CONVERGING SEQUENCES OF p-ADIC GALOIS REPRESENTATIONS AND DENSITY THEOREMS

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Abstract. We compare different notions of convergence for sequences of finite-dimensional representations over a valued field. We then consider convergent sequences of Galois representations over C_p, prove a control result for the component groups of the algebraic envelopes of the representations, and deduce a uniform Cebotarev-type density theorem for the representations in the sequence.

In this paper we prove density theorems for *converging* sequences of continuous representations $\rho_n\colon G_F \to \mathrm{GL}_d(\mathbb{C}_p)$, with G_F the absolute Galois group of a number field F and $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$.

There is some play in what one means by *convergence* which is studied in the first part of the paper. We work in a more general context : let $\rho_n\colon G \to \mathrm{GL}_d(K)$ be a sequence of representations of an arbitrary group G on a complete valued field K (of characteristic zero, or $p > d$).

One might consider *trace-convergence*: the sequence of traces, and hence the sequence of characteristic polynomials, of $\rho_n(g)$ converges for each $g \in G$. Then the limit of the trace is a well-defined K-valued pseudo-character of G, which by theory of pseudo-representations initiated by Wiles and developed by Taylor [Tay], is the trace of a representation of G defined over a finite extension K' of K, and unique up to semi-simplification. The stronger notion of *physical convergence* means that we can conjugate each ρ_n by an element of $\mathrm{GL}_d(K)$ (depending on n) so that the resulting homomorphisms converge entry by entry. Then there is at least one limit representation ρ defined over K, and well-defined up to semi-simplification. Of course physical convergence implies trace-convergence. We are interested in results in the other direction, as in applications (e.g. to Galois representations) the sequences which arise naturally are only trace-convergent (e.g. the sequence of Galois representations given by congruences between characters of a Hecke algebra), but sometimes the results we can prove about them (see the remarks at end of section 3) need physical convergence.

Before stating our results, we have to introduce a second distinction in the notions of convergence (both trace and physical). We may ask that the trace function (resp. the entries) converge simply or uniformly in G.

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So we consider in fact four notions of convergence, namely : *simple trace-convergence, uniform trace-convergence, simple physical convergence*, and *uniform physical convergence*.

Our first result (Theorem 1.2) states that if the ρ_n simply trace-converge, and if the limit pseudo-character is absolutely irreducible, then it is the trace of a representation defined over K which is the physical limit of the ρ_n. We show by an example that we cannot omit the hypothesis that the limit pseudo-character is absolutely irreducible, as long as we only assume simple trace-convergence. But our second result (Theorem 1.4) says that if the ρ_n uniformly trace-converges, and if the ρ_n are irreducible and the limit pseudo-character is a sum of *distinct* absolutely irreducible pseudo-characters, then it is the trace of a semisimple representation defined over K which is a uniform physical limit of the ρ_n.

Note that Theorem 1.2 was known in the case $K = \mathbb{C}$, G finitely generated. Its proof used invariant theory. Our argument is completely different. Theorem 1.4 is new, as far as we know.

We also consider analogues of these questions for integral models of ρ_n which are no longer unique but of which there are only finitely many up to isomorphism. Here we assume that the field K is non-archimedean, with ring $\mathcal{O}$. We prove (Proposition 1.9) that in case of uniform convergence, when the limit representation has a stable lattice, then the ρ_n also have one for n big enough, and there is physical convergence in bases which are $\mathcal{O}$-bases of stable lattices.

In the second part of this paper, we specialize to the case of sequences of representations of a compact group taking values in $\mathbb{C}_p$. We consider the algebraic envelopes of the representations, i.e., the Zariski-closures of their images. Our goal is to control the component groups of the resulting algebraic groups. In particular, we prove that the order of these groups is bounded in a uniformly trace-convergent sequence of representations and that in the case when a limit representation is irreducible, all but finitely many of these component groups are quotients of the component group of the limit representation.

Control of component groups is needed in the last part of the paper, where we study density theorems for uniformly trace-convergent sequences of *Galois* representations. Here our point of view is that sequences of converging representations behave like one big representation with given specialisations, and one particularly interesting specialisation which corresponds to a limit representation which controls the behaviour of almost all elements of the sequence. The main theorems are Theorem 3.6, Theorem 3.7 and Theorem 3.8. Cebotarev density theorems for a *single* p-adic Galois representation were proved by Serre in [S2] and density 0 results about ramified primes in a single semisimple representation were proved in [Kh-Raj] and [KLR]. Thus the density theorems we prove in the last section may be regarded as generalisations of these results to the situation where we have at hand converging sequences of Galois representations rather than just a single representation.

Limits of Galois representations were previously studied in [Kh] in the residually irreducible case in which case the convergence results needed were available because of the results of Carayol ([Ca]).

1. Limits of representations

1.1. **The limit representation.**

Let G be a group, K be a complete, non-discrete, valued field and $d \geq 1$ an integer. If K has finite characteristic $p > 0$, we assume that $d < p$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of d-dimensional representations of G over K.

Definition 1.1. *i) We say that (ρ_n) is* trace-convergent *if for all $g \in G$, the sequence $(\mathrm{tr}(\rho_n(g)))$ converges in K. Moreover, if the functions $\mathrm{tr}(\rho_n(.))$ converge uniformly on G, we will say that (ρ_n) is* uniformly trace-convergent.

ii) We say that (ρ_n) is physically convergent *if for all n, there exists a K-basis of ρ_n such that the matrix coefficients $c^n_{i,j}$ in this basis satisfy:*

$$\forall g \in G,\ (c^n_{i,j}(g))_{n\geq 0} \text{ converges in } K.$$

The equivalence class of the representation $G \to \mathrm{GL}_n(K)$, $g \mapsto \lim c_{i,j}(g)$ is called a physical limit *of (ρ_n). Moreover, if the functions $c^n_{i,j}$ converge uniformly on G, we will say that (ρ_n) is* uniformly physically convergent.

Suppose (ρ_n) is trace-convergent, and let T be the K-valued function on G defined by

$$T(g) := \lim_{n\to\infty} \mathrm{tr}(\rho_n(g)).$$

Then T is a K-valued pseudo-character on G in the sense of [Rou]. We claim it is d-dimensional. Indeed, if $d' := \dim(T)$, then $d' \leq d$ and, by [Rou, Prop. 2.4] we have $d' \equiv d \bmod \mathrm{char}(K)$, so that $d' = d$.

By loc. cit. Lemma 4.1, T is the trace of a semisimple representation $\rho\colon G \to \mathrm{GL}_d(\overline{K})$, which is unique by the Brauer-Nesbitt theorem. Here, $\overline{K}$ is an algebraic closure of K. We will call ρ the limit representation of (ρ_n). It is a priori defined over a finite extension of K.

Note that if G is topological, and T is continuous (this happens e.g. if each ρ_n is continuous and (ρ_n) is uniformly trace-convergent), then ρ is continuous. (This is quite easy: for a proof see, e.g., [BC, lemma 7.1].)

1.2. **Simple convergence and irreducible limit.**

Theorem 1.2. *Assume that (ρ_n) is trace-convergent and that ρ is irreducible. Then the representations ρ_n are absolutely irreducible for n big enough, (ρ_n) is physically convergent, and ρ is defined over K.*

If, moreover, (ρ_n) is uniformly trace-convergent, then it is uniformly physically convergent.

Let A be the K-algebra of sequences $(x_n)_{n\in\mathbb{N}}$ of elements of K, such that x_n converges in K. Let $r \in \mathbb{N}$, and $f_r \in A$ be the sequence such that $(f_r)_n = 0$ for $n < r$ and 1 for $n \geq r$. Let $A_r := A_{f_r}$. The natural map $A \to A_r$ is surjective, with kernel the ideal of sequences (x_n) with $x_n = 0$ for $n \geq r$. Let $\mathfrak{m} \subset A$ be the ideal of sequences converging to 0.

For the basic results and definitions concerning Azumaya algebras, we refer to [G, §5.1]. The reader should note that our rings A, A_r, A_m are by no means noetherian.

Lemma 1.3. (a) *The maximal ideals of A are exactly $\mathfrak{m}$ and, for $i \geq 0$, $\mathfrak{m}_i := \{(x_n) \in A, x_i = 0\}$.*

(b) *The canonical maps $A_r \to A_{\mathfrak{m}}$ induce an isomorphism $\varinjlim_r A_r \xrightarrow{\sim} A_{\mathfrak{m}}$. Hence $A_{\mathfrak{m}}$ is the ring of germs at ∞ of converging sequences.*

(c) *$A_{\mathfrak{m}}$ is a local Henselian ring.*

(d) *If B is an Azumya algebra over A, then $B \otimes_A A_r$ is isomorphic to $D \otimes_K A_r$ for r big enough, where $D := B/mB$.*

In particular, if B/m_iB is trivial for an infinite number of integers i, then D and $B \otimes_A A_r$ are also trivial.

Proof. (a) The ring A equipped with the sup. norm is a K-Banach algebra, so that each maximal ideal is closed. Let I be such an ideal. If I is not in $\mathfrak{m}_i$, it contains a sequence δ_i such that $(\delta_i)_n = 0$ if and only if $n \neq i$. So if I is none of the $\mathfrak{m}_i$, I contains all the finite sequences, which are dense in $\mathfrak{m}$.

(b) Let $f \in A\backslash\mathfrak{m}$, then $f_n \neq 0$ for all $n \geq r$ for r big enough. Fix such an r, then the natural map $A_f \to A_{\mathfrak{m}}$ does factor through $A_r \to A_{\mathfrak{m}}$.

(c) We must show that if a sequence of monic polynomials $P_n \in K[T]$ of a fixed degree converges to P, and P has a simple root x, then for all n big enough, there exists a root $x_n \in K$ of P_n, such that $x_n \to x$. Suppose first that K is non-archimedean, then for n big enough, $|P_n(x)| < |P_n'(x)|^2$ and Newton's method gives a root $x_n \in K$ of P_n such that $|x - x_n| \leq |P_n(x)|/|P_n'(x)|$, and we are done. If $K = \mathbb{R}$, then for each $\varepsilon > 0$ small enough and n big enough, $P_n(x+\varepsilon)P_n(x-\varepsilon) < 0$. In particular, P_n has a real root for n big enough, and we can choose x_n to be one of the closest to x. If $K = \mathbb{C}$ this is simply the continuity of roots of polynomials.

(d) Let B be an Azumaya algebra over A. We call $B_r := B \otimes_A A_r$, $B_\infty := B \otimes_A A_m$. By (b), B_∞ is the inductive limit of B_r when r grows. As A_m is henselian by (c) and Azumaya theorem [G, thm. 6.1], B_∞ is isomorphic to $D \otimes_K A_m$ where $D := B/mB$. For $r \in \mathbb{N} \cup \{\infty\}$, let $C_r := D \otimes_K A_r$. As B and C are finitely presented as A-modules and by (b), any A_m-module isomorphism $\varphi_\infty : B_\infty \to C_\infty$ comes from an A_r-module isomomorphism $\varphi_r : B_r \to C_r$ for r big enough. If we assume moreover that φ_∞ is an ring homomorphism then for $r' > r$ big enough, $\varphi_r \otimes_{A_r} A_{r'} : B_{r'} \to C_{r'}$ is also a ring homomorphism. Indeed, there are only a finite number of products to check by linearity, and we are done by (b). $\square$

We now prove Theorem 1.2. Let $\mathrm{Tr}\colon G \to A$ be the function defined by $\mathrm{Tr}(g)_n := \mathrm{tr}(\rho_n(g))$, which is an element of A by assumption. We note first that ρ_n is absolutely irreducible for all n big enough.

By assumption, ρ is absolutely irreducible. By the non-degeneracy of the trace in $M_d(\overline{K})$ we can find d^2 elements $g_s \in G$ such that $\det(\mathrm{tr}(\rho(g_s g_t))) \in K^*$. By continuity, $\det(\mathrm{tr}(\rho_n(g_s g_t)))$ is also non-zero for n big enough, i.e. the $(\rho_n(g_s))_s$ form a K-basis of $M_d(K)$, as we wanted.

So we can assume that all the ρ_n are absolutely irreducible. By hypothesis, ρ also is absolutely irreducible, so that by the lemma (a), Tr mod I is absolutely irreducible for each maximal ideal I of A. We can then apply Rouquier's theorem ([Rou, Theorem 5.1]) that there exists an Azumaya algebra B over A and a surjective A-algebra homomorphism $A[G] \to B$ whose reduced trace coincide with Tr on G. As ρ_n is absolutely irreducible (and defined over K by hypothesis), the K-algebra B/m_nB is then isomorphic to $M_d(K)$ for all n. By lemma (d), is follows that B_r is isomorphic with $M_d(A_r)$ for some r. This concludes the first point of the proof.

Consider the representation $\rho'\colon G \to \mathrm{GL}_d(A_r)$, whose trace is Tr, constructed in the previous paragraph. We know that the induced morphism $A_r[G] \to M_d(A_r)$ is surjective. It implies that the A_r-dual of $M_d(A_r)$ is generated as A_r-module by linear forms of the form: $x \mapsto \mathrm{Tr}(xh)$, for some h in G. Applying this to the (i,j)-matrix coefficient $c_{i,j}$, we get that there exists a finite number of $a_k \in A$ and $g_k \in G$ such that

$$\forall g \in G, c_{i,j}(g) = \sum_k a_{k,i,j} \mathrm{Tr}(g g_k).$$

As sequence of functions on G, this implies that $c_{i,j}$ converges uniformly. $\square$

Remark: As the above proof shows, the result holds in the context of representations of A-algebras: if R is any A-algebra equipped with a faithful d-dimensional pseudo-character $T = (T_n) : R \to A$ such that $\lim T$ is absolutely irreducible, then for r big enough, $R \otimes_A A_r$ is isomorphic to $M_d(A_r)$ as A_r-algebra.

Remark: (i) Assume that G is a topological group, that the ρ_n are continuous, uniformly trace-convergent, and that ρ is irreducible, then ρ is also continous by the theorem.

(ii) When (ρ_n) is trace-convergent but ρ is reducible, (ρ_n) need not converge physically in general, as the following example shows.

Let A be the ring introduced in section 1.2, $\mathfrak{m}$ its maximal ideal of sequences converging to zero, and $A' \supset A$ the ring of bounded, K-valued, sequences. We have $\mathfrak{m}A' \subset \mathfrak{m}$ and $(A \backslash \mathfrak{m}) + \mathfrak{m} \subset (A \backslash \mathfrak{m})$. We can thus consider the following group $G \subset \mathrm{GL}_2(A')$ of matrices:

$$\begin{pmatrix} A \backslash \mathfrak{m} & A' \\ \mathfrak{m} & A \backslash \mathfrak{m} \end{pmatrix} \cap \mathrm{GL}_2(A')$$

Let $\rho'\colon G \to \mathrm{GL}_2(A')$ be the canonical representation, and ρ_n its nth-coordinate projection, $\rho_n\colon G \to \mathrm{GL}_2(K)$. Then ρ_n is trace-convergent by construction, but not physically.

Here is a proof of this last fact. If a $\rho'\colon G \to \mathrm{GL}_2(A)$ commutes with trace, we can conjugate it such that the constant element $(-1,1)$ acts diagonally by $(-1,1)$. In that base, because of the trace identity, every diagonal matrix maps to itself. If $\rho'_n\colon G \to \mathrm{GL}_2(K)$ denotes the projection of ρ' on the nth coordinate, ρ'_n has the same trace as the irreducible representation ρ_n, so that it factors through the nth coordinate $G \to \mathrm{GL}_2(K)$, which is surjective. Call the induced map $\mathrm{GL}_2(K) \to \mathrm{GL}_2(K)$ the nth-component of ρ'. As ρ_n is absolutely irreducible, the nth component of ρ' is an inner embedding $\mathrm{GL}_2(K) \to \mathrm{GL}_2(K)$, which is the identity on all diagonal matrices. It is therefore a diagonal conjugation and preserves the standard upper and lower Borels by multiplying the coordinate by a non-zero element, say $x_n \in K^*$ for the upper, and so by x_n^{-1} for the lower. We get then a map $A' \to A$ given by $(b_n) \mapsto (x_n b_n)$. Taking $(b_n) = (1)$ and (b'_n) with $b'_{2n} = 1$, $b'_{2n+1} = 0$ implies that x_n converges to 0. But we get also a map $m \to A$, $(c_n) \mapsto (x_n^{-1} c_n)$. Let (c_n) be given by $c_{2n} = 0$ and $c_{2n+1} = x_{2n+1}$, we get a contradiction.

1.3. **Uniform convergence and multiplicity-free limit.**

Theorem 1.4. *Assume that the representations ρ_n are absolutely irreducible for n big enough, that (ρ_n) is uniformly trace-convergent and that $\mathrm{tr}(\rho)$ is a sum of pairwise distinct, K-valued, absolutely irreducible pseudo-characters. Then ρ is defined over K and (ρ_n) is uniformly physically convergent to ρ.*

Remark: It is easy to give examples where (ρ_n) is physically convergent to several non-isomorphic representations (which have of course isomorphic semi-simplifications). The above theorem asserts that we can make the (ρ_n) physically converge, and what is more, to converge to the semisimple limit. The methods of the proof below are close in spirit to those of [BG].

We now begin the proof of Theorem 1.4. Let A, A_r and $\mathfrak{m}$ be as in section 1.2. Let $S := K^{\mathbb{N}} \supset A$ be the K-algebra of all K-valued sequences, S_r be the quotient of S by the ideal of sequences which are zero after r, and let $B \supset A_{\mathfrak{m}}$ be the K-algebra of germs at ∞ of elements of S, that is $B = \varinjlim_r S_r$. The representations ρ_n of the assumption altogether give rise to a representation $\rho' :\ G \to \mathrm{GL}_d(S)$ whose trace is A-valued. Let $(T_i)_{i=1,\ldots,s}$ be the pairwise distinct, absolutely irreducible, K-valued, pseudo-characters of the assumption, and $d_i := \dim(T_i)$.

Lemma 1.5. *For r big enough, there are s orthogonal idempotents $e_1, \ldots, e_s$ in $\rho'(A_r[G]) \subset M_d(S_r)$ satisfying:*

i) $e_1 + \cdots + e_s = 1$,

ii) for each i, $\mathrm{tr}(e_i) = d_i$,

iii) for each $i \neq j$, and $x, y \in \rho'(A_r[G])$, $\mathrm{tr}(e_i x e_j y e_i) \in \mathfrak{m}$,

iv) for each i, $e_i \rho'(A_r[G]) e_i$ is isomorphic as A_r-algebra to $M_{d_i}(A_r)$.

Proof. Let $R := \rho'(A_{\mathfrak{m}}[G]) \subset M_d(B)$, $T = \mathrm{tr}(\rho')$, $\overline{R} := R/\mathfrak{m}R$, $\overline{T} := T \bmod \mathfrak{m}$ and

$$\mathrm{Ker}(\overline{T}) := \{x \in \overline{R},\ \forall y \in G, \overline{T}(xy) = 0\}.$$

Let $\pi : K[G] \to \overline{R}$ be the surjective K-algebra morphism which sends g to the reduction of $\rho'(g)$. By hypothesis, $\overline{T} \circ \pi = \sum_{i=1}^{s} T_i$. Now choose (cf. [Rou, thm 4.2]) an irreducible representation $\bar{\rho}_i : K[G] \to M_{d_i}(\bar{K})$ whose trace is T_i, and let $\bar{\rho} = \oplus_{i=1}^{s} \bar{\rho}_i$. Because the $\bar{\rho}_i$ are pairwise non-isomorphic, the image $\bar{\rho}(K[G])$ is $\oplus_{i=1}^{s} R_i$, with $R_i = K[G]/(\mathrm{Ker}(T_i))$ a central simple algebra over K. Because $\bar{\rho}$ is semisimple we have $\mathrm{Ker}(\overline{T} \circ \pi) = \mathrm{Ker}(\bar{\rho})$ by [Tay, Th. 1.1.]. Then $\bar{\rho}$ induces an isomorphism $\overline{R}/(\mathrm{Ker}(\overline{T})) \simeq \oplus_{i=1}^{s} R_i$ such that the reduced trace of R_i is T_i.

We call ϵ_i, for $i = 1, \dots, s$, the unit of R_i, seen as a central idempotent of $\overline{R}/(\mathrm{Ker}(\overline{T}))$. As $A_{\mathfrak{m}}$ is local henselian and R is integral over $A_{\mathfrak{m}}$ by Cayley-Hamilton theorem, [Bki, III, §4, exercice 5(b)] implies that there exist orthogonal idempotents $f_i \in R$, $i = 1, \dots, d$ lifting ϵ_i. By construction, we have $\mathrm{tr}(f_i) \equiv d_i \bmod \mathfrak{m}$. Note that if $f \in M_d(B)$ is an idempotent, then its trace is the germ of a sequence of integers between 0 and d. If moreover $\mathrm{tr}(f) \in A_{\mathfrak{m}}$, then this sequence is eventually constant. In particular, $\mathrm{tr}(f_i) = d_i$, and so $f_1 + \cdots + f_s = 1$.

Now, fix $e_i \in A[\rho'(G)]$ be a lift of f_i. For r big enough, we have $e_i e_j = \delta_{i,j} e_i$ and $\sum_i e_i = 1$ in $M_d(B_r)$, and also $\mathrm{tr}(e_i) = d_i \in A_r$. This proves i) and ii). For iii), it suffices to prove that the image of $(e_i x e_j y e_i)$ is zero in $\overline{R}/\ker(\overline{T}) \simeq \oplus_{i=1}^{s} R_i$. But this image is $(\epsilon_i \overline{x} \epsilon_j \overline{y} \epsilon_i)$ which is obviously zero. It remains to prove iv). Let $R' := e_i R e_i \subset M_{d_i}(S)$, T' the restriction of T to R', then $T' \bmod \mathfrak{m} = T_i$ is absolutely irreducible, and T' is faithful if r is big enough so that all the representations ρ_n, $n \geq r$, are absolutely irreducible. By Theorem 1.2 (see the remark immediately following the proof of the theorem), R' is isomorphic as A_r-algebra to $M_{d_i}(A_r)$ for r big enough. □

In particular, assertion iv) implies that each irreducible factor of ρ is defined over K. Forgetting the first r terms of our sequence, we may assume $r = 0$ in the preceding lemma, so we drop the r in S_r and A_r.

For the convenience of the reader, we first prove the theorem in the case where ρ is a sum of pairwise distinct *one-dimensional characters*, i.e $s = d$, $d_i = 1$, $\forall i$. We will return to the general case, which requires no new idea but a great deal of additional notation, at the end of the proof.

¿ From i) and ii) of the previous lemma, we can choose an S-basis (E_i) of S^d such that for each n, $K.E_i^n = e_i^n(K^d)$. For $y \in M_d(S)$, we note $y_{i,j}$ the (i,j)-component of y. We note $E_{i,j}$ the matrix whose (i,j)-coefficient is one and others are zero. Note that $E_{i,i} = e_i$ and that for $y \in M_d(S)$, $e_i y e_j = y_{i,j} E_{i,j}$. Now for each $i, j \in \{1, \dots, d\}$ and $n \in \mathbb{N}$ we define

$$x_{i,j}^n := \inf_{g \in G} v(\rho'(g)_{i,j}^n).$$

Here v is a fixed $\mathbb{R}$-valued valuation of K. Note that for each $n \in \mathbb{N}$, and each i, j we have $x_{i,j}^n \in \mathbb{R} \cup \{-\infty\}$ because ρ_n is absolutely irreducible. As a consequence, it makes sense to add and compare those numbers.

Lemma 1.6. *There exists a real number N such that for each i, j, k pairwise distinct in $\{1, \dots, d\}$, and each $n \in \mathbb{N}$, we have*

$$x_{i,j}^n \leq x_{i,k}^n + x_{k,j}^n + N$$

Proof. We can write all idempotents e_i as finite sums of elements of $\rho'(G)$ with coefficients in A : there is an l such that for each i

$$e_i = \sum_{s=1}^{l} a_{i,s} \rho'(h_{i,s}) \tag{1}$$

and all the coefficients $a_{i,s}$ are convergent (hence bounded) sequences in K. We define $-N$ as

$$-N := v(l) + \inf_{i,s,n \in \mathbb{N}} v(a_{i,s}^n).$$

Now we fix an $n \in \mathbb{N}$, and $i, j, k \in \{1, \dots, d\}$ and choose a real $\varepsilon > 0$. We choose g and g' such that $v(g_{i,k}^n) \leq x_{i,k}^n + \varepsilon$ and $v({g'}_{k,j}^n) \leq x_{k,j}^n + \varepsilon$. We have

$$g_{i,k} g_{k,j} E_{i,j} = e_i g e_k g' e_j = \sum_{k=1}^{l} a_{k,s} (g h_{i,s} g')_{i,j} E_{i,j},$$

so

$$x_{i,k}^n + x_{k,j}^n + 2\varepsilon \geq v(g_{i,k}^n g_{k,j}^n) = v(\sum_{k=1}^{l} a_{k,s}^n (g h_{i,s} g')_{i,j}^n).$$

Now

$$|\sum_{k=1}^{l} a_{k,s}^n (g h_{i,s} g')_{i,j}^n| \leq l \sup_{k,s,n} |a_{k,s}^n| \sup_{g \in G} |g_{i,j}^n|,$$

and so $v(\sum_{k=1}^{l} a_{k,s}^n (g h_{i,s} g')_{i,j}^n) \geq -N + x_{i,j}$. This concludes the proof. □

We now use uniform trace-convergence to prove the following lemma

Lemma 1.7. *For each $i \neq j$, $x_{i,j}^n + x_{j,i}^n$ goes to infinity with n.*

Proof. By hypothesis there is a sequence $\delta_n \in \mathbb{R} \cup \{-\infty\}$ which goes to infinity, such that for each $g \in G$, we have $v(\mathrm{tr}(\rho_n(g)) - \lim \mathrm{tr}(\rho(g))) > \delta_n$.

We have by (1) (applied twice to e_i and once to e_j)

$$e_i \rho'(g) e_j \rho'(g') e_i = \sum_{s,s',s''} a_{i,s} a_{j,s'} a_{i,s''} \rho'(h_{i,s} g h_{j,s'} g' h_{i,s''})$$

which proves that there exists a sequence δ'_n which goes to infinity such that for all $g, g' \in G$, $i, j \in \{1, \dots, d\}$

$$v(\mathrm{tr}(e_i \rho_n(g) e_j \rho_n(g') e_i)) - \lim(\mathrm{tr}(e_i \rho'(g) e_j \rho'(g') e_i))) > \delta'_n.$$

But by Lemma 1.5, (iii), we have $\lim(\mathrm{tr}(e_i\rho'(g)e_j\rho'(g')e_i)) = 0$. Hence

$$v(\mathrm{tr}(e_i\rho_n(g)e_j\rho_n(g')e_i)) > \delta'_n,$$

that is

$$v(g^n_{i,j}g'^n_{j,i}) > \delta'_n.$$

Taking inf on g and g', we get

$$x^n_{i,j} + x^n_{j,i} > \delta'_n.$$

□

In particular, for n big enough, all the $x^n_{i,j}$ are true real numbers, so that we can assume that $x^n_{i,j} \in \mathbb{R}$ for all n, i, j. The following lemma is a simple matter of real inequalities.

Lemma 1.8. *Let $x_{i,j}$, $i \neq j \in \{1, \ldots, d\}$, be a family of sequences of real numbers, and a real N, such that $x_{i,j} + x_{j,i}$ goes to infinity and $x_{i,j} \leq x_{i,k} + x_{k,j} + N$. Then there exist sequences of integers u_i, $i = 1, \ldots, d$, such that for each $i \neq j$, $x_{i,j} - (u_i - u_j)$ goes to infinity.*

Proof. First we may assume that $N = 0$. Indeed let $x'_{i,j} = x_{i,j} + N$, we have $x'_{i,j} \leq x'_{i,k} + x'_{k,j}$, and the other hypothesis as well as the conclusion remain unchanged. Note that in the conclusion we may also choose the numbers u_i to be real instead of integer, for the integer parts of the u_i will also work. We may also assume that $x^n_{i,j} + x^n_{j,i} \geq 0$ for all i, j, n.

Choose $n \in \mathbb{N}$. For each $l \in \{1, \ldots, d\}$, we define $u^n(l) \in \mathbb{R}^d$ by $u^n(l)_i = -x^n_{l,i}$ if $i \neq l$, and $u^n_l(l) = 0$. For each $i \neq j \in \{1, \ldots, d\}$, we check easily that

$$u^n(l)_i - u^n(l)_j \leq x^n_{i,j}.$$

Now we consider u the barycenter of the $u(l)$'s, that is

$$u^n = \frac{1}{d}\sum_{l=1}^{d} u^n(l)$$

We have

$$\begin{aligned} x^n_{i,j} - (u^n_i - u^n_j) &= \frac{1}{d}\sum_{l=1}^{d}(x^n_{i,j} - (u^n(l)_i - u^n(l)_j)) \\ &= \frac{1}{d}\Big(\sum_{l=1, l\neq j}^{d}(x^n_{i,j} - (u^n(l)_i - u^n(l)_j))\Big) + \frac{1}{d}(x^n_{i,j} - u^n(j)_i) \\ &\geq \frac{1}{d}(x^n_{i,j} + x^n_{j,i}) \end{aligned}$$

This last inequality makes clear that $x^n_{i,j} - (u^n_i - u^n_j) \underset{n\to\infty}{\longrightarrow} \infty$. □

Now we can finish the proof of the theorem. We may assume that v takes the value 1, and choose an element $\varpi \in K^*$ such that $v(\varpi) = 1$. We choose now u_i as in the preceding lemma, and consider the following new basis of S^d : $F_i := (\varpi^{u_i^n})_n E_i$. In this basis, the (i,j) coefficient of $\rho'(g)$, is equal to $\varpi^{u_i - u_j} \rho'(g)_{i,j}$, whose nth term has valuation greater or equal than $x_{i,j}^n - (u_i^n - u_j^n)$. If $i \neq j$, this shows that the (i,j)-coefficient of $\rho'(g)$ converges to zero uniformly in g. Moreover, the diagonal coefficients, which are still $\rho'(g)_{i,i}$, are uniformly convergent, as they are equal to $\mathrm{tr}(e_i \rho(g) e_i)$. We thus have shown that the sequences of matrices of $\rho'(g)$, in the basis (F_i), converge uniformly to the diagonal matrix $(\chi_i(g))$, which is what we wanted.

We now return to the general case with $d_i \geq 1$, by indicating only the modifications of the proof. We choose first an S-basis $(F_\alpha)_{1 \leq \alpha \leq d}$ of S^d adapted to the idempotents e_i. That means that $\forall \alpha \in \{1, \dots, d\}$ there exists i (necessarily unique) such that $e_i F_\alpha = F_\alpha$, we then say that α belongs to i. Define the

$$x_{\alpha,\beta}^n := \inf_{g \in G} v(\rho'(g)_{\alpha,\beta}^n),$$

and $x_{i,j}^n = \inf_{\alpha,\beta} x_{\alpha,\beta}^n$ where α (resp. β) belongs to i (resp. j.). As a consequence of lemma 1.5 iv), there exists a constant $N' \in \mathbb{R}$ such that for any $n \in \mathbb{N}$, any i, j, and α belonging to i, β belonging to j, then $x_{\alpha,\beta} \leq x_{i,j} + N'$. Then, it is easy to see that lemmas 1.6 and 1.7 hold for the $x_{i,j}^n$ with the same proof (note however that 1.7 does not hold for the $x_{\alpha,\beta}^n$'s.) . We conclude as above. $\square$.

Remarks: As we have already seen, the uniform convergence hypothesis cannot be omitted. Moreover, we can not omit the hypothesis that the ρ_n are absolutely irreducible, as shown by this counterexample :

Let K be non-archimedean, let A be as before, and let A_u be the subring of A of sequence x_i such that $v(x_i) \geq 0$ and $v(x_i - \lim x_i) \geq i$ for all $i \in \mathbb{N}$. Let G be the group

$$\begin{pmatrix} A_u^* & A \\ 0 & A_u^* \end{pmatrix}$$

Let $\rho' \colon G \to \mathrm{GL}_2(A)$ be the canonical representation, and ρ_n its nth-coordinate projection, $\rho_n \colon G \to \mathrm{GL}_2(K)$. Then ρ_n is uniformly trace-convergent by construction, but not uniformly physically. Moreover the ρ_n are simply physically convergent, but have no semisimple physical limit. (We leave the proofs to the reader.)

We do not know whether the hypothesis that the limit pseudo-character is multiplicity free is really necessary. We believe it is not, but that a new idea would be needed to remove it.

1.4. **Lattices.** Assume that K is non-archimedean, and denote by $\mathcal{O}$ its valuation ring and m its maximal ideal. Let τ be a representation of G on a finite dimensional K-vector space V. A *stable lattice* of τ is a finitely generated sub-$\mathcal{O}$-module of V which is stable by $\tau(G)$ and generates V as

K-vector space. As $\mathcal{O}$ is a valuation ring, such a lattice is automatically free as $\mathcal{O}$-module, of rank $\dim_K(V)$.

If τ has a stable lattice, then $\tau(G)$ is bounded and $\mathrm{tr}(\tau(G)) \subset \mathcal{O}$. Conversely, we can ensure that τ admits at least a stable lattice in the following three cases:

i) $\tau(G)$ has compact closure in $\mathrm{GL}(V)$,

ii) K is discretely valued, τ is absolutely semisimple and $\mathrm{tr}(\tau(G)) \subset \mathcal{O}$.

iii) $\mathrm{tr}(\tau(G)) \subset \mathcal{O}$ and $\mathrm{tr}(\tau(.))$ mod m is an absolutely irreducible pseudo-character.

Remark: Let $\mathcal{O}_p$ denote the ring of integers in $\mathbb{C}_p$ and G the subgroup of $\mathrm{GL}_2(\mathbb{C}_p)$ of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,\, d \in \mathcal{O}_p^*,\ v(b) \geq -\sqrt{2},\ v(c) \geq \sqrt{2}.$$

Then G is bounded, $\mathrm{tr}(G) \subset \mathcal{O}_p$, G acts irreducibly on $\mathbb{C}_p^2$ but there is no stable lattice.

Proposition 1.9. *Suppose (ρ_n) is uniformly physically convergent to ρ, and assume that ρ and has a stable lattice. Then, for n big enough, there exists a K-basis of ρ_n which generates over $\mathcal{O}$ a stable lattice and such that the matrix coefficients $c_{i,j}^n$ in this basis converges uniformly. In particular, ρ_n has a stable lattice for n big enough.*

Proof. By assumption, we can assume that there is a representation $\rho' \colon G \to GL_d(A)$ whose nth-coordinate is ρ_n, and whose (i,j)-coefficients are uniformly converging as sequence of functions on G. We choose a basis of a stable lattice of the limit representation $\rho = \lim \rho'$, and fix $P \in \mathrm{GL}_d(K)$ so that $P\rho(G)P^{-1} \subset \mathrm{GL}_d(\mathcal{O})$. If P' in $\mathrm{GL}_d(A_r)$ is the constant sequence of matrices $(P, P, P, \dots)$, then conjugating ρ' by P' allows us to assume that $\rho(G) \subset GL_d(\mathcal{O})$.

Now, by the first sentence of the proof, we can choose an integer N such that

$$\forall n \geq N,\ \forall i,\ j,\ \sup_{g \in G} |c_{i,j}^n(g) - c_{i,j}^\infty(g)| < 1.$$

As K is non-archimedean, we have $c_{i,j}^n(g) \in \mathcal{O}$ for all $n \geq N$, $g \in G$. □

Remark: Assume that (ρ_n) is uniformly physically convergent to ρ and that (ρ_n) has a stable lattice for all n. Then it does not follow that ρ has a stable lattice, as the following counter-example shows.

Let $(\beta_n)_{n \geq 0}$ and $(\gamma_n)_{n \geq 0}$ be strictly decreasing sequences of real numbers converging to $-\sqrt{2}$ and $\sqrt{2}$ respectively. Fix $K = \mathbb{C}_p$ and A as in the first section. Consider the subset $G \subset GL_2(A)$ of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that: $\lim a \in \mathcal{O}^*$ and $v(a_n - \lim a) > n$, $\lim d \in \mathcal{O}^*$ and $v(d_n - \lim d) > n$, $v(b_n) > \beta_n$ and $v(b_n - \lim b) > n - \sqrt{2}$, $v(c_n) > \gamma_n$ and $v(c_n - \lim c) > n + \sqrt{2}$.

It is easy to check this is indeed a subgroup. Let $\rho' \colon G \to \mathrm{GL}_2(A)$ be the canonical representation, and ρ_n its nth-coordinate projection, $\rho_n \colon G \to \mathrm{GL}_2(\mathbb{C}_p)$. Then ρ_n is uniformly physically convergent by construction. We see easily that the image of ρ_n is the subgroup of $\mathrm{GL}_2(\mathbb{C}_p)$ of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, $d \in \mathcal{O}^*$ and $v(b) > \beta_n$, $v(c) > \gamma_n$. As $\beta_n + \gamma_n > 0$, ρ_n has a stable lattice. Moreover, the image of the limit representation ρ is the group of the above remark, which has no stable lattice.

1.5. **Simple versus uniform physical convergence.**

Proposition 1.10. *Assume that K is non-archimedean, G is a compact group, and the representations ρ_n are continuous, with a simple physical limit ρ. Then ρ_n converges uniformly to ρ in each of the following two cases:*

(i) *the group G is topologically finitely generated ;*
(ii) *the limit representation ρ is continuous.*

Proof. First suppose we are in case (i). Up to conjugation, we may assume $\rho(G) \subset \mathrm{GL}_d(\mathcal{O})$. Let $g_1, \ldots, g_k$ be a family of topological generators of G. For all n big enough, and all $i \in \{1, \ldots, k\}$, each $\rho_n(g_i)$ is in $\mathrm{GL}_d(\mathcal{O})$ and the sequence. Moreover for all integer N, and for all i, the sequence $\rho_n(g_i) \pmod{\varpi^N}$ is eventually constant. Choose an n_0 such that for all i, the $\rho_n(g_i) \pmod{\varpi^N}$'s are constant for $n \geq n_0$. Then it is clear that the sequences $\rho_n(g) \pmod{\varpi^N}$ are constant for all $n \geq n_0$, $g \in G$. That is, ρ_n is uniformly convergent.

Now suppose we are in case (ii). Failure of uniform convergence means there exists an open neighborhood of the identity U in $\mathrm{GL}_n(K)$ such that for all integers N there exists x_N in G and $i, j > N$ such that $\rho_i(x_N)\rho_j(x_N)^{-1} \notin U$. Fix an open neighborhood V of the identity so that $V^3 \subset U$. As the topology at 1 is generated by open subgroups, we may assume V a subgroup. Let x be the limit of a convergent subsequence of x_N in G. Pointwise convergence at x means that there exists M such that for all $i, j > M$, $\rho_i(x)\rho_j(x)^{-1} \in V$. Choose $N > M$ for which x_N belongs to our subsequence converging to x. Then there exist $i, j > N > M$ such that either $\rho_i(x_N)\rho_i(x)^{-1}$ or $\rho_j(x)\rho_j(x_N)^{-1} \notin V$. Either way, there exist $k > N$ such that $\rho_k(xx_N^{-1}) \notin V$. We can therefore extract a subsequence of the representations ρ and a subsequence of terms of the form $y_N x x_N^{-1}$ which violates the following lemma. □

Lemma 1.11. *Let G be a compact topological group, H any topological group, V an open subgroup of H, $\rho_i : G \to H$ a sequence of continuous* *homomorphisms converging pointwise to the continuous homomorphism* ρ *and* $y_i \in G$ *converging pointwise to the identity, then* $\rho_i(y_i) \in V$ *for some* i.

Proof. We iteratively construct a strictly monotone sequence $a_1, a_2, \dots$ of positive integers such that

1) $\rho(y_{a_i}) \in V$ for all i.
2) $\rho_{a_n}(y_{a_i}) \in V$ for all $i < n$.
3) $\rho_{a_i}(y_{a_n}) \in V$ for all $i < n$.

Note that as long as (1) holds for $i < n$, (2) holds for all sufficiently large a_n by pointwise convergence of the representation sequence; and (3) holds for all sufficiently large a_n by continuity of each ρ_i. Replacing ρ_i and y_i by subsequences, therefore, we can arrange that $\rho_i(y_j) \in V$ if and only if $i \neq j$. Now let $z_n = y_1 \cdots y_n$. This gives a sequence of points such that $\rho(z_n) \in V$ and $\rho_i(z_n) \notin V$ for all i. Let z be a limit point of this sequence. Then $\rho_i(z) \notin V$ for all i, and $\rho(z) \in V$, contrary to pointwise convergence. □

2. Component groups for algebraic envelopes

Throughout this section, $\pi_0(G)$ denotes the group G/G° of connected components of a linear algebraic group G.

2.1. Preliminary lemmas.

Lemma 2.1. *Let* $G \subset \mathrm{GL}_d$ *be a linear algebraic group defined over an algebraically closed field* K *of characteristic zero. Let* $g \in G(K)$ *be an element of* G *whose image in* $\pi_0(G)$ *has order* m. *Then the subgroup of* $K^\times$ *generated by the eigenvalues of* g *contains a primitive* m*th root of unity.*

Proof. Let $g = g_s g_u$ denote the Jordan decomposition of g, and let C (resp. C_s, C_u) denote the Zariski-closure of the cyclic group $\langle g \rangle$ (resp. $\langle g_s \rangle$, $\langle g_u \rangle$). By [Bor, 4.7], $C_s, C_u \subset C$, and the product map gives an isomorphism $C_s \times C_u \cong C$. The map $t \mapsto \exp(t \log(g_u))$ gives an isomorphism from the additive group $\mathbb{G}_a$ to C_u, so $C_u \subset C \subset G$ implies $C_u \subset G^\circ$. The same observations apply to powers of g. If C^k (resp. C_s^k) denotes the Zariski-closure of $\langle g^k \rangle$ (resp. $\langle g_s^k \rangle$), then we have equivalences

$$g_s^k \in G^\circ \Leftrightarrow C_s^k \subset G^\circ \Leftrightarrow C^k \subset G^\circ \Leftrightarrow g^k \in G^\circ \Leftrightarrow k \in m\mathbb{Z}.$$

There is a natural surjection from $\mathbb{Z}/m\mathbb{Z}$ to C_s/C_s^m sending 1 to the class represented by g_s. It is an isomorphism because $g_s^k \in C_s^m$ implies $k \in m\mathbb{Z}$. By [Bor, 8.4], C_s^m and C_s are diagonalizable groups. As K is of characteristic zero, the functor $C \mapsto X^*(C)$ is an equivalence of categories [Bor, 8.3], so the inclusion $C_s^m \to C_s$ corresponds to a surjection $X^*(C_s) \to X^*(C_s^m)$ with kernel cyclic of order m; let χ be an element of $X^*(C_s)$ lying in this kernel. Then $\chi^k(g_s) = 1$ if and only if $\chi^k(C_s) = 1$, and the latter condition is equivalent to $k \in m\mathbb{Z}$. Finally, the inclusion of C_s in $\mathbb{G}_m^d \subset \mathrm{GL}_d$ gives a surjective homomorphism $\mathbb{Z}^d \to X^*(C_s)$, so χ corresponds to a d-tuple of integers $(a_1, \dots, a_d)$. If g_s maps to the diagonal matrix with entries

$(\lambda_1, \ldots, \lambda_d)$, then $(\lambda_1^{a_1} \cdots \lambda_d^{a_d})^k = 1$ if and only if $k \in m\mathbb{Z}$, so $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$ is a primitive mth root of unity. □

Lemma 2.2. *There exists a function $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that if G is a closed subgroup of GL_d defined over an algebraically closed field K of characteristic 0, and every element of $\pi_0(G)$ has order $\leq m$, then $|\pi_0(G)| \leq f(m,d)$.*

Proof. By [Mos], there exists a Levi decomposition $G = MN$, where N is the unipotent radical of G and therefore $\pi_0(G) \cong \pi_0(M)$. Without loss of generality, therefore, we may assume G is reductive. Up to K-isomorphism there are only finitely many possibilities for G° given d, and each such G° admits only finitely many equivalence classes of d-dimensional representation. Conjugation by any element $g \in N_{\mathrm{GL}_d}(G^\circ)$ induces an automorphism of G° which is inner if and only if $g \in Z_{GL_d}(G^\circ)G^\circ$. It follows that the quotient $N_{\mathrm{GL}_d}(G^\circ)/Z_{GL_d}(G^\circ)G^\circ$ is contained in the outer automorphism group of the reductive Lie group G° and is therefore discrete. As it is a linear algebraic group, it is finite. We have a homomorphism from $\pi_0(G)$ to this quotient, so to bound the order of the former it suffices to bound the order of the kernel of the homomorphism, i.e.,

$$\begin{aligned}(G \cap Z_{GL_d}(G^\circ)G^\circ)/Z(G^\circ)G^\circ &\subset Z_{GL_d}(G^\circ)G^\circ/G^\circ \\ &= Z_{GL_d}(G^\circ)/Z_{GL_d}(G^\circ) \cap G^\circ \\ &= Z_{GL_d}(G^\circ)/Z(G^\circ).\end{aligned}$$

This latter group is determined by G° together with its ambient representation, for which there are only finitely many possibilities. For any fixed linear group $Z_{GL_d}(G^\circ)/Z(G^\circ)$ in characteristic 0, Jordan's theorem gives an upper bound to the order of a finite subgroup whose elements all have bounded order. □

Lemma 2.3. *Let $\lambda_1, \ldots, \lambda_d \in \mathbb{C}_p^\times$ be units and $F \subset \mathbb{C}_p$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$. Suppose that the elementary symmetric polynomials of $\lambda_1, \ldots, \lambda_d$ have values in $\mathcal{O}_F + p\mathcal{O}_p$. Then for every element λ of the multiplicative group $\langle \lambda_1, \ldots, \lambda_d \rangle$, there exists a monic polynomial $P_\lambda(x) \in \mathcal{O}_F[x]$ such that $P_\lambda(\lambda)$ is divisible by p and $\deg(P_\lambda) = d!$.*

Proof. We write λ as $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$ and define $Q_\lambda(x)$ to be the monic polynomial whose roots are $\lambda_{\sigma(1)}^{a_1} \cdots \lambda_{\sigma(n)}^{a_d}$, as σ ranges over S_d. The elementary symmetric polynomials in these roots lie in the ring generated by the elementary symmetric polynomials in $\lambda_1, \ldots, \lambda_d$ together with $(\lambda_1 \cdots \lambda_d)^{-1}$. By hypothesis, the elementary symmetric polynomials in $\lambda_1, \ldots, \lambda_d$ lie in the ring $\mathcal{O}_F + p\mathcal{O}_p$. The same is true of $(\lambda_1 \cdots \lambda_d)^{-1}$ since $\lambda_1 \cdots \lambda_d$ can be written $u_F + pe$, where u_F is a *unit* in $\mathcal{O}_F$. Thus $Q_\lambda(x)$ is monic with coefficients in $\mathcal{O}_F + p\mathcal{O}_p$. It follows that there exists a monic polynomial $P_\lambda(x)$ of the same degree with coefficients in $\mathcal{O}_F$ which is congruent to $P_\lambda(x)$ (mod p). Thus p divides $P_\lambda(\lambda)$. □

2.2. Variation in $\pi_0(G_n)$ for a convergent sequence of representations.

Theorem 2.4. *Let Γ be a compact group, and let $\rho_n\colon \Gamma \to \mathrm{GL}_d(\mathbb{C}_p)$ denote a uniformly trace-convergent sequence of continuous representations. Let G_n denote the Zariski closure of $\rho_n(\Gamma)$. Then $|\pi_0(G_n)|$ is bounded.*

Proof. We know that the representations ρ_n uniformly trace converge to a continuous representation $\rho : \Gamma \to GL_d(\mathbb{C}_p)$ by using the theory of pseudo-representations as in Section 1. Further by [KLR, Lemma 2.2] all the representations ρ_n and ρ may be assumed to be valued in $\mathrm{GL}_d(\mathcal{O}_p)$. In its reduction (mod p) under ρ, Γ has finite image, so that all its entries lie in $\mathcal{O}_F/p$ for some finite extension $F/\mathbb{Q}_p$. For large enough n, the mod p characteristic polynomials of ρ_n and ρ_n agree. Thus to prove the theorem we may assume without any loss of generality that they do so for all n. If $\rho_n(g)$ lies in $G_n(\mathbb{C}_p) \setminus G_n^\circ(\mathbb{C}_p)$, by Lemma 2.1, some non-trivial root of unity ζ lies in the group generated by the eigenvalues of $\rho_n(g)$. We claim that there exists an upper bound on the order of ζ depending only on n and F. If ζ has order $p^k m$, ζ^{p^k} and ζ^m both lie in the group generated by eigenvalues of $\rho_n(g)$. It suffices, therefore, to prove that the order of ζ is bounded in the case that this order is prime to p and in the case that it is a power of p.

By Lemma 2.3, if the order of ζ is prime to p, the reduction of ζ modulo the maximal ideal of $\mathcal{O}_p$ must satisfy a polynomial equation of degree less than or equal to $d!$ over the residue field of $\mathcal{O}_F$. This gives a bound on the order. If ζ is of prime power order, p^k, then the valuation of $\lambda := 1 - \zeta$ is $\frac{1}{p^k - p^{k-1}}$ times the valuation of p. If the ramification degree of F over $\mathbb{Q}_p$ is e, then $1, \lambda, \lambda^2, \ldots, \lambda^m$ are linearly independent over $\mathcal{O}_F/p\mathcal{O}_F$ as long as $em < p^k - p^{k-1}$. Thus, if $p^k - p^{k-1} > ed!$, ζ cannot satisfy a monic degree $d!$ polynomial equation (mod p) with coefficients in $\mathcal{O}_F$.

By Lemma 2.1, there exists m such that for all $n \gg 0$, every element of $\pi_0(G_n)$ has order less than m. Thus Lemma 2.2 gives an upper bound of $f(m, d)$ on $|\pi_0(G_n)|$ for $n \gg 0$ which proves the theorem. ☐

In general, as the examples in section 2.3 illustrate, there is little that can be said about the relation between the $\pi_0(G_n)$ and $\pi_0(G)$. In the irreducible case, however, we have the following theorem which makes crucial use of Theorem 2.4:

Theorem 2.5. *Let Γ be a compact group, and let $\rho_n\colon \Gamma \to \mathrm{GL}_d(\mathbb{C}_p)$ denote a uniformly trace-convergent sequence of continuous representations. Then we know that there is a continuous semisimple representation $\rho\colon \Gamma \to \mathrm{GL}_d(\mathbb{C}_p)$ such that the representations ρ_n uniformly trace converge to ρ. Let G_n (resp. G) denote the Zariski closure of $\rho_n(\Gamma)$ (resp. $\rho(\Gamma)$), regarded as a subgroup of GL_d. Suppose that ρ is irreducible. Then for all $n \gg 0$ there exists a surjective homomorphism $\pi_0(G) \to \pi_0(G_n)$.*

Proof. By Theorem 1.2 we know that ρ_n is irreducible for large enough n, and thus in proving the theorem we may assume without any loss of generality that ρ_n are irreducible for all n. As ρ_n and ρ are irreducible, the identity components G_n° and G° are reductive. (By results of Section 1 we also know that (ρ_n) uniformly physically converges to ρ, although we will not need this in the proof.) Let $V = \mathbb{C}_p^d$, regarded as a representation space of G_n. (In what follows we several times use the hypothesis of irreducibility of the limit ρ without explicit mention, and examples of section 2.3 show why this hypothesis is necessary.)

We would like to prove that there exists a finite set $S \subset \mathbb{Z}^d$ and an integer N (independent of n) such that for all $g \in G_n(\mathbb{C}_p) \setminus G_n^\circ(\mathbb{C}_p)$ with eigenvalues $\lambda_1, \dots, \lambda_d$, there exists $(a_1, \dots, a_d) \in S$ such that $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$ is a non-trivial root of unity of order less than or equal to N. The dimension data, consisting of the number of distinct irreducible representations, their dimension, and their multiplicity, arising from the decomposition of V as a G_n° representation (Clifford theory) admits only finitely many possibilities as n varies. Thus by partitioning the given sequence into finitely many subsequences we may assume that the dimension data is independent of n.

We consider two cases according to whether or not g preserves every summand in the decomposition of V as G_n° representation. If not, g induces a non-trivial permutation action on the isotypic components $V_1^e, \dots V_k^e$ of $V|_{G_n^\circ}$. If a linear transformation T cyclically permutes r independent subspaces of order n, then T and $\zeta_r T$ are conjugate, for ζ_r a primitive rth root of unity, and therefore the eigenvalues of T form a homomogeneous space under the action of the group of rth roots of unity. Thus we can take S to consist of all vectors in $\mathbb{Z}^d$ obtained by permuting the coordinates of $(1, -1, 0, \dots, 0)$ and $N = k$.

If g preserves each V_i^e, then without loss of generality, we may assume its image in $\mathrm{GL}(V_1^e)$ does not lie in the image H_n of $G_n^\circ(\mathbb{C}_p) \to \mathrm{GL}(V_1^e)$. Let D_n denote the derived group of H_n, so D_n is connected and semisimple, and $H_n = D_n Z_n$, where Z_n is either $\{1\}$ or the group $\mathbb{G}_m$ of scalar matrices in $\mathrm{GL}(V_1^e)$. By classification, there are only finitely many isomorphism classes of semisimple groups of dimension less than d^2 over $\mathbb{C}_p$ and finitely many equivalence classes of representations of dimension less than or equal to d for each; so up to conjugation in $\mathrm{GL}(V_1^e)$ there are finitely many possibilities for D_n and therefore for H_n. Without loss of generality, therefore, we may pass to an infinite subsequence of (ρ_n) such that the vector spaces V_1^e are all isomorphic, the H_n mutually isomorphic, and the representations of H_n on V_1^e equivalent. By a well-known theorem [DMOS], there exist non-negative integers m_1 and m_2 (independent of n) such that H_n is the pointwise stabilizer of

$$W_n := ((V_1^e)^{\otimes m_1} \otimes (V_1^e)^{*\otimes m_2})^{G_n^\circ}$$

in GL_d. As the image $\bar{g}$ of g in $\mathrm{GL}(V_1^e)$ normalizes H_n, it stabilizes W_n, and acts non-trivially on it, but its $|\pi_0(G_n)|$th power must act trivially. But now as $|\pi_0(G_n)|$ is bounded independently of n by Theorem 2.4, it follows that $\bar{g}$ has an eigenvector with eigenvalue which is a non-trivial root of unity of some bounded order N (with N independent of g). Furthermore this eigenvalue can be written $\lambda_1^{a_1}\cdots\lambda_d^{a_d}$, with $-m_2 \le a_1,\dots,a_d \le m_1$ and λ_i the eigenvalues of $\bar{g}$.

Let $\Gamma^\circ = \rho^{-1}G^\circ(\mathbb{C}_p)$, and let $X \subset G^\circ$ denote the closed subset consisting of elements with eigenvalues $\lambda_1,\dots,\lambda_d$ such that $\lambda_1^{a_1}\cdots\lambda_d^{a_d}$ is a non-trivial root of unity of order $\le N$ for some $(a_1,\cdots,a_d) \in S$. By Proposition 3.5 below, which again uses crucially Theorem 2.4, the Zariski closure of $\rho_n(\Gamma^\circ)$ is connected. We can therefore define a surjective homomorphism from $\pi_0(G)$ to $\pi_0(G_n)$ by lifting to Γ, mapping by ρ_n to $G_n(\mathbb{C}_p)$, and projecting onto $\pi_0(G_n)$. □

2.3. **Examples.** The examples in this section are intended to give some perspective on Theorems 2.4 and 2.5.

Let p be an odd prime. Let $e\colon \mathbb{Z}_p \to \mathbb{C}_p$ denote the exponential map $\exp(pz)$ given by the (convergent) power series

$$e(z) := \sum_{i=0}^{\infty} \frac{p^i}{i!} z^i.$$

Let $a_1, a_2, \dots$ denote a sequence in $\mathbb{Z}$ which converges p-adically to an irrational element $a \in \mathbb{Z}_p \setminus \mathbb{Q}$. Let $\Gamma = \mathbb{Z}_p$ and

$$\rho_n(z) = \begin{pmatrix} e(z) & 0 \\ 0 & e(a_n z) \end{pmatrix}$$

The Zariski closure of $\rho_n(z)$ is $\mathbb{G}_m$ embedded in GL_2 as

$$\begin{pmatrix} t & 0 \\ 0 & t^{a_n} \end{pmatrix}.$$

The limit representation ρ is given by

$$\rho(z) = \begin{pmatrix} e(z) & 0 \\ 0 & e(az) \end{pmatrix}$$

whose envelope is $\mathbb{G}_m^2$, the group of all invertible diagonal matrices, because $a \notin \mathbb{Q}$.

In this case, the envelopes G_n and G are all connected, but because the dimension jumps for the limit representation, we can easily modify the example either to prevent $|\pi_0(G_n)|$ from converging at all as $n \to \infty$ or to allow convergence to a value different from $|\pi_0(G)|$. For instance, we may set $\Gamma = \mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}$ and define

$$\rho_n(z,k) = (-1)^k \begin{pmatrix} e(z) & 0 \\ 0 & e(a_n z) \end{pmatrix},\ \rho(z,k) = (-1)^k \begin{pmatrix} e(z) & 0 \\ 0 & e(az) \end{pmatrix}.$$

Then G_n has 1 or 2 components depending on whether a_n is odd or even. Since $p > 2$, the parity of a p-adically convergent sequence of integers need not stabilize. If all the a_n are even, then $|\pi_0(G_n)| = 2$ for all n, but $|\pi_0(G)| = |\pi_0(\mathbb{G}_m^2)| = 1$.

We also remark that the isomorphism class of G_n° need not stabilize as $n \to \infty$, and even if it does stabilize, it need not coincide with that of G°. For example, if $\Gamma = \mathbb{Z}_p^2$, and a_n is a sequence of p-adic integers converging to 0, we can set

$$\rho_n(z_1, z_2) = \begin{pmatrix} e(z_1) & 0 \\ 0 & e(a_n z_2) \end{pmatrix}, \ \rho_n(z_1, z_2) = \begin{pmatrix} e(z_1) & 0 \\ 0 & 1 \end{pmatrix}.$$

In this example, G_n is isomorphic to $\mathbb{G}_m$ whenever $a_n \neq 0$ and otherwise to $\mathbb{G}_m^2$, and of course G is isomorphic to $\mathbb{G}_n$.

Finally, it may even happen that G_n is reductive for infinitely many values of n and unipotent for infinitely many values. For example, let $\Gamma = \mathbb{Z}_p$ and a_n be a sequence of p-adic integers converging to 0. Let

$$\rho(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

and let

$$\rho_n(z) = \begin{cases} \begin{pmatrix} 1 & \frac{e(a_n z)-1}{e(a_n)-1} \\ 0 & e(a_n z) \end{pmatrix} & \text{if } a_n \neq 0, \\ \rho(z) & \text{if } a_n = 0. \end{cases}$$

Thus G_n is isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$ depending on whether a_n is or is not equal to zero, and G is isomorphic to $\mathbb{G}_a$.

3. Density theorems for converging sequences

In this section we consider only continuous Galois representations to $\mathrm{GL}_n(\mathbb{C}_p)$. We fix the following situation and notation for all of this section.

Let X be a subvariety of GL_d defined by a finite set $\{f_1, \ldots, f_t\}$ of the coordinate ring A of GL_d that we assume can be chosen so that each f_i is conjugation invariant. We call such a X a characteristic subvariety. Consider a compact subgroup Γ of $\mathrm{GL}_d(\mathbb{C}_p)$, that by [KLR, Lemma 2.2] we can assume to be in $\mathrm{GL}_d(\mathcal{O}_p)$ (by conjugating), and consider a Haar measure μ on Γ. By saying that a point γ of Γ *lands inside X mod p^m* (or Γ is in X mod p^m, or is in a *tubular neighborhood of X of radius p^{-m}*), we will mean that $v(f_i(\gamma)) > m$ for $1 \leq i \leq t$ (γ lands inside X means this should hold for all m!). Here v is the valuation of $\mathbb{C}_p$ normalised so that $v(p) = 1$. We say that X is *thin* with respect to Γ if for every finite subset $\{g_1, \ldots, g_m\}$ of the coordinate ring A of GL_d such that $V(g_1, \ldots, g_m) \cap G = X$, we have

$$\lim_{\alpha \to \infty} \mu(\{\gamma \in \Gamma \mid \forall i \ v(g_i(\gamma)) > \alpha\}) = 0.$$

We recall [KLR, Proposition 2.3], slightly reformulated, as we will repeatedly use it below:

Proposition 3.1. *Let Γ denote a compact subgroup of $\mathrm{GL}_d(\mathbb{C}_p)$, μ Haar measure on Γ, G the Zariski closure of Γ in GL_d, and X a subvariety of*

GL_d *that intersects all components of* G *with positive codimension. Then* X *is thin with respect to* Γ*.*

Let F be a number field and write G_F for its absolute Galois group. We consider uniformly trace-convergent sequence of continuous representations $\rho_n\colon G_F \to \mathrm{GL}_d(\mathbb{C}_p)$ which by the theory of pseudo-representations (see section 1.1) is uniformly trace-convergent to a continuous, semisimple representation $\rho\colon G_F \to \mathrm{GL}_d(\mathbb{C}_p)$. (This also implies that in fact the characteristic polynomials of the $\rho_n(g)$ converge to those of $\rho(g)$ uniformly in g.)

3.1. **Frobenius polynomials at almost all ramified primes.** By [KLR, Lemma 2.2] we may assume that all the representations ρ_n, and ρ itself, are valued in $\mathrm{GL}_d(\mathcal{O}_p)$.

Proposition 3.2. *There exists a finite set* S *of places of* F *such that for all primes outside* S*, the ramification of all* ρ_n *is tame and unipotent (or trivial).*

Proof. We first exclude from discussion the finite set of places of F of residue characteristic p.

¿From the assumptions on ρ_n it follows that the number of residual mod p representations that arise from reducing any integral model of ρ_n modulo the maximal ideal of $\mathcal{O}_p$ and semisimplifying is finite (for all i at once). If ρ_n is wildly ramified at any remaining place q of F, then $\bar{\rho}_n$ and indeed $\bar{\rho}_n^{ss}$ is already wildly ramified at q. Thus we see that the number of places of F whose ramification in any of the ρ_n is wild is finite. We exclude this set from discussion.

Let q be a place of F that has not been excluded. Then the image of a decomposition group D_q at q under any ρ_n factors through its tame quotient, which is topologically generated by σ_q and τ_q with the relation

$$\sigma_q \tau_q \sigma_q^{-1} = \tau_q^{\|q\|}. \tag{2}$$

Here σ_q induces the qth power map on residue fields and τ_q is a (non-canonical) generator of tame inertia. From this it follows that the eigenvalues of $\rho_n(\tau_q)$ for any n are roots of unity. It is easy to see that for some $m \gg 0$ that depends only on d, if ζ is a root of unity such that $(\zeta-1)^d$ is 0 mod p^m, then $\zeta = 1$. Thus we see that if $\rho_n(\tau_q)$ has the same characteristic polynomial mod p^m as that of the identity, then $\rho_n(\tau_q)$ is unipotent. We fix N such that the characteristic polynomials of $\rho_i(g)$ and $\rho_j(g)$ are congruent to one another mod p^m for all $i, j \geq N$ and $g \in G_F$. Taking $g = \tau_q$, we see that if for some i ρ_i fails to have unipotent ramification at q, then for some $j \leq N$, $\rho_j(\tau_q)$ is not congruent to the identity (mod p^m). As the mod p^m reductions of the representations ρ_i have finite images, the set of possible places q is finite.

□

The utility of Proposition 3.2 is that given uniformly trace-converging ρ_n, for a place q outside the finite set of places excluded in its statement, one can define the *characteristic polynomial of* $\rho_n(Frob_q)$ as that of $\rho_n(\sigma_q)$ for any σ_q that lifts the qth power map on residue fields. Using unipotence of $\rho_n(\tau_q)$ and the tame inertia relation above we see that this is independent of choice of σ_q (proof: from this relation we see that σ_q preserves the kernel of $(\tau_q - 1)^i$ for any i, and thus as $\rho(\tau_q)$ is unipotent it follows easily that $\rho(D_q)$ can be conjugated into upper triangular matrices over an algebraic closure with τ_q mapped to strictly upper triangular element.) Thus given a characteristic subvariety X of GL_d, we can with some abuse talk of $\rho_n(\mathrm{Frob}_q)$ landing in X, as this condition will depend only on the characteristic polynomial of $\rho_n(\mathrm{Frob}_q)$. In fact, one can prove slightly more: the conjugacy class of every element in the Frobenius coset is the same:

Corollary 3.3. *Let $I_q \subset D_q$ denotes the inertia group at q. For $q \notin S$, $\rho_n(\sigma_q I_q)$ lies in a single $\mathrm{GL}_d(\mathbb{C}_p)$-conjugacy class for all n.*

Proof. As we are working in characteristic zero, the log and exp maps give mutually inverse bijections between the variety of unipotent elements in GL_d and the variety of nilpotent elements in M_d. For fixed n and a fixed choice of τ_q, let $N_\tau = \log \rho_n(\tau_q)$. Then (2) implies that

$$\rho_n(\tau_q)^{\|q\|}\rho_n(\sigma_q)\rho_n(\tau_q)^{-\|q\|} = \rho_n(\sigma_q)\rho_n(\tau_q)^{\|q\|-1},$$

and therefore, for all $t \in \mathbb{C}_p$,

$$\rho_n(\tau_q)^{\|q\|}\rho_n(\sigma_q)\exp(tN_\tau)\rho_n(\tau_q)^{-\|q\|} = \rho_n(\sigma_q)\exp((t + \|q\| - 1)N_\tau),$$

If $O \cong \mathbb{G}_a$ denotes the Zariski-closure of

$$\{\rho_n(\sigma_q)\exp(tN_\tau) \mid t \in \mathbb{C}_p\},$$

this implies that conjugation by $\rho_n(\tau_q)^{\|q\|}$ acts on O without points of finite order. Any orbit of this action is Zariski-dense in O, and it follows that every GL_d conjugacy class in O is Zariski-dense. As O is connected, this implies that there is a single orbit. As ρ_n is tamely ramified at q and τ_q is a topological generator of the tame inertia group, $\rho_n(\sigma_q I_q) \subset O(\mathbb{C}_p)$. □

Corollary 3.4. *With notations as above:*

- $d > 1$*: For any n, and $q \notin S$, if ρ_n is ramified at q, then $\rho_n(\sigma_q)$ has two eigenvalues with ratio $\|q\|$.*
- $d = 1$*: The union of the ramifying sets for all ρ_n is finite.*

Proof. See [KLR, Lemma 2.6]. □

In the rest of the paper we will implicitly exclude from the discussion the finite set S in Proposition 3.2.

3.2. **Density theorems.** As in section 2.2, G_n and G denote the Zariski-closures of ρ_n and ρ respectively. The following proposition is key to proving the density theorems below.

Proposition 3.5. *Let X, a characteristic subvariety of GL_d, intersect all the components of G with positive codimension. Then for $n \gg 0$, X intersects all the components of G_n with positive codimension.*

Proof. Assume the contrary. Let Γ be the image of the limiting representation ρ. We get a contradiction by proving the following claim: if Γ_m is the (finite) reduction of Γ mod p^m, then there exists $\alpha > 0$ independent of m such that at least $\alpha|\Gamma_m|$ elements of Γ_m land inside X mod p^m.

This will contradict the hypothesis that X has proper intersection with all components of G as, under the assumptions Proposition 3.1 proves that X is *thin* with respect to Γ.

We prove the claim by observing that: (i) by assumption for infinitely many n, X contains a connected component of G_n, (ii) for a given tubular neighborhood U of the *characteristic* subvariety X the image of $g \in G_F$ under ρ is in U if and only if $\rho_n(g) \in U$ for $n \gg 0$, (iii) the number of connected components of G_n is bounded by some number r by Theorem 2.4.

¿From these three facts we see that we can in fact take α to be $\frac{1}{r}$. □

Theorem 3.6. *Let X be a characteristic subvariety of GL_d such that X intersects all the components of G with positive codimension. Then there is a $N \gg 0$, such that the set of places q of F such that $\rho_n(Frob_q) \in X$ for even one $n > N$ is of Dirichlet density zero.*

Proof. Choose N such that X does not contain any component of G_n for $n > N$ using Proposition 3.5 above. Let Γ be the image of ρ and μ a Haar measure on it. We want to show that the upper density of primes q such that $\rho_n(\mathrm{Frob}_q)$, for even one $n > N$, lands in X can be made $< \epsilon$ for any given $\epsilon > 0$. As before we claim this follows easily using Proposition 3.1, which proves that X is thin with respect to Γ. Namely, using loc. cit. and the classical Cebotarev density theorem for finite Galois extensions of number fields (the reader may also look at [KLR, Th. 2.4] for similar concluions, and proof of [KLR, Th. 2.5] for a similar argument), we get that for $m \gg 0$, the upper density of q such that $\rho(\mathrm{Frob}_q)$ lands in a tubular neighborhood U_m of X of radius p^{-m} is $< \epsilon$. Now as (ρ_n) uniformly trace-converges to ρ, and as X is characteristic, we see that there is a $N' \gg 0$ such that $\rho_n(g) \in U_m$ for $n > N'$ if and only if $\rho(g) \in U_m$. Further the density of q such that $\rho_n(\mathrm{Frob}_q)$ lands in X for any $N < n < N'$ is of density 0 as X intersects all components of G_n with positive codimension and then we can again use Proposition 3.1 and the classical Cebotarev density therorem. Thus treating the cases $N < n < N'$ and $n > N'$ separately we see that the upper density of primes q such that $\rho_n(\mathrm{Frob}_q)$, for even one $i > N$, lands in X can be made $< \epsilon$. □

Remark: In the case when all the representations ρ_n are unramified outside a fixed finite set of places, we do not know (even assuming that the representations are $\mathrm{GL}_d(\mathbb{Q}_p)$ valued) if there are quantitative refinements of the theorem above like the ones for a single representation proved in Théorème 10 of [S2].

We now prove a result about density of primes that ramify in uniformly trace-converging sequences, which has a precursor in [Kh], and is close to the proof of [KLR, Theorem 2.5] which proves the statement for a single representation (when the representation is valued in a $\mathrm{GL}_d(K)$, with K a finite extension of $\mathbb{Q}_p$, the result for a single representation goes back to [Kh-Raj]).

Theorem 3.7. *If $\rho_n \colon G_F \to \mathrm{GL}_d(\mathbb{C}_p)$ is a sequence of irreducible representation which trace-converges uniformly to an irreducible $\rho \colon G_F \to \mathrm{GL}_d(\mathbb{C}_p)$, then the union over n of the sets of primes ramified in ρ_n has Dirichlet density zero.*

Proof. We can exclude the case of $d = 1$ by Corollary 3.4. Let ε denote the p-adic cyclotomic character. Consider the direct sum representations $\rho_n \oplus \varepsilon \colon G_F \to \mathrm{GL}_d \times \mathrm{GL}_1$ and $\rho \oplus \varepsilon \colon G_F \to \mathrm{GL}_d \times \mathrm{GL}_1$. These again trace-converge uniformly, and it suffices to prove the theorem for them instead. The proof would follow from Theorem 3.6, Proposition 3.2 and Corollary 3.4, if we knew the following:

Claim: Let H denote the Zariski closure of $\rho\oplus\varepsilon(G_F)$. Thus $H \subset G\times\mathrm{GL}_1$, with G the Zariski closure of ρ, and H projects onto each factor. Let $X \subset H$ denote the subvariety of pairs $(g, c) \in H$ such that g and gc have at least one eigenvalue in common. The claim is that X is of codimension greater than one in each component of H.

The X of the claim is characteristic in $\mathrm{GL}_d \times \mathrm{GL}_1$ and hence Theorem 3.6 applies to it. Thus we are done once the claim is proved.

To check the claim as ρ is irreducible and thus centralised only by scalars, we can go modulo the centre of G, as this does not change anything and thus assume that G is semisimple. By Goursat's lemma, H is the pullback of the graph of an isomorphism between a quotient of G and a quotient of GL_1. Every quotient of GL_1 is a torus and G admits no non-trivial toric quotient, so $H = G \times \mathrm{GL}_1$. For each g there are only finitely many possible values of c such that g and gc have an eigenvalue in common, so X is of codimension ≥ 1 in each component of H.

□

We also have the following result about equidistribution of Frobenius elements in groups of connected components for converging sequences that is a simple consequence of Theorem 2.5 and the classical Cebotarev density theorem.

Proposition 3.8. *Assume that ρ is irreducible. Let $\rho_n^\circ \colon G_F \to \pi_0(G_n)$ (resp. $\rho^\circ \colon G_F \to \pi_1(G)$) denote the homomorphisms obtained by composing*

$\rho_n\colon G_F \to G_n(\mathbb{C}_p)$ (resp. $\rho\colon G_F \to G(\mathbb{C}_p)$) with the quotient map by $G_n^\circ(\mathbb{C}_p)$ (resp. $G^\circ(\mathbb{C}_p)$), and let K_n (resp. K) be the fixed fields of their kernels. For $n \gg 0$, each K_n is contained in K. For a conjugacy class C of $\mathrm{Gal}(K/F)$ we denote by C_n its image in $\mathrm{Gal}(K_n/F)$ ($n \gg 0$). The density of places whose Frobenius under ρ_n° lie in C_n for all $n \gg 0$ and in C under ρ°, is $|C|/|\mathrm{Gal}(K/F)|$.

Proof. By considering the restriction $\rho|_{G_K}$ and using Theorem 2.5 we see that all the Zariski closures of the images of $\rho_n|_{G_K}$ for $n \gg 0$ are connected, and thus for $n \gg 0$, the fields K_n are contained in K. The rest follows from a direct application of the classical Cebotarev density theorem. □

Remarks:

1. Because of results of section 1, Proposition 3.5 and Theorem 3.6 also work when one simply assumes that X is a conjugation invariant subvariety and the limit ρ is multiplicity-free as then by Theorem 1.4 the sequence ρ_n uniformly physically converges.

2. Even if a characteristic X intersect all components of G_n with positive codimension, it might still happen that the density of Frobenius elements that land in X under ρ_n for any $n \gg 0$ is 1. Of course such a putative X will contain some component of G because of the results of this section.

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