_{\mathrm{disc}}(\mathrm{SO}_n)$ for the arithmetician.

We now go back to Question 0. An obstacle that we faced immediately, at least when we started working on this question, is that very few results were known about $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ with $n > 2$, even if we restrict ourselves to algebraic representations. [10] For instance, assuming there was a representation π in $\Pi_{\mathrm{disc}}(\mathrm{SO}_{24})$ satisfying $\pi_\infty = 1$, and such that one of the associated π_i was in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_{n_i})$ with n_i big, then it is very likely that we would never be able to say anything interesting neither about this π, nor about the p-neighbor problem in dimension 24. Observe that we always have $n_i \leq 24$, but also $\mathrm{w}(\pi_i) \leq 24$, by an inspection of $\mathrm{c}_\infty(\pi)$.

One of our main results will be the proof of a classification of the automorphic representations π in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$, with $n \geq 1$ arbitrary, which are algebraic of motivic weight $\mathrm{w}(\pi) \leq 22$. We will see that there are only 11 such representations, and that they all appear (as π_i) in the answer to Question 0. We have furthermore $n \leq 4$ in all cases, with exactly four of them in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_4)$. These four representations, which actually *come* from certain vector valued Siegel cuspform of genus 2, will play an important role in this memoir.

The scope of the classification above is broader: G being arbitrary, the

[10] The situation is very different by now, thanks to the works [CR12] and [Tai14]. Note that although these works have been published before the present memoir, they have actually been entirely motivated by it. A lot of important questions remain, for example we do not know the number of algebraic π in $\Pi_{\mathrm{cusp}}(\mathrm{GL}_3)$ with a given Archimedean component π_∞.

Arthur-Langlands conjecture suggests that any representation π in $\Pi_{\mathrm{disc}}(G)$ with π_∞ is the discrete series, and such that $\mathrm{c}_\infty(\pi)$ is "small enough", is *built* from the 11 above automorphic representations. For example, we will see how to use this approach to determine the dimension of the space of Siegel modular cuspforms of weight ≤ 12 for $\mathrm{Sp}_{2g}(\mathbb{Z})$, for any genus $g \geq 1$.

It seems reasonable to end this preface here, and to leave the reader the pleasure to immerse themselves in the actual introduction of the memoir.

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I. Introduction

1. Even unimodular lattices

Let $n \geq 1$ be an integer and consider the Euclidean space $\mathbb{R}^n$ equipped with its standard inner product $(x_i) \cdot (y_i) = \sum_i x_i y_i$. An even unimodular lattice of rank n is a lattice $L \subset \mathbb{R}^n$ of covolume 1 such that $x \cdot x$ is an even integer for all x in L. The set $\mathcal{L}_n$ of those lattices is equipped with an action of the Euclidean orthogonal group $\mathrm{O}(\mathbb{R}^n)$, and we denote by

$$\mathrm{X}_n \ := \ \mathrm{O}(\mathbb{R}^n)\backslash\mathcal{L}_n$$

the set of isometry classes of even unimodular lattices of rank n. To each L in $\mathcal{L}_n$, one can attach a quadratic form

$$\mathrm{q}_L : L \to \mathbb{Z}, \ \ x \mapsto \frac{x \cdot x}{2},$$

whose associated bilinear form $x \cdot y$ has determinant 1. The map $L \mapsto \mathrm{q}_L$ induces a bijection between X_n and the set of isomorphism classes of positive definite quadratic forms of rank n over $\mathbb{Z}$ whose determinant is 1.

As is well known, the set X_n is finite. It is not empty if and only if we have $n \equiv 0 \bmod 8$. A standard example of an element of $\mathcal{L}_n$ is the lattice

$$\mathrm{E}_n \ := \ \mathrm{D}_n + \mathbb{Z}e$$

where $\mathrm{D}_n = \{(x_i) \in \mathbb{Z}^n, \sum_i x_i \equiv 0 \bmod 2\}$, $e = \frac{1}{2}(1, 1, \ldots, 1)$, and $n \equiv 0 \bmod 8$. Let us explain this notation. To each element L of $\mathcal{L}_n$ is associated a root system (of type ADE)

$$\mathrm{R}(L) \ := \ \{x \in L, x \cdot x = 2\},$$

whose rank is $\leq n$. The root system $\mathrm{R}(\mathrm{E}_8)$ is then of type $\mathbf{E}_8$ and generates over $\mathbb{Z}$ the lattice E_8. For $n > 8$, $\mathrm{R}(\mathrm{E}_n)$ is of type $\mathbf{D}_n$ and generates D_n.

The set X_n has only been determined so far in dimension $n \leq 24$. Mordell and Witt have respectively proved

$$\mathrm{X}_8 = \{\mathrm{E}_8\} \ \text{ and } \ \mathrm{X}_{16} = \{\mathrm{E}_{16}, \mathrm{E}_8 \oplus \mathrm{E}_8\}.$$

The two lattices E_{16} and $\mathrm{E}_8 \oplus \mathrm{E}_8$ will play an important role in this memoir. They are at the same time easy and hard to distinguish : their root systems are different, but they represent each integer exactly the same number of times. This last and rather famous property leads for instance to the isospectral tori discovered by Milnor.

The elements of X_{24} have been classified by Niemeier [Ni73], who proved in particular $|\mathrm{X}_{24}| = 24$. Before saying more about those lattices, let us mention that for $n \geq 32$ the Minkowski-Siegel-Smith mass formula shows that the size of X_n explodes. As an example, we have $|\mathrm{X}_{32}| > 8 \cdot 10^6$ [Ser70]; there is even more than a billion of elements in X_{32} according to King [Kin03].

An element of $\mathcal{L}_{24}$ is called a *Niemeier lattice*. The most famous Niemeier lattice is the Leech lattice. Up to isometry, it is the unique element L in $\mathcal{L}_{24}$ such that $\mathrm{R}(L) = \emptyset$ (Conway). If L is a Niemeier lattice which is not isomorphic to the Leech lattice, a remarkable fact is that its root system $\mathrm{R}(L)$ has rank 24 and all its irreducible components share the same Coxeter number. A simple proof of this fact has been given by Venkov [Ven80]. The miracle is then that the map $L \mapsto \mathrm{R}(L)$ induces a bijection between $\mathrm{X}_{24} - \{\mathrm{Leech}\}$ and the set of isomorphism classes of root systems R of rank 24 whose irreducible components are of type ADE and share the same Coxeter number $\mathrm{h}(R)$. The proof is a rather tedious case-by-case verification.

R	$\mathbf{D}_{24}$	$\mathbf{D}_{16}\mathbf{E}_8$	$3\mathbf{E}_8$	$\mathbf{A}_{24}$	$2\mathbf{D}_{12}$	$\mathbf{A}_{17}\,\mathbf{E}_7$	$\mathbf{D}_{10}\,2\mathbf{E}_7$	$\mathbf{A}_{15}\,\mathbf{D}_9$
h(R)	46	30	30	25	22	18	18	16
R	$3\mathbf{D}_8$	$2\mathbf{A}_{12}$	$\mathbf{A}_{11}\,\mathbf{D}_7\,\mathbf{E}_6$	$4\mathbf{E}_6$	$2\mathbf{A}_9\mathbf{D}_6$	$4\mathbf{D}_6$	$3\mathbf{A}_8$	$2\mathbf{A}_7\,2\mathbf{D}_5$
h(R)	14	13	12	12	10	10	9	8
R	$4\mathbf{A}_6$	$4\mathbf{A}_5\,\mathbf{D}_4$	$6\mathbf{D}_4$	$6\mathbf{A}_4$	$8\mathbf{A}_3$	$12\mathbf{A}_2$	$24\mathbf{A}_1$	
h(R)	7	6	6	5	4	3	2	

Table I.1: The 23 equi-Coxeter root systems of type ADE and rank 24

The results mentioned in this paragraph are explained in Chapter II, whose main aim is to recall some classical results. We first develop some prerequisites of bilinear and quadratic algebra. They are necessary to understand the constructions of quadratic forms alluded above, as well as others that we shall need throughout this memoir. In particular, we recall Venkov's theory and explain the construction of certain Niemeier lattices. We also take the opportunity to recall some basic facts about the classical group schemes over $\mathbb{Z}$ that we shall use in the sequel. Appendix B contains in particular a variant of the results of Chapter II: we study the even Euclidean lattices with determinant 2 as well as the corresponding theory of quadratic forms over $\mathbb{Z}$ (in odd dimensions).

2. Kneser neighbors

Let p be a prime. The notion of p-neighbors has been introduced by M. Kneser. It can be viewed as a tool to construct a collection of even unimodular lattices from a given one and the prime p. In Chapter III, we study several variations of this notion and give many examples.

Following Kneser, we say that two lattices L, M in $\mathcal{L}_n$ are *p-neighbors* if $L \cap M$ has index p in L (hence in M). It is easy to construct all the p-neighbors of a given even unimodular lattice L. Indeed, to each isotropic line ℓ in $L \otimes \mathbb{F}_p$, say generated by an element x in L such that $\mathrm{q}_L(x) \equiv 0 \bmod p^2$, let us associate the even unimodular lattice[1]

$$\mathrm{vois}_p(L;\ell) \;:=\; H + \mathbb{Z}\,\frac{x}{p},$$

where $H = \{y \in L, x \cdot y \equiv 0 \bmod p\}$ (this lattice does not depend on the choice of x). The map $\ell \mapsto \mathrm{vois}_p(L;\ell)$ induces a bijection between $\mathrm{C}_L(\mathbb{F}_p)$ and the set of p-neighbors of L, where C_L denotes the projective (and smooth) quadric over $\mathbb{Z}$ defined by $\mathrm{q}_L = 0$. This quadric turns out to be hyperbolic over $\mathbb{F}_p$ for each prime p, thus the number of p-neighbors of L is exactly $|\mathrm{C}_L(\mathbb{F}_p)| = 1 + p + p^2 + \cdots + p^{n-2} + p^{\frac{n}{2}-1}$, a number that we shall also denote by $\mathrm{c}_n(p)$.

Consider for instance the element $\rho = (0, 1, 2, \ldots, 23)$ of E_{24}. It generates an isotropic line in $\mathrm{E}_{24} \otimes \mathbb{F}_{47}$ because of the congruence $\sum_{i=1}^{23} i^2 \equiv 0 \bmod 47$. It is not very difficult to check that $\mathrm{vois}_{47}(\mathrm{E}_{24}; \rho)$ has no root; we thus have an isometry

$$\mathrm{vois}_{47}(\mathrm{E}_{24}; \rho) \simeq \mathrm{Leech}.$$

This especially simple construction of the Leech lattice is attributed to Thompson in [CS99], we shall come back to it later. It illustrates the slogan claiming that many constructions of lattices are special cases of constructions of neighbors.

Going back to the general setting, we obtain for each L in $\mathcal{L}_n$ a partition of the finite quadric $\mathrm{C}_L(\mathbb{F}_p)$ given by the isometry class of the associated p-neighbor of L. One of the aims of this memoir is to study this partition in dimension $n \leq 24$. For instance, can we determine the number $\mathrm{N}_p(L, M)$

[1] The notation *vois* comes from the french word *voisin* for *neighbor*.

of p-neighbors of L which are isometric to another given M in $\mathcal{L}_n$? The first interesting case of this question occurs of course in dimension $n = 16$. In order to formulate the result, it will be convenient to introduce the linear map $\mathrm{T}_p : \mathbb{Z}[\mathrm{X}_n] \to \mathbb{Z}[\mathrm{X}_n]$ defined by $\mathrm{T}_p\,[L] = \sum [M]$, the sum being over all the p-neighbors M of L.

Theorem A. *In the basis* $\mathrm{E}_8 \oplus \mathrm{E}_8,\ \mathrm{E}_{16}$ *the matrix of* T_p *is*

$$\mathrm{c}_{16}(p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1+p+p^2+p^3)\,\frac{1+p^{11}-\tau(p)}{691} \begin{bmatrix} -405 & 286 \\ 405 & -286 \end{bmatrix},$$

where τ is Ramanujan's function, defined by $q \prod\limits_{m\geq 1} (1-q^m)^{24} = \sum_{n\geq 1} \tau(n) q^n$.

For instance, this statement asserts that for each prime p we have the formula $\mathrm{N}_p(\mathrm{E}_8 \oplus \mathrm{E}_8,\, \mathrm{E}_{16}) = \frac{405}{691}(1+p^{11}-\tau(p))\frac{p^4-1}{p-1}$. Although Theorem A was probably known to specialists, we have not found it stated this way in the literature. Throughout this memoir, we shall give several proofs of it. Given the theory of theta series and modular forms for $\mathrm{SL}_2(\mathbb{Z})$, the presence of $\tau(n)$ in the statement may look quite classical at first sight. For instance, if we set $\mathrm{r}_L(n) = |\{x \in L, x\cdot x = 2n\}|$, it is easy to show $\mathrm{r}_{\mathrm{Leech}}(p) = \frac{65520}{691}(1+p^{11}-\tau(p))$ for each prime p, a formula that looks similar to the one of the theorem. Nevertheless, the presence of the term $\tau(p)\frac{p^4-1}{p-1}$ in the counting problem above appears to be quite subtler; it will turn out to be equivalent to a non trivial case of the Arthur-Langlands functoriality conjecture.[2]

Our main theorem is a statement similar to the one of Theorem A but concerning the Niemeier lattices. It would be possible to give a formulation of it in the same style as the one of this theorem, namely an explicit formula for a matrix of T_p on $\mathbb{Z}[\mathrm{X}_{24}]$, although it would look very indigestible. This explicit formula actually involves rational coefficients with such big denominators that it appears quite remarkable from their inspection that $\mathrm{N}_p(L, M)$ may be an integer ! We will state a conceptual (and equivalent) version of our result in §I.4 below (Theorem E). A remarkable feature is that all the cuspidal modular forms of weight $k \leq 22$ for the group $\mathrm{SL}_2(\mathbb{Z})$, as well as 4 vector-valued Siegel modular forms for $\mathrm{Sp}_4(\mathbb{Z})$, appear in the statement. Let us discuss from now on some consequences concerning the Niemeier lattices that will follow *in fine* from an analysis of our formulas.

[2] Incidentally, the comparison between theorem A and the formula above for $\mathrm{r}_{\mathrm{Leech}}(p)$ leads to the "purely quadratic" relation $\mathrm{N}_p(\mathrm{E}_8 \oplus \mathrm{E}_8, \mathrm{E}_{16}) = \frac{9}{1456} \cdot \mathrm{r}_{\mathrm{Leech}}(p) \cdot \frac{p^4-1}{p-1}$, that we don't know how to prove directly.

Consider the graph $\mathrm{K}_n(p)$ whose set of vertices is X_n, and with an edge between two different classes L, M in X_{24} if and only if we have $\mathrm{N}_p(L, M) \neq 0$. Kneser has proved that $\mathrm{K}_n(p)$ is connected, for all n and p, as a consequence of his celebrated strong approximation theorem. This nice result implies that we can theoretically reconstruct X_n from the single lattice E_n and a prime p. This was actually useful to Niemeier in his computation of X_{24} using 2-neighbors.

The graph $\mathrm{K}_{16}(p)$ is the connected graph with 2 vertices, thanks to Kneser. This is of course compatible with the estimate $|\tau(p)| < 2p^{\frac{11}{2}}$ (Deligne-Ramanujan) and with the formula for $\mathrm{N}_p(\mathrm{E}_8 \oplus \mathrm{E}_8, \mathrm{E}_{16})$ given by Theorem A. On the other hand, the graph $\mathrm{K}_{24}(2)$, determined by Borcherds [CS99], is not trivial at all. It has diameter 5 and this Wikipedia page http://en.wikipedia.org/wiki/Niemeier_lattice gives a lovely representation of it, also due to Borcherds. Our results allow for instance to determine the graph $\mathrm{K}_{24}(p)$ for each prime p (§X.2). [3]

Theorem B. *(i) Let L be a Niemier lattice with roots. Then L is a p-neighbor of the Leech lattice if and only if $p \geq \mathrm{h}(\mathrm{R}(L))$.*

(ii) The graph $\mathrm{K}_{24}(p)$ is complete if and only if $p \geq 47$.

Let us make some remarks about this statement. The first point concerns the constructions of the Leech lattice as a p-neighbor of a Niemeier lattice with roots. For instance, we observe on the Borcherds graph $\mathrm{K}_{24}(2)$ that Leech is at distance 5 from E_{24}, and that it is only linked with the lattice with root system $24\mathbf{A}_1$ (which is the Niemeier lattice with roots whose construction is the most delicate, as it uses the Golay code, §II.3). This last property is quite simple to understand: if the Leech lattice is a 2-neighbor of a Niemeier lattice L (with roots), then L has an index 2 subgroup without roots. In particular, $\mathrm{R}(L)$ has the property that the sum of two roots is not a root, so that its irreducible constituents must have rank 1, which forces $\mathrm{R}(L) = 24\mathbf{A}_1$. Among the root systems of table I.1, this root system is the unique one whose Coxeter number is 2, which fits assertion (i) of Theorem B.

The most elementary part of Theorem B, proved in §III.4 and generalizing the previous observations, amounts to show the inequality $p \geq \mathrm{h}(\mathrm{R}(L))$ if Leech is a p-neighbor of L. This is an analog of a result of Kostant [Kos59] asserting that the minimal order of a finite order regular element in a connected, compact, adjoint Lie group, coincides with the Coxeter number of its

[3] A list of those graphs is available at the address http://gaetan.chenevier.perso.math.cnrs.fr/niemeier/niemeier.html.

root system. On the other hand, the proof of the other assertions requires the use of Theorem E as well as a number of Ramanujan style inequalities; it will be completed only in Chapter X (§X.2 and §X.3).

In Chapter III, we also study the limit cases of the assertion (i) of Theorem B (the arguments are direct, *i.e.* they do not rely on Theorem E). For that purpose, we proceed to a detailed analysis of the elements c of $\mathrm{C_L}(\mathbb{F}_p)$ satisfying $\mathrm{vois}_p(L;c) \simeq \mathrm{Leech}$, where L is a Niemeier lattice with non-empty root system $R = \mathrm{R}(L)$. It is necessary, for the pertinence of the statements, to study more generally the d-neighbors of L, where $d \geq 1$ is a non-necessarily prime integer (§III.1). We show that if ρ is a Weyl vector of R, and if we set $h = \mathrm{h}(R)$, then we have isometries (Theorem III.4.2.10)

$$\mathrm{vois}_h(L;\rho) \simeq \mathrm{vois}_{h+1}(L;\rho) \simeq \mathrm{Leech}. \tag{2.1}$$

This makes sense since we have $\rho \in L$ (Borcherds) and $\mathrm{q}_L(\rho) = h(h+1)$ (Venkov). This statement contains for instance the aforementioned observation of Thompson. Actually, these 23 (or 46) constructions of the Leech lattice are nothing else than the famous *holy constructions* due to Conway and Sloane [CS82]. However, we give a new proof of the isometries (2.1) using the theory of neighbors, and we show as well the identities

$$\mathrm{N}_h(L, \mathrm{Leech}) = \frac{|W|}{\varphi(h)g} \quad \text{and} \quad \mathrm{N}_{h+1}(L, \mathrm{Leech}) = \frac{|W|}{h+1}, \tag{2.2}$$

where W denotes the Weyl group of R, and g^2 is the index of connexion of R in the sense of Bourbaki. We conclude Chapter III by a study of $\mathrm{vois}_2(L;\rho)$ inspired by results of Borcherds (see Figure III.1).

3. Theta series and Siegel modular forms

Let us come back to the determination of the operator T_p on $\mathbb{Z}[\mathrm{X}_n]$. We start with some simple observations: the T_p commute each others and are self-adjoint for a suitable inner product on $\mathbb{R}[\mathrm{X}_n]$ [NV01] (§III.2). We have thus to find a basis of common eigenvectors of the T_p, as well as the associated collections of eigenvalues. There is only one obvious stable line, generated by $\sum_{L \in \mathrm{X}_n} \frac{[L]}{|\mathrm{O}(L)|}$, on which the operator T_p has the "trivial" eigenvalue $\mathrm{c}_n(p)$.

As already seen in the preface, we are actually in the presence of a disguised problem belonging to the spectral theory of automorphic forms. Indeed, if $G = \mathrm{O}_n$ denotes the orthogonal group scheme over $\mathbb{Z}$ defined by the quadratic form $\mathrm{q}_{\mathrm{E}_n}$, and if $\mathbb{A}$ denotes the adèle ring of $\mathbb{Q}$, genus theory leads to an isomorphism of $G(\mathbb{R})$-sets $\mathcal{L}_n = G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}})$ (§II.2,

§IV.1). It follows that the dual of $\mathbb{R}[\mathrm{X}_n]$ is naturally isomorphic to the space of real-valued functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which are invariant under the action of $G(\mathbb{R}) \times G(\widehat{\mathbb{Z}})$ by right translations. In this description, the operator T_p is induced by a specific element of the ring $\mathrm{H}(G)$ of Hecke operators of G.

Those classical observations are recalled in Chapter IV. Although we shall be mostly interested in the automorphic forms of the $\mathbb{Z}$-group O_n, our statements and proofs will require the introduction of several variants of them (automorphic forms for SO_n, PGO_n and PGSO_n), modular forms for $\mathrm{SL}_2(\mathbb{Z})$, vector-valued Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$, and even, by the use of Arthur's results, some automorphic forms for PGL_n. It will thus be necessary to adopt from the beginning a sufficiently general point of view embracing all these objects (§IV.3). The reader will find in §IV.1 and §IV.2 an elementary exposition of Hecke operators. The emphasis is placed on the examples given by the classical groups and their variants (Hecke, Satake, Shimura); they lead to a deeper point of view on p-neighbors and their generalizations. In §IV.4 and §IV.5, we review some properties of automorphic forms of O_n and of Siegel modular forms. Let us emphasize that the writing in this chapter is intended for non-specialists, and pretends to little originality.

An approach to study the $\mathrm{H}(\mathrm{O}_n)$-module $\mathbb{Z}[\mathrm{X}_n]$ is to examine the Siegel theta series $\vartheta_g(L)$ of each genus $g \geq 1$ of the elements L of $\mathcal{L}_n$. For all $n \equiv 0 \bmod 8$ and all $g \geq 1$, they allow us to define a linear map

$$\vartheta_g : \mathbb{C}[\mathrm{X}_n] \to \mathrm{M}_{\frac{n}{2}}(\mathrm{Sp}_{2g}(\mathbb{Z})), \quad [L] \mapsto \vartheta_g(L),$$

where $\mathrm{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ denotes the space of Siegel modular forms of weight $k \in \mathbb{Z}$ for $\mathrm{Sp}_{2g}(\mathbb{Z})$ (§V.1). The relevance of this map for our problem comes from the (generalized) Eichler commutation relations; they assert that ϑ_g intertwines each element of $\mathrm{H}(\mathrm{O}_n)$ with a certain "explicit" element of $\mathrm{H}(\mathrm{Sp}_{2g})$ (Eichler, Freitag, Yoshida, Andrianov, §V.1). The map ϑ_g is trivially injective for $g \geq n$. It seems however quite difficult in general to determine the structure of the $\mathrm{H}(\mathrm{Sp}_{2g})$-module $\mathrm{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$, especially when g grows. Nevertheless, we will develop in Chapter IX a strategy allowing to solve some new cases of this problem, that uses among other things recent results of Arthur [Art13].

The map ϑ_g has been studied by several authors. Its kernel, which decreases when g grows, describes the linear relations between the genus g theta series of the elements of $\mathcal{L}_n$, and the description of its image is an instance of Eichler's famous *basis problem*. More precisely, ϑ_g induces an injective map

$$\mathrm{Ker}\,\vartheta_{g-1}/\mathrm{Ker}\,\vartheta_g \longrightarrow \mathrm{S}_{\frac{n}{2}}(\mathrm{Sp}_{2g}(\mathbb{Z})) \tag{3.1}$$

where $\mathrm{S}_k(\mathrm{Sp}_{2g}(\mathbb{Z})) \subset \mathrm{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ denotes the subspace of cuspforms (see §V.1 or the footnote below for the convention on ϑ_0), and Eichler asks whether this map is surjective. An important result of Böcherer [Boc89] gives a necessary and sufficient condition for an eigenform for $\mathrm{H}(\mathrm{Sp}_{2g})$ to be in its image, in terms of the vanishing at a certain integer of an associated L-function (§VII.2).

The case $n = 16$.

The case $n = 16$ is the subject of a famous story, recalled in §V.2. Indeed, a classical result of Witt and Igusa asserts that we have

$$\vartheta_g(\mathrm{E}_8 \oplus \mathrm{E}_8) = \vartheta_g(\mathrm{E}_{16}) \text{ if } g \leq 3. \tag{3.2}$$

These remarkable identities mean that $\mathrm{E}_8 \oplus \mathrm{E}_8$ and E_{16} represent each positive integral quadratic forms of rank ≤ 3 exactly the same number of times. This is well known in genus $g = 1$, as a consequence of the vanishing $\mathrm{S}_8(\mathrm{SL}_2(\mathbb{Z})) = 0$ (and leads to the aforementioned Milnor's isospectral tori). This shows incidentally[4] that "the" non-trivial eigenvector in $\mathbb{Z}[\mathrm{X}_{16}]$ is $[\mathrm{E}_{16}] - [\mathrm{E}_8 \oplus \mathrm{E}_8]$. The difficulty in genera 2 and 3 is that although the vanishing of $\mathrm{S}_8(\mathrm{Sp}_{2g}(\mathbb{Z}))$ still holds, it is more complicated to prove. In Appendix A, we present a different and ingenious proof of the identities (3.2) due to Kneser, which does not rely on any such vanishing results.

The Siegel modular form $J = \vartheta_4(\mathrm{E}_8 \oplus \mathrm{E}_8) - \vartheta_4(\mathrm{E}_{16})$, which is nothing else than the famous *Schottky form*, is easily shown to be non-zero. Following Poor and Yuen [PY96], we even know that it generates $\mathrm{S}_8(\mathrm{Sp}_8(\mathbb{Z}))$. Theorem A follows then from the resolution by Ikeda [Ike01] of the Duke-Imamoḡlu conjecture [BK00]. Indeed, applied to Jacobi's modular form Δ in $\mathrm{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$, Ikeda's theorem shows the existence of a non-zero Siegel modular form in $\mathrm{S}_8(\mathrm{Sp}_8(\mathbb{Z}))$, which is an eigenform for $\mathrm{H}(\mathrm{Sp}_8)$, whose Hecke eigenvalues are explicitly determined by the $\tau(p)$. Ikeda's proof is quite difficult, and our main contribution to Theorem A in this memoir is to have found a very different proof of Ikeda's result in this specific case.

The important result is the following. For any map $f : \mathcal{L}_n \to \mathbb{C}$, we define a map $\mathrm{T}_p(f) : \mathcal{L}_n \to \mathbb{C}$ by setting, for any $L \in \mathcal{L}_n$, $\mathrm{T}_p(f)(L) = \sum_M f(M)$, the sum being over all the p-neighbors M of L. If $1 \leq g \leq n/2$, we denote by $\mathrm{H}_{d,g}(\mathbb{R}^n)$ the space of polynomials $(\mathbb{R}^n)^g \to \mathbb{C}$ which are harmonic for the Euclidean Laplace operator on $(\mathbb{R}^n)^g$, and which satisfy $P \circ \gamma = (\det \gamma)^d\, P$ for all $\gamma \in \mathrm{GL}_g(\mathbb{C})$ (§V.4). This space is equipped with a linear representation of $\mathrm{O}(\mathbb{R}^n)$ in a natural way.

[4] This assertion can be proved in a much more direct manner. Indeed, if ϑ_0 denotes the linear map $\mathbb{C}[\mathrm{X}_n] \to \mathbb{C}$ sending to 1 the class of any element of $\mathcal{L}_n$, the obvious equality $\vartheta_0 \circ \mathrm{T}_p = \mathrm{c}_n(p)\, \vartheta_0$ shows that $\operatorname{Ker} \vartheta_0$ is stable by T_p.

Theorem C. *Let $q + \sum_{n\geq 2} a_n q^n$ be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ which is an eigenform for the Hecke operators, and let $d = \frac{k}{2} - 2$. There exists a map $f : \mathcal{L}_8 \to \mathbb{C}$ such that :*

(i) for each prime p, we have $\mathrm{T}_p(f) = p^{-d}\,\frac{p^4-1}{p-1}\,a_p\,f$,

(ii) f generates under $\mathrm{O}(\mathbb{R}^8)$ a representation isomorphic to $\mathrm{H}_{d,4}(\mathbb{R}^8)$.

The paragraph V.4 is mostly devoted to the proof of the special case $k = 12$ of this theorem. It leads to a complete and rather elementary proof of Theorem A. The general case will be addressed, and made more precise, in §VII.2.

Let us give an idea of the proof. The first step is to realize the modular form we are starting from as a theta series $\sum_{x\in \mathrm{E}_8} P(x)\, q^{\frac{x\cdot x}{2}}$, where $P : \mathbb{R}^8 \to \mathbb{C}$ is a suitable harmonic polynomial. In the case of the modular form Δ, we check that any non-zero harmonic polynomial of degree 8 which is invariant under the Weyl group $\mathrm{W}(\mathbf{E}_8)$ does the trick, and in general we invoke a result of Waldspurger [WAL79]. This construction defines a subspace of functions $\mathcal{L}_8 \to \mathbb{C}$ which are eigenvectors for the Hecke operators in $\mathrm{H}(\mathrm{O}_8)$, with eigenvalues related to the a_p by the Eichler commutation relations, and which generate under $\mathrm{O}(\mathbb{R}^8)$ a representation isomorphic to $\mathrm{H}_{8,1}(\mathbb{R}^8)$. The main idea is to apply to them, at the source, an automorphism of order 3 of $\mathcal{L}_8$ arising from triality. Such an automorphism is constructed from a structure of Coxeter octonions on the E_8 lattice, and from an isomorphism $\mathcal{L}_8 \simeq G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}})$ where $G = \mathrm{PGSO}_8$. The resulting functions satisfy the theorem: we refer to §V.4 for the details.

Condition (ii) of the statement implies that the function f gives birth to a Siegel theta series of genus 4 (with "pluriharmonic" coefficients). When this series does not vanish, we observe that is a substitute for the Ikeda lift of genus 4 of the modular form we started from. As it is not difficult to check this non-vanishing when $k = 12$, Theorem A follows.

Let us mention that we shall prove later the vanishing of $\mathrm{S}_8(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for all $g \neq 4$ (Theorem IX.5.10). When $g = 5, 6$, this had already been obtained by Poor and Yuen [PY07], by different methods. As a consequence, the map $\vartheta_g : \mathbb{C}[\mathrm{X}_{16}] \to \mathrm{M}_8(\mathrm{Sp}_{2g}(\mathbb{Z}))$ is surjective for any genus $g \geq 1$.

THE CASE $n = 24$

This case has been the subject of remarkable works by Erokhin [ERO79], Borcherds-Freitag-Weissauer [BFW98] and Nebe-Venkov [NV01] (§V.3). Erokhin has showed the vanishing $\mathrm{Ker}\,\vartheta_{12} = 0$, and the three authors of [BFW98] have proved that Ker ϑ_{11} has dimension 1. Nebe and Venkov have studied *loc. cit.* the whole filtration of $\mathbb{Z}[\mathrm{X}_{24}]$ given by the sequence of $\mathrm{Ker}\,\vartheta_g$ for $g \geq 1$. Their starting point is an explicit computation of the operator T_2 on $\mathbb{Z}[\mathrm{X}_{24}]$ that they deduce from results of Borcherds (§III.3.3). They observe that its eigenvalues are distinct integers, which allows them to exhibit an explicit basis of $\mathbb{Q}[\mathrm{X}_{24}]$ made of common eigenvectors for all the T_p. They propose as well a conjecture for the dimension of the image of the map (3.1) for each integer $1 \leq g \leq 12$, that they prove in many cases, but not all. We establish their conjecture and even show that Eichler's basis problem has a positive answer in dimension $n = 24$ for each genus $1 \leq g \leq 23$ (Theorem IX.5.2 and Corollary IX.5.7).

Theorem D. *The map* $\vartheta_g : \mathbb{C}[\mathrm{X}_{24}] \to \mathrm{M}_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ *is surjective, and induces an isomorphism* $\mathrm{Ker}\,\vartheta_{g-1}/\mathrm{Ker}\,\vartheta_g \overset{\sim}{\to} \mathrm{S}_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$*, for each integer* $g \leq 23$*. The dimension of* $\mathrm{S}_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ *for* $g \neq 24$ *is given by the table:*

g	1	2	3	4	5	6	7	8	9	10	11	12	> 12
$\dim \mathrm{S}_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$	1	1	1	2	2	3	3	4	2	2	1	1	0

We will give an idea of the proof in §I.6, the most difficult point being the first assertion. As a consequence of the theorem, we obtain a complete description of the filtration $(\mathrm{Ker}\,\vartheta_g)_{g\geq 1}$ of $\mathbb{Z}[\mathrm{X}_{24}]$. Let us mention that Eichler's basis problem has a negative answer in dimension $n = 32$ and genus $g = 14$, as we observe in Corollary VII.3.5.

4. Automorphic forms of classical groups

Siegel modular forms, as well as automorphic forms for O_n, can be studied from the perspective of the recent results of Arthur [ART13]. If only to formulate Arthur's results, we are forced to recall the most basic features of Langlands point of view on the theory of automorphic forms [LAN70] [BOR77]. They are gathered in Chapter VI. The outlines of this point of view have already been discussed in the preface. We shall recall briefly here some of its aspects.

Let G be a semisimple[5] group scheme over $\mathbb{Z}$. We denote by $\Pi_{\mathrm{disc}}(G)$ the set of topologically irreducible subrepresentations of the space of square-integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}})$, for the natural actions of $G(\mathbb{R})$ and of the (commutative) ring $\mathrm{H}(G)$ of Hecke operators of G (§IV.3). The Satake isomorphism asociates to each $\pi \in \Pi_{\mathrm{disc}}(G)$, and to each prime p, a semisimple conjugacy class $\mathrm{c}_p(\pi)$ in $\widehat{G}(\mathbb{C})$, where $\widehat{G}$ denotes the complex semisimple algebraic group which is dual to $G_{\mathbb{C}}$ in the sense of Langlands (§VI.1, §VI.2). This enlightening point of view on the eigenvalues of Hecke operators, due to Langlands, is made explicit in §VI.2.8 in the case of classical groups and of the Hecke operators of interest in this memoir, following Gross's article [Gro98]. Similarly, we recall how the Harish-Chandra isomorphism allows to view the infinitesimal character of the Archimedean component π_∞ of π as a semisimple conjugacy class $\mathrm{c}_\infty(\pi)$ in the Lie algebra of $\widehat{G}$ (§VI.3).

As already explained in the preface, a central and structuring conjecture, originally due to Langlands "in the tempered case" and extended by Arthur in general [Art89], asserts that those collections of conjugacy classes can all be expressed in terms of the similar data relative to the elements of $\Pi_{\mathrm{disc}}(\mathrm{PGL}_m)$ for all $m \geq 1$. This conjecture is discussed in §VI.4.4. Let us give here an alternative way to state it, in terms of L-functions, which is especially easy to state. Fix $\pi \in \Pi_{\mathrm{disc}}(G)$ and $r : \widehat{G}(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ an algebraic representation. Following Langlands, the Euler product

$$\mathrm{L}(s, \pi, r) = \prod_p \det(1 - p^{-s}\, r(\mathrm{c}_p(\pi)))^{-1}$$

converges for $\mathrm{Re}\, s$ big enough. When G is the $\mathbb{Z}$-group PGL_m and r is the tautological representation of $\widehat{G} = \mathrm{SL}_m$, we simply write $\mathrm{L}(s,\pi) = \mathrm{L}(s,\pi,r)$. The Arthur-Langlands conjecture for the couple (π, r) predicts the existence of an integer $k \geq 1$, and for $i = 1, \ldots, k$ of a representation [6] π_i in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i})$ and an integer $d_i \geq 1$, such that the following equality holds (§VI.4.4)

$$\mathrm{L}(s, \pi, r) = \prod_{i=1}^{k} \prod_{j=0}^{d_i - 1} \mathrm{L}(s + j - \frac{d_i - 1}{2}, \pi_i). \tag{4.1}$$

[5] The following discussion does not apply verbatim to certain non-connected group schemes which are natural to consider here, such as O_n or PGO_n. We will be more precise in the text about the necessary modifications needed to deal with them as well, but we shall ignore this detail in this introduction.

[6] As is customary, we denote by $\Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G)$ the subset of representations appearing in the subspace of cuspforms [GGPS66] (§IV.3).

In a slightly abusive way, the collection of conjugacy classes $r(\mathrm{c}_v(\pi))$ will be called the *Langlands parameter* of (π, r), and we shall denote it by $\psi(\pi, r)$. When the equality (4.1) holds, we shall also write it symbolically as follows[7]

$$\psi(\pi, r) = \oplus_{i=1}^{k} \pi_i[d_i].$$

If G is a *classical group over* $\mathbb{Z}$ (§VI.4.7, §VIII.1), its dual group $\widehat{G}$ is a complex classical group (that is special orthogonal, or symplectic). In particular, $\widehat{G}$ possesses a natural or "tautological" representation, called the *standard representation*, and denoted by St. An important result proved by Arthur in [Art13] asserts that the Arthur-Langlands conjecture holds for (π, St) for all π in $\Pi_{\mathrm{disc}}(G)$ if G is either Sp_{2g} or a split special orthogonal group over $\mathbb{Z}$.

In Chapter VII, we shall illustrate these ideas by giving numerous examples of specific cases of the Arthur-Langlands conjecture stated above, concerning automorphic forms for SO_n or Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$. They do not rely on Arthur's results, but rather on more classical constructions of theta series. We recall Rallis' point of view on Eichler's commutation relations (§VII.1) as well as important results of Böcherer [Boc89] and of Ikeda [Ike01]. We prove Theorem C and give other applications of triality to the construction of certain elements in $\Pi_{\mathrm{disc}}(\mathrm{SO}_8)$ (§VII.2). One ingredient of the proofs is a slight refinement of Rallis' identities to the pair $(\mathrm{PGO}_n, \mathrm{PGSp}_{2g})$ (§VII.1.4). *In fine*, our analysis covers enough constructions to allow us to determine $\psi(\pi, \mathrm{St})$ for 13 of the 16 "first" representations π in $\Pi_{\mathrm{disc}}(\mathrm{SO}_8)$ (§VII.4).

We are now able to state the analog for X_{24} of Theorem A; we refer to §X.1 for a formulation of the same theorem in terms of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, in the spirit of our announcement [CL11]. We shall need a few extra notations:

– The representation Δ_w, for $w \in \{11, 15, 17, 19, 21\}$, will denote the element of $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)$ generated by the one dimensional space $\mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z}))$ of cuspforms of weight $w + 1$ for the group $\mathrm{SL}_2(\mathbb{Z})$.

– The representation $\mathrm{Sym}^2\Delta_w$ is the Gelbart-Jacquet symmetric square of Δ_w [GJ78]. It is the unique element of $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_3)$ satisfying the equality $\mathrm{c}_v(\mathrm{Sym}^2\Delta_w) = \mathrm{Sym}^2\,\mathrm{c}_v(\Delta_w)$ for all places v of $\mathbb{Q}$.

[7] In the strict sense, we shall include in this equality the natural corresponding identity at the Archimedean place (§VI.4.4). Moreover, the "summand" $\pi_i[d_i]$ will be simply denoted by $[d_i]$ (resp. π_i) when $n_i = 1$ (resp. $d_i = 1$). Those conventions are in force in Table I.2.

– Let (w,v) be one of the 4 couples $(19,7)$, $(21,5)$, $(21,9)$ or $(21,13)$. We will denote by $\Delta_{w,v}$ a certain representation in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_4)$ defined and studied in §IX.1. The eigenvalues of its infinitesimal character $\mathrm{c}_\infty(\Delta_{w,v})$, which is by definition a semisimple conjugacy class in $\mathrm{M}_4(\mathbb{C})$, are[8] $\pm\frac{w}{2}$ and $\pm\frac{v}{2}$; we shall eventually show that this property characterizes $\Delta_{w,v}$ uniquely.

Theorem E. *The parameters $\psi(\pi,\mathrm{St})$ of the 24 representations π in $\Pi_{\mathrm{disc}}(\mathrm{O}_{24})$ generated by the maps $\mathrm{X}_{24}\to\mathbb{C}$ which are eigenvectors for $\mathrm{H}(\mathrm{O}_{24})$ are :*

$[23]\oplus[1]$	$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{17}[4]\oplus\Delta_{11}[2]\oplus[9]$
$\mathrm{Sym}^2\Delta_{11}\oplus[21]$	$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{15}[6]\oplus[9]$
$\Delta_{21}[2]\oplus[1]\oplus[19]$	$\Delta_{15}[8]\oplus[1]\oplus[7]$
$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{19}[2]\oplus[17]$	$\Delta_{21}[2]\oplus\Delta_{17}[2]\oplus\Delta_{11}[4]\oplus[1]\oplus[7]$
$\Delta_{21}[2]\oplus\Delta_{17}[2]\oplus[1]\oplus[15]$	$\Delta_{19}[4]\oplus\Delta_{11}[4]\oplus[1]\oplus[7]$
$\Delta_{19}[4]\oplus[1]\oplus[15]$	$\Delta_{21,9}[2]\oplus\Delta_{15}[4]\oplus[1]\oplus[7]$
$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{19}[2]\oplus\Delta_{15}[2]\oplus[13]$	$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{19}[2]\oplus\Delta_{11}[6]\oplus[5]$
$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{17}[4]\oplus[13]$	$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{19,7}[2]\oplus\Delta_{15}[2]\oplus\Delta_{11}[2]\oplus[5]$
$\Delta_{17}[6]\oplus[1]\oplus[11]$	$\Delta_{21}[2]\oplus\Delta_{11}[8]\oplus[1]\oplus[3]$
$\Delta_{21}[2]\oplus\Delta_{15}[4]\oplus[1]\oplus[11]$	$\Delta_{21,5}[2]\oplus\Delta_{17}[2]\oplus\Delta_{11}[4]\oplus[1]\oplus[3]$
$\Delta_{21,13}[2]\oplus\Delta_{17}[2]\oplus[1]\oplus[11]$	$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{11}[10]\oplus[1]$
$\mathrm{Sym}^2\Delta_{11}\oplus\Delta_{19}[2]\oplus\Delta_{15}[2]\oplus\Delta_{11}[2]\oplus[9]$	$\Delta_{11}[12]$

Table I.2: The standard parameters of the $\pi\in\Pi_{\mathrm{disc}}(\mathrm{O}_{24})$ such that $\pi_\infty=1$.

Let us stress that in a remarkable work [Ike06], Ikeda had been able to determine 20 of the 24 parameters above, namely the ones which do not contain any representation of the form $\Delta_{w,v}$.

Given the importance of the role played by those $\Delta_{w,v}$ in this memoir, let us say a bit more about their origin. Let (j,k) be one of the 4 couples $(6,8),(4,10),(8,8)$ or $(12,6)$. A dimension formula due to R. Tsushima shows that the space of vector-valued Siegel modular cuspforms for $\mathrm{Sp}_4(\mathbb{Z})$, with coefficients in the representation $\mathrm{Sym}^j\mathbb{C}^2\otimes\det^k$ of $\mathrm{GL}_2(\mathbb{C})$, has dimension 1 [Tsu83]. We will give a concrete generator of this space using a construction of theta series with "pluriharmonic" coefficients based of the E_8 lattice. If $\pi_{j,k}$

[8] In a similar way, the eigenvalues of $\mathrm{c}_\infty(\Delta_w)$ are $\pm\frac{w}{2}$.

denotes the element of $\Pi_{\rm cusp}(\mathrm{PGSp}_4)$ generated by this eigenform, then we have the relation $\psi(\pi_{j,k}, \mathrm{St}) = \Delta_{w,v}$ with $(w, v) = (2j+k-3, j+1)$. Observe that PGSp_4 is isomorphic to the split classical group $\mathrm{SO}_{3,2}$ over $\mathbb{Z}$, whose dual group is Sp_4 over $\mathbb{C}$, so that Arthur's theory does apply to $(\pi_{j,k}, \mathrm{St})$.

Theorem E will be proved in §X.4.3, using a method that we shall describe in §I.6. Nevertheless, we will first give two conditional proofs of this result in §X.2.10 and §X.2.11. These proofs, obtained by applying Arthur's *multiplicity formula*, will eventually be the most natural ones, but they depend at the moment on some refinements of Arthur's results which are expected but not yet available.

In Chapter VIII we thus come back to the general results of Arthur [Art13], that we specify to the case of classical groups over $\mathbb{Z}$ and of their automorphic forms which are "unramified everywhere"; such an analysis had already been mostly carried out in [CR12, §3], that we develop and complete. Most of Chapter VIII is devoted to make explicit the famous aforementioned multiplicity formula. This formula gives a necessary and sufficient condition on a given collection of $(\pi_i, d_i)'s$ to "arise" as the standard parameter of some π in $\Pi_{\rm disc}(G)$ having furthermore a prescribed Archimedean component π_∞ (§VIII.3); the first part of the assertion simply means that we ask the equality $\psi(\pi, \mathrm{St}) = \oplus_{i=1}^{k} \pi_i[d_i]$. We limit ourselves to the case where π_∞ is a discrete series representation of $G(\mathbb{R})$ and explain in concrete terms the way those representations are parameterized by Shelstad, which is the ingredient needed to understand Arthur's formula (§VIII.4). At the moment, the version of the formula that we give is only proved if G is split over $\mathbb{Z}$ and if all the integers d_i are equal to 1. Nevertheless, we discuss it in general, and we make precise the conjectures on which various specific cases depend (§VIII.4.21), because it greatly enlightens several constructions studied in this memoir. In particular, we draw very concrete and explicit formulas in the specific cases of Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$ and automorphic forms for SO_n (§VIII.5). We check that those formulas are compatible with the results of Chapter VII and with the results of Böcherer on the image of the ϑ maps (3.1) (§VIII.6). As promised, we show in §IX.2 that they lead to a rather miraculous, but simple, conditional proof of Theorem E.

5. Algebraic automorphic representations of small weight

Let $n \geq 1$ be an integer. We say that a representation π in $\Pi_{\rm cusp}(\mathrm{PGL}_n)$ is *algebraic* if the eigenvalues λ_i of $\mathrm{c}_\infty(\pi)$ satisfy $\lambda_i \in \frac{1}{2}\mathbb{Z}$ and $\lambda_i - \lambda_j \in \mathbb{Z}$ for all i, j. The largest difference between two eigenvalues of $\mathrm{c}_\infty(\pi)$ is then called the *motivic weight* of π; it is a nonnegative integer denoted by $\mathrm{w}(\pi)$. As already seen in the preface[9], these algebraic cuspidal automorphic representations are interesting in their own rights: they are precisely the ones which are related to ℓ-adic "geometric" Galois representations according to the yoga of Fontaine-Mazur and Langlands (§VIII.2.16). The reason we are interested in these representations here is a little different, and explained by the following observation.

Let G be a semisimple group scheme over $\mathbb{Z}$, let π in $\Pi_{\mathrm{disc}}(G)$ be such that π_∞ has the same infinitesimal character as an algebraic finite dimensional representation V of $G(\mathbb{C})$, and let $r : \widehat{G}(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ be an algebraic representation. Assume that we have $\psi(\pi, r) = \oplus_{i=1}^k \pi_i[d_i]$ following Arthur and Langlands. Then the representations π_i are algebraic and their motivic weight is bounded in terms of the highest weights of V and r (§VIII.2). For instance, if $G = \mathrm{Sp}_{2g}$ and if π in $\Pi_{\mathrm{cusp}}(\mathrm{Sp}_{2g})$ is generated by a Siegel modular eigenform of weight k for $\mathrm{Sp}_{2g}(\mathbb{Z})$ (say with $k > g$, but this condition can be weakened), then we may write $\psi(\pi, \mathrm{St}) = \oplus_{i=1}^k \pi_i[d_i]$ thanks to Arthur, and each π_i is algebraic of motivic weight $\leq 2k-2$. An important ingredient in our proofs is the following classification statement, which is of independent interest, and proved in §IX.3.

Theorem F. *Let $n \geq 1$ and let π in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ be algebraic of motivic weight ≤ 22. Then π belongs to the following list of 11 representations :*

$$1,\ \Delta_{11},\ \Delta_{15},\ \Delta_{17},\ \Delta_{19},\ \Delta_{19,7},\ \Delta_{21},\ \Delta_{21,5},\ \Delta_{21,9},\ \ \Delta_{21,13},\ \ \mathrm{Sym}^2\Delta_{11}.$$

In motivic weight < 11, this theorem states that we have $n = 1$ and that π is the trivial representation, a result which was already known to Mestre and Serre (up to the language, see [Mes86, §III, Remarque 1]). In this very specific case, it provides us (among other things!) with an "automorphic" analog of a classical theorem of Minkowski asserting that every number field different from $\mathbb{Q}$ contains at least one ramified prime (case $\mathrm{w}(\pi) = 0$), and also of a conjecture of Shaffarevic, independently proved by Abrashkin and Fontaine, according to which there is no abelian variety over $\mathbb{Z}$ (case $\mathrm{w}(\pi) = 1$). As far as we know, the special case $\mathrm{w}(\pi) = 11$ of the theorem is already new. Let us emphasize that there is no assumption on n in the statement of Theorem F, and that it implies $n \leq 4$.

Our proof of Theorem F, in the spirit of the works of Stark, Odlyzko and Serre on lower bounds of discriminants of number fields, relies on an analogue

[9] The definition of algebraic given in the preface, apparently less restrictive, is actually equivalent to the one given above: see Remark VIII.2.14. The motivic weight $\mathrm{w}(\pi)$ is also twice the largest eigenvalue of $\mathrm{c}_\infty(\pi)$.

in the framework of automorphic L-functions of the so-called *explicit formulas* of Riemann and Weil in the theory of prime numbers. This analogue has been developed by Mestre [Mes86] and applied by Fermigier to the standard L-function $\mathrm{L}(s,\pi)$ for π in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ to show the nonexistence of certain π's [Fer96]. We shall apply it more generally to the "Rankin-Selberg L-function" of an arbitrary pair $\{\pi,\pi'\}$ of cuspidal automorphic representations of PGL_n and $\mathrm{PGL}_{n'}$ (Jacquet, Piatetski-Shapiro, Shalika).

In the specific case where π' is the dual of π, this method had already been successfully employed by Miller [Mil02]; however our study contains some novelties which deserve to be mentioned. First of all, we have discovered that certain symmetric bilinear forms, which are defined on the Grothendieck ring K_∞ of the Weil group of $\mathbb{R}$ and real-valued, and which naturally occur in a formulation of the explicit formulas, are positive definite on rather large subgroups of K_∞. It is this phenomenon which is responsible for the finiteness of the list of the statement of Theorem F. Moreover, we establish some simple but efficient criterions, for instance involving only π_∞ and π'_∞, that prevent the simultaneous existence of π and π'. We refer to §IX.3 for more precise statements.

6. Proofs of Theorems D and E

Let us sketch the main steps in the proof of Theorem E (§IX.4.3). Let π in $\Pi_{\mathrm{disc}}(\mathrm{O}_{24})$ be such that π_∞ is the trivial representation. The results of Erokhin and Borcherds-Freitag-Weissauer recalled in §I.3 show that π admits a "ϑ-correspondent" π' in $\Pi_{\mathrm{cusp}}(\mathrm{Sp}_{2g})$, generated by a Siegel modular form of weight 12 and genus ≤ 11 for $\mathrm{Sp}_{2g}(\mathbb{Z})$ (§VII.1), except for a single π satisfying $\psi(\pi,\mathrm{St})=\Delta_{11}[12]$ already determined by Ikeda. Arthur's theorem applied to π', and the point of view of Rallis on Eichler's relations, imply that (π,St) satisfies the Arthur-Langlands conjecture. A simple combinatorial argument, relying only on Theorem F, shows that there are at most 24 possibilities for $\psi(\pi,\mathrm{St})$, namely the ones given in Table I.2. On the other hand, there are at least 24 possibilities for $\psi(\pi,\mathrm{St})$, as the Hecke operator T_2 has distinct eigenvalues on $\mathbb{C}[\mathrm{X}_{24}]$ according to Nebe and Venkov: this concludes the proof.

This method allows to study more generally the elements of $\Pi_{\mathrm{cusp}}(\mathrm{Sp}_{2g})$ generated by a Siegel modular form of weight $k\leq 12$ for the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ (§IX.5). The statement of Theorem D is the result of our study in the special case $k=12$. We find exactly 23 Siegel modular forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$ which are eigenvectors for $\mathrm{H}(\mathrm{Sp}_{2g})$, of weight 12 and genus $g\leq 23$, and we even give their standard parameters (Table C.1). In the case of forms of weight $k \leq 11$, we prove the following theorem, generalizing previous results of [DI98] and [PY07] (Theorem IX.5.10).

Theorem G. *Let $g \geq 1$ be an integer and let $k \in \mathbb{Z}$.*

(i) If $k \leq 10$ then $\mathrm{S}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ vanishes, unless (k,g) is among

$$(8,4), \qquad (10,2), \qquad (10,4), \qquad (10,6), \qquad (10,8),$$

in which case $\mathrm{S}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ has dimension 1. The standard parameters of the 5 elements of $\Pi_{\mathrm{cusp}}(\mathrm{Sp}_{2g})$ generated by thoses spaces are respectively

$$\Delta_{11}[4]\oplus[1], \quad \Delta_{17}[2]\oplus[1], \quad \Delta_{15}[4]\oplus[1], \quad \Delta_{17}[2]\oplus\Delta_{11}[4]\oplus[1] \quad \text{and} \quad \Delta_{11}[8]\oplus[1].$$

(ii) If $k = 11$ then $\mathrm{S}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ vanishes, except maybe if $g = 6$.

Let us mention some difficulties in the proofs of Theorems D and G which do not appear in the one of Theorem E. Let π in $\Pi_{\mathrm{cusp}}(\mathrm{Sp}_{2g})$ be generated by a Siegel modular form of weight $k \leq 12$ and genus $g < 12+k$. Theorem F allows to show that $\psi(\pi, \mathrm{St})$ belongs to an explicit finite list of possibilities. Contrary to the situation of Theorem E, some elements of this list should actually not occur, as an inspection of the multiplicity formula shows. We bypass the use of this formula by relying instead on results of Böcherer [Boc89], Ikeda [Ike01, Ike13], as well as on various theta series constructions. We expect that $\mathrm{S}_{11}(\mathrm{Sp}_{12}(\mathbb{Z}))$ vanishes but we can't prove it unconditionally. The cases $g \geq k$ are more delicate (we don't even know the explicit form of Arthur's multiplicity formula in these cases); they are excluded in a rather *ad hoc* manner by using the work of S. Mizumoto [Miz91] on the poles of the L-function $\mathrm{L}(s, \pi, \mathrm{St})$ (§VIII.7).

7. Some applications

Thanks to Theorem E, the original problem of determining the numbers $\mathrm{N}_p(L, M)$ for L, M in X_{24} and p prime becomes equivalent to the one of determining the eigenvalues of the Hecke operators in $\mathrm{H}(\mathrm{PGSp}_4)$ on the 4 genus 2 vector-valued Siegel modular eigenforms mentioned in §I.4. We explain in §X.3 a method to compute those eigenvalues that we discovered, which relies on our analysis of the p-neighbors of the Leech lattice made in §III.4.

Let (j,k) be one of the 4 couples considered in §I.4, that is $(6,8)$, $(4,10)$, $(8,8)$ or $(12,6)$. Let (w,v) denote the corresponding couple $(2j+k-3, j+1)$. If q is of the form p^k where p is a prime and k an integer ≥ 1, we set

$$\tau_{j,k}(q) \ := \ q^{\frac{w}{2}} \operatorname{trace} \mathrm{c}_p(\Delta_{w,v})^k.$$

This complex number is actually in $\mathbb{Z}$.

Theorem H. *Let (j,k) be one of the 4 couples $(6,8), (4,10), (8,8)$ or $(12,6)$. The integers $\tau_{j,k}(p)$ with p a prime ≤ 113, and $\tau_{j,k}(p^2)$ with p a prime ≤ 29, are respectively given by the tables C.3 and C.4.*

These results confirm and extend previous computations by Faber et van der Geer [FvdG04] [vdG08, §25] for $p \leq 37$ by very different methods. Our computations allow to determine the exact value of $\mathrm{N}_p(L,M)$ for all L, M in X_{24} and all primes $p \leq 113$.

Theorem F shows that the computation of $\tau_{j,k}(q)$ is perhaps not as futile as it may seem. Indeed, it suggests a parallel classification, yet to be proved on the "ℓ-adic side", of the effective pure motives over $\mathbb{Q}$, with good reduction everywhere, whose motivic weight is ≤ 22. For instance, it imposes a remarkable conjectural constraint on the Hasse-Weil zeta function of the Deligne-Mumford stack $\overline{\mathcal{M}_{g,n}}$ classifying the stable curves of genus g equipped with n marked points, say when $g \geq 2$, $n \geq 0$ and $3g-3+n \leq 22$: this zeta function should be expressible in terms of the $\tau_{j,k}(q)$ and of the coefficients of the normalized cuspforms of weight ≤ 22 for $\mathrm{SL}_2(\mathbb{Z})$. This confirms certain results (resp. conjectures) of Bergström, Faber and van der Geer [FvdG04, Fab13, BFG14] when $g = 2$ (resp. $g = 3$).

In §X.4 we use Theorem E to prove some congruences satisfied by the integers $\tau_{j,k}(p)$, say with p prime. They are obtained as a consequence of a study of the eigenvectors of T_2 in the natural basis of $\mathbb{Z}[\mathrm{X}_{24}]$ and by some Galois representations arguments. Among these congruences, we prove the following one conjectured by Harder in [Har08].

Theorem I. (Harder's conjecture) *For each prime p we have the congruence*

$$\tau_{4,10}(p) \ \equiv \ \tau_{22}(p) + p^{13} + p^8 \bmod 41,$$

where $\tau_{22}(p)$ denotes the le p-th coefficient of the normalized cuspform of weight 22 for the group $\mathrm{SL}_2(\mathbb{Z})$.

Let us go back to the proof of Theorem E sketched in §I.6. It relies on the equality $|X_{24}| = 24$, which is a consequence of Niemeier's classification. However, we explain in §IX.6 how a combination of the ideas above and of Arthur's multiplicity formula (including the conjectures 4.22 and 4.25 formulated in §VIII) allows to bypass the use of this equality, and even to give a new proof of it "without any Euclidean lattice computation". Even better, we not only recover the fact that there are exactly 24 isometry classes of Niemeier lattices, but also that there is a unique such lattice with no isometry of determinant -1.

Is it reasonnable to hope for a fine estimate of the cardinality of X_{32} using such a method ? The question remains open, but the example of the dimension 24 shows that this approach, dear to the first author, is not totally absurd. A necessary ingredient in this project is the knowledge of the algebraic representations (say "selfdual, regular") in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ whose motivic weight is ≤ 30: some progresses in this direction have recently been obtained in [CR12] and [Tai14].

*

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To end this introduction, let us say a word about the use in this memoir of the recent results of Arthur. They rely on an impressive collection of difficult works, whose last pieces have been established only very recently (see [Art13], [MW14, Wal14], as well as the discussion in §VIII.1). We thus found it useful to indicate throughout the memoir by a star $*$ the statements that depend on the results of Arthur's book [Art13]. In this introduction, it concerns the proofs of Theorems B, D, E, F[10], G, H and I. However, let us emphasize that contrary to our initial announcements in [CL11] and [Che13], our final proofs neither rely on the results announced by Arthur *loc. cit.* in his last chapter 9 concerning inner forms, nor on the conjectural properties of Arthur's packets of the type of the ones studied by Adams and Johnson.

[10] We actually prove a version of Theorem F which is almost as strong and which does not rely on Arthur's results: see Theorem IX.3.2.