# Unimodular hunting

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## I. Introduction

$n \geq 1$ $\qquad \mathcal{L}_n = \{ \underset{\text{latt.}}{L \subset \mathbb{R}^n}, \ L \text{ integral \& unimodular} \}$

<u>Ex:</u> $I_n = \mathbb{Z}^n$, $E_8$ $(n=8)$, Leech $(n=24)$ ...

<u>Main pb:</u> understand $\quad X_n = O(\mathbb{R}^n) \backslash \mathcal{L}_n$ (isom. classes)

<u>known</u> $X_n$ finite, $\quad \text{mass}(\mathcal{L}_n) = \sum_{[L] \in X_n} 1/|O(L)|$ ("mass formula")

Representatives of $X_n$ known up to $n=25$ by works of

Lagrange, Gauss, Mordell, Witt, Kneser, Niemeier, Conway & Sloane, Borcherds

| $n$ | $\leq 7$ | $\leq 11$ | $\leq 13$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\# X_n$ | 1 | 2 | 3 | 4 | 5 | 8 | 9 | 13 | 16 | 18 | 40 | 68 | 117 | 297 | 665 |

**Theorem A** $\# X_{26} = 2566$, $\# X_{27} = 17059$, $\# X_{28} = 374\,062$

(with Bill Allombert)

with concrete representatives in each case

$\longrightarrow$ see my homepage for lists!

* Use computer : 1 month, 1 year & 72 years. To check lists are complete : 5 h., 40 h., 27 d.

<u>how?</u> . use invariants $R_i(L) = \{ v \in L \mid v \cdot v = i \}$ and $R_{\leq i}(L)$ for $i \leq 3$. Enough to distinguish all lattices (not obvious)

. use Plesken - Souvignier to compute $|O(L)|$ and check $\text{mass}(\mathcal{L}_n) = \sum 1/|O(L_i)|$

<u>Method</u> all lattices given as Kneser neighbors of $I_n$ (in spirit of Bacher - Venkov)

## II Kneser neighbors

**Def:** Fix $L \in \mathcal{L}_n$ and $d \geq 1$. A (cyclic) $d$-neighbor of $L$ is an $N \in \mathcal{L}_n$ with $L/L\cap N \simeq \mathbb{Z}/d$

**Fact:** set $C_L(\mathbb{Z}/d) = \{ \ell \subset L\otimes\mathbb{Z}/d \mid \ell\cdot\ell \equiv 0 \ (d) \text{ or } (2d) \}$, $\ell \simeq \mathbb{Z}/d$

finite quadric

case $d$ odd

case $d$ even

For $\ell \in C_L(\mathbb{Z}/d\mathbb{Z})$, $M_d(\ell) = \{ v \in L \mid v\cdot\ell \equiv 0 \ (d)\} \underset{d}{\subset} L$

may choose $w \in L$ s.t. $w \cdot w \equiv 0 \,(d^2)$, $\ell = \mathbb{Z}/d\, \overline{w}$

<u>then</u> $N_d(\ell) = M_d(\ell) + \mathbb{Z}\, w/d$ d-neighb. of $L$

<u>Fact</u>: (i) does not depend on $w$ for $d$ odd, 2 values for $d$ even $\ell$

(ii) each d-neighb. of $L$ is $N_d(\ell)$ for unique $\ell$

<u>2 applications</u>

- Kneser's neighb. method (strong approx.)
- beautiful & compact forms for lattices

Ex: $L = I_n$ $\quad \ell \longleftrightarrow (x_i) \in (\mathbb{Z}/d\mathbb{Z})^n$ $\quad \sum x_i^2 \equiv 0 \ (d), (2d)$

$\gcd(\{x_i\}, d) = 1$

$N_2(1^n)$ for $n \equiv 0 \mod 4$, $\quad N_2(1^8) \simeq E_8$

$N_{94}(1,3,\dots,47) \simeq \text{Leech}$ (Thompson) $\quad 1^2 + 2^2 + \dots + 23^2 \equiv 0 \ (47)$

$(n = 24)$ $\quad 2 \times 94 = 4 \times 47$

Explanation $\quad R_2(L) = \{ v \in L \mid v \cdot v = 2 \}$ ADE root system

any integral latt.

Leech is "unique" even $L \in \mathcal{L}_{24}$ st. $R_2(L) = \emptyset$

<u>situation</u>

$$I_n \underset{d}{\supset} M_d(x) \underset{d}{\subset} N_d(x)$$

$$R_2(I_n) \supset R_d(M_d(x)) \subset R_d(N_d(x))$$

$$R_2(I_n) = \{\pm \varepsilon_i \pm \varepsilon_j\}_{i \neq j} \simeq D_n$$

$\pm\varepsilon_i \pm \varepsilon_j \in R_d(M_d(x)) \iff \pm x_i \pm x_j \equiv 0\ (d)$

$R_d(N_d(x))$ ? "visible root system of $N_d(x)$"

$\rightsquigarrow$ easily leads to Thompson's example, which actually work!

<u>Rmk:</u> $O(I_n) = \{\pm 1\}^n \rtimes S_n$, w.m.a. $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq d/2$

# III Statistics

Fix $\mathcal{G}$ any (spinor) genus of int. lattices $L$ in $\mathbb{R}^n$, $n>2$

Ex: $\mathcal{G} = \mathcal{L}_n^{odd}$ or $\mathcal{L}_n^{even}$

Thm (Hsia–Jöchner) $\forall L, L' \in \mathcal{G}$, $\forall p \gg 0$ prime, $L$ has a p-neighb. isometric to $L'$.

Theorem B

$\forall L, L' \in \mathcal{G}$

$$\frac{\#\left\{\begin{array}{c}\text{p-neighb. } N \text{ of } L\\ \text{with } N \simeq L'\end{array}\right\}}{\underbrace{\# C_L(\mathbb{Z}/p\mathbb{Z})}_{p^{n-2} \simeq}} = \frac{1/|O(L')|}{\text{mass } \mathcal{G}} + O\Big(\frac{1}{\sqrt{p}}\Big) \quad p \to \infty$$

for $n \geq 5$, $1/p$

Meaning: the probability to find the isometry class of a given odd $L$ as a $p$-neighbor of $I_n$ is proportional to

$$\operatorname{mass}(L) := 1/|O(L)|$$

$\rightarrow$ lattices with small mass are harder to find / to construct

Pf: A natural consequence of Arthur's classification of autom. rep. of classical groups. $2^{nd}$ proof in the spirit of Clozel-Oh-Ullmo equidistribution results $\square$

$\rightarrow$ many variants!

# IV Proof of theorem A : an example

* Induction $n$, we may assume $R_1(L) = \emptyset$
* Split the classification "root system by root system"

O. King : For $n \le 30$, $\forall$ ADE r.s. $R$ rk $\le n$, we know mass of $L \in \mathcal{L}_n$ with $R_1(L) = \emptyset$ & $R_2(L) \simeq R$

| $n$ | 23 | 24 | 25 | 26 | 27 | 28 |
|---|---|---|---|---|---|---|
| # $R$ | 49 | 149 | 327 | 1086 | 2796 | 4722 |
| non zero mass | | 156 | 368 | 1901 | 14493 | ~~374062~~ 357003 |

Concrete example: $n = 26$ $\quad R \simeq 10 A_1$

reduced mass $:= 2^{11}$ mass $= 4\,424\,507 / 58\,060\,800$

<u>prescribed visible root system</u>: $8 A_1$ (best!)

↳ restrict to $(x_i) \in (\mathbb{Z}/d)^{26}$, 8 pairs $x_i$ equal, 10 other coord. $\neq$

$\Rightarrow d \geq 36$

| $d$ | # matching / # iso lines | new lattices | reduced masses |
|---|---|---|---|
| 36 | 98 / 216 | 4 | $1/32$ $1/48$ $1/48$ $1/320$ |
| 37 - 38 | 1243 / 2852 | 0 | |
| 39 | 820 / 1821 | 1 | $1/6144$ |
| 40 - 49 | several millions | 0 | |

reduced mass remaining = $^{17}/_{58\,060\,800}$

→ probability to find new lattices $< 3 \cdot 10^{-6}$ (certainly less)

need billions of lines! need <u>other ideas</u>

<u>Rmk</u> thm B useful to guess if we missed some lattice or not

<u>One example of ideas used</u> if $R(L) \simeq 10 A_1$, ah. of

$2A_1 \hookrightarrow L$ has rk 24, r.s $8A_1 \hookrightarrow$ u. lat. rk 24 but $L := N_{35}$ (11 2 3 44 5 6 77 8 99 10 11 1212 13 14 1515 16 1717)

hardest 24 dim, with root system $8A_1$ (visible $7A_1$)

$\longrightarrow$ compute 2 neighb. of $L \oplus I_2$ with vis. $8A_1$

$d = 70$ | 3632 / 16384 only! | 2 new lattices | 1/3686400 & 1/46448640

$\longrightarrow$ 7 lattices, QED

Thank you!

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