Unimodular hunting

February 1^{st} 2021

Gaëtan Chenevier

CNRS, ORSAY

"Moduli spaces and modular forms"
Oberwolfach

gaetan . chenevier @ math . cnrs . fr

I) Introduction

euclidean

$$\mathcal{L}_n = \{ L \subset \mathbb{R}^n , \ L \text{ unimodular integral lattice} \}$$

$O(\mathbb{R}^n)$

$X_n = O(\mathbb{R}^n) \backslash \mathcal{L}_n$ isom. classes of i.u.l

<u>known</u> . X_n finite, known mass $\sum_{[L] \in X_n} 1/|O(L)|$

. representatives for $n \leq 25$

n	≤ 7	...	22	23	24	25
$\# X_n$	1	...	68	117	297	665

(Borcherds, Conway-Sloane, Niemeier, Kneser, Witt...)

<u>Theorem</u> A: $\# X_{26} = 2566$, $\# X_{27} = 17059$, $\# X_{28} = 374062$

+ representatives in each cases. (with Bill Allombert)

<u>Remarks</u>

- lists on my homepage
- use computer $\simeq$ 1m. , 1y. and 72y! use (P.S.)
- a posteriori check : check mass formula and that all lattices we give have $\neq$ configurations of vectors v with $v \cdot v \leq 3$. <u>Time</u>: 5h, 40h, 27d.
- some lattices known to Bacher & Venkov

II Method: Kneser neighbours

Recall for $L \in \mathcal{L}_n$, $d \geq 1$, a d-neighbour of L is an $N \in \mathcal{L}_n$ s.t. $L/_{L \cap N} \simeq \mathbb{Z}/d$ ← cyclic

Construction: $C_L(\mathbb{Z}/d) := \{ \underset{\mathbb{Z}/d}{\underset{\cup}{\ell}} \subset L \otimes \mathbb{Z}/d \mid \ell \cdot \ell \equiv 0 \ (ed) \}$

with $e = 1$ (d odd) or 2 (d even)

for $\ell \in C_L(\mathbb{Z}/d)$ set $M_d(\ell) = \{ v \in L \mid v \cdot \ell \equiv 0 \ (d) \}$

choose $w \in L$ gen. ℓ & $w \cdot w \equiv 0 \ (d^2)$
then $N_d(\ell) := M_d(\ell) + \mathbb{Z}\, w/d$ is a d-neighb. of L with $N_d(\ell) \cap L = M_d(\ell)$.

<u>Fact</u> $N_d(\ell)$ does not depend on w for d odd, defines 2 $\neq$ lattices for d even, exhaust d-neighb. of L.

$\rightsquigarrow$ <u>Neighbouring method</u> (Kneser, strong approximation)
algo. to find all isom. classes in a genus of lattices.

Not only very efficient, but give beautiful and "compact" lattice constructions

Examples with $L = I_n = \mathbb{Z}^n$

$N_2(1^n)$ for $n \equiv 0 \ (4)$; $E_8 \simeq N_2(1^8)$

$N_{94}(1, 3, 5, 7, \ldots, 47) \simeq$ Leech (Thompson)

$2 \cdot 94 = 4 \cdot 47$

$1^2 + 2^2 + 3^2 + \cdots + 23^2 \equiv 0 \ (47)$

III Statistics of p-neighbors

More generally, fix $\mathcal{G}$ any spinor genus of integral lattices in $\mathbb{R}^n$.

Assume $n > 2$. E.g. $\mathcal{G} = \mathcal{L}_n^{\text{even}}$ or $\mathcal{L}_n^{\text{odd}}$.

Hsia–Jöchner ('97): Fix $L, L' \in \mathcal{G}$, then for all $p \gg 0$

L has a p-neighbor isom. to L'.

Theorem B (Ch.) $\forall L, L' \in \mathcal{G}$

$$\frac{\#\left\{\begin{array}{l} p\text{-neigh. } N \text{ of } L \\ \text{with } N \simeq L' \end{array}\right\}}{\# C_L(\mathbb{Z}/p)} = \frac{1/|O(L')|}{\text{mass } \mathcal{G}} + O\left(\frac{1}{\sqrt{p}}\right)$$

$p^{n-2} \simeq \# C_L(\mathbb{Z}/p)$

$p \to \infty$

- <u>Meaning</u>: probability to find $[L']$ as p-neighb. of L is proportionnal to $1/|O(L')|$ (mass of L')

$\longrightarrow$ lattices with small mass are harder to find / construct.

- <u>Pf</u>: use quite deep results from theory of automorphic representations (Arthur's theorem, Jacquet–Shalika estimates toward Ramanujan conj.). We give several variants ... we can replace $1/\sqrt{p}$ by $1/p$ for $n \geq 5$.

IV Proof of theorem A

Several ingredients.

① Induction n, may assume $R_1(L) = \{v \mid v.v=1\} = \emptyset$

② split the classification "root system by root system"

for $L \in \mathcal{L}_n$, $R_2(L) = \{ v \in L \mid v.v = 2 \}$ (ADE r.s.)

(King) $\underset{ADE}{\forall R}$, $n \leq 30$, know mass $\left\{ L \in \mathcal{L}_n \;\middle|\; \begin{array}{l} R_1(L) = \emptyset \\ R_2(L) \simeq R \end{array} \right\}$

For $n \le 23$ unique lattice at most with given root system

n	24	25	26	27	28
$\# R$ nonzero mass	149	327	1086	2797	4722
	156	368	1901	14493	357003

③ choose certain invariant of lattices and <u>bet</u> that they will be enough (only use conf. of vectors of norm ≤ 3)

$\to$ ④ Enumerate all d-neighbors of I_n, $d = 2, 3, \dots$

<u>Key fact</u>: $O(I_n) = \{\pm 1\}^n \rtimes S_n$ big, enoug to restrict to $(x_i) \in \mathbb{Z}^n$ with $1 \le x_1 \le x_2 \le \dots \le x_n \le d/2$

few such vectors for small d $\left(\binom{d/2+n-1}{n} \simeq \frac{d^n}{2^n n!} \right)$

compute R_1, R_2, invariants, mass .... find many lattices!

but not all $\longrightarrow$ need other arguments for small mass / high d

⑤ the "visible root system":

$$R_2(N_d(x)) \supset R_2(M_d(x)) \subset R_2(I_n)$$

but $R_2(I_n) = \{\pm \varepsilon_i \pm \varepsilon_j\}_{i \neq j}$ and $\varepsilon_i \pm \varepsilon_j \in M_d(x) \Leftrightarrow x_i \equiv \mp x_j \ (d)$

so $R_2(M_d(x)) \simeq A_{a_1-1} A_{a_2-1} \ldots A_{a_s-1} D_x$ concrete!

Ex: if we want $R_2(N_d(x)) = \emptyset$ (e.g. $N_d(x) \simeq$ leech)

must have $R_2(M_d(x)) = \emptyset$ i.e. $x_i \neq x_j \ \forall i \neq j$

$\rightsquigarrow$ leads to discover thompson's obs.

Key idea: biaise statistics by imposing visible root system.

(reach much higher d this way)

⚠ visible root system can't be equal to $R_2(N_d(x))$ in general, e.g. it is always saturated in $N_d(x)$

for d prime $\gg 0$

V Example $n = 26$, root system $10A_1$

reduced mass $= 4\,424\,507 / 58\,060\,800$

$(= 2^{11} \times \text{mass})$

prescribed visible root system: $8A_1$ (restrict to $x = (x_i)$ with 8 pairs of x_i equal, 10 other coordinates $\neq$ $\Rightarrow d \geq 36$)

d	matching / #iso lines	new lattices	reduced masses
36	98 / 216	4	1/32, 1/48, 1/48, 1/320
37 & 38	1243 / 2852	0	
39	820 / 1821	1	1/6144
40 → 49	several millions		0

reduced mass remaining = 17 / 58 060 800

→ probability to find new lattice $< 3 \cdot 10^{-6}$ (certainly less)

need billions of lines! too much, need other ideas.

Here, consider

$L = N_{35}(1\ 1\ 2\ 3\ 4\ 4\ 5\ 6\ 7\ 7\ 8\ 9\ 9\ 10\ 11\ 12\ 12\ 13\ 14\ 15\ 15\ 16\ 17\ 17)$

(one of the "last" lattices in X_{24}, root. sys. $8A_1$, visible $7A_1$)

compute 2-neighbors of $L \oplus I_2$ with visible root. system $8A_1$

$d = 70$, $3632 / 16384$ (only!), 2 new lattices, $1/3686400$ & $1/46448640$

→ 7 lattices QED.

THANK YOU!