Unimodular lattices of rank 29 and applications

MIT number theory seminar 02/24

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Joint work with Olivier Taïbi

Plan

I. Classification of unimodular lattices

II. Applications to aut. forms for $GL_n(\mathbb{Z})$

I. Unimodular lattices

$n \geq 1$, euclidean $\mathbb{R}^n$

$$\mathcal{L}_n = \left\{ \text{lattice } L \subset \mathbb{R}^n \;\middle|\; \begin{array}{l} x \cdot y \in \mathbb{Z} \quad \forall x, y \in L \\ \text{covol } L = 1 \end{array} \right\} \circlearrowleft O(n)$$

Ex: $I_n = \mathbb{Z}^n \in \mathcal{L}_n$

$X_n = O(n) \backslash \mathcal{L}_n$

general facts:
- finitely many isom. classes
- mass $\mathcal{L}_n$ known
- $\mathcal{L}_n = \mathcal{L}_n^{\text{odd}} \coprod \mathcal{L}_n^{\text{even}}$ $\neq \emptyset \iff n \equiv 0 \ (8)$

known 80's

n	≤ 7	8-11	12-13	14	15	16	17	18	19	20	21	22	23	24	25
$\#X_n$	1	2^1	3	4	5	8^2	9	13	16	28	40	68	117	297^{24}	665

(Lagrange, Gauss, Mordell '38, Witt, Kneser '47, Niemeier, Conway-Sloane '82, Borcherds '84)

Thm:

n	26	27	28	29
$\lvert X_n \rvert$	2566	17059	374 062	38 592 290
	Ch.	Ch.	Allombert-Ch.	Ch.-Taïbi

Goals
- ideas of method used with Olivier
- motivations / applications

Rmk: Configuration vectors of norm i

$R_i(L) = \{ v \in L \mid v.v = i \}$ & $R_{\leq i}(L)$ sim.

- wma $R_1(L) = \emptyset$
- $R_2(L)$ = root system of L (ADE) : sharp on X_n for $n \leq 23$!

King: mass of $X_n^R = \left\{ [L] \in X_n \;\middle|\; \begin{array}{l} R_1(L) = \emptyset \\ R_2(L) \simeq R \end{array} \right\}$ for $n \leq 30$

$\Downarrow$

good l.b. on $\# X_n$!

Thm $n \leq 29$. $L \longrightarrow R_{\leq 3}(L)$ sharp invariant on X_n

(better $L \to BV(L)$ sharp)

$\rightsquigarrow$ given Gram matrices + invariant BV + mass formula $\Rightarrow$ completeness of lists!

Ideas & methods

- starting point: unimodular hunting method (Bacher-Venkov, Ch.) variant Kneser neighb.

$d \geq 1$ $\{$ cyclic d-neighbor of $I_n\}$ $\overset{\sim}{\to}$ $\{$ isotropic $\ell \simeq \mathbb{Z}/d \subset I_n \otimes \mathbb{Z}/d \}$

2:1 d even $\qquad$ $\ell \cdot \ell \equiv 0 \ (2d)$

Concretely $\quad \ell = \mathbb{Z}/d\mathbb{Z}\, x \,, \quad \sum_{i=1}^{n} x_i^2 \equiv 0 \ (d^2)$

$$N_d(\ell) = I_n \cap \ell^{\perp} + \mathbb{Z}\frac{x}{d} \in \mathcal{L}_n$$

Ex: $E_8 \simeq N_2(1^8)$, Leech $= N_{94}(1,3,5,\dots,47)$ (Thompson)

$\longrightarrow$ compact def. of latt.

Thm (Ch.) for $L \in \mathcal{L}_n^{odd}$

$$\#\left\{ d\text{-neighb. } N \text{ of } I_n \text{ with } N \simeq L \right\} \Big/ \underbrace{\text{tot \# } d\text{-neighb.}}_{= d^{n-2}} \xrightarrow[d \text{ odd}]{d \to \infty} \frac{1/|O(L)|}{\text{mass } \mathcal{L}_n^{odd}}$$

Cor:

- any L can be obtained (but small mass. lat. are rare)
- "non uniform coupon collector problem"
- much more general

Unimodular Hunting I & II : methods for clever choices of isotropic lines to bias statistics ("visible root system", "visible isometries") in enumeration

Efficient for $n \leq 28$ but lengthy : 72 years of CPU time $n = 28$ (All.-Ch.)

For $n = 29$, we only use this method for determining X_{29}^{R} with $R = \emptyset$ (10092 All. Ch), $R = A_1$ (59105) & $R = A_2$ (37141)

[Improvement of lat. algorithms : we also det. X_{30}^{R} for $R = \emptyset, A_1, A_2$

e.g. $\# X_{30}^{\emptyset} = 82\ 323\ 107$]

$\underline{n=29}$ (remaining lat.) all have a pair $\{\alpha, \beta\}$ of orthogonal roots
$(\alpha.\alpha = 2 = \beta\beta \quad \alpha.\beta = 0)$

<u>elem. observation</u> $\forall n \geq 1 \ \exists$ natural groupoid eq. between

pairs (L, e) $L \in \mathcal{L}_n$ with $e \in L/2L$ $e.e \equiv 2\ (4)$

and

pairs $(U, \{\alpha, \beta\})$ $U \in \mathcal{L}_{n+2}$ & α, β pair of $\perp$ roots $\left(\frac{\alpha+\beta}{2} \notin U\right)$

<u>idea</u> . $(U, \{\alpha, \beta\}) \longrightarrow M := U \cap \alpha^{\perp} \cap \beta^{\perp}$ $\quad (M^{\#}/M \simeq \mathbb{Z}/2 \perp \mathbb{Z}/2)$

has index 2 in unique $L \in \mathcal{L}_n$

$= \{v \in L \mid v.e \equiv 0\ (2)\}$ unique $e \in L/2L$

. $(L, e) \rightsquigarrow M \rightsquigarrow L = (M \perp \mathbb{Z}\alpha \perp \mathbb{Z}\beta) + \mathbb{Z}\frac{e+\alpha}{2} + \mathbb{Z}\left(\frac{e+\beta}{2} + f\right)$

$L = M + \mathbb{Z}f$

$\leadsto$ exhaustion of X_{m+2} (with 1 roots) by first listing orbits of mod 2 vectors in L in X_m

$\oplus$: efficient algo for orbits of mod 2 vec.

$\ominus$: each U obtained as many times as $O(U)$ - orbit of $\{\alpha,\beta\}$ in U

worst cases $R_2(U) \simeq a A_1 \; b A_2 \quad a+b$ big ($>$ 10 millions sublat.)

numerology: $(m=29)$ 1.3 billions of $(U, \{\alpha,\beta\})$ up to iso ($\simeq$ 1 month CPU)

computation of all $BV(U)$ $\simeq 2000\, d.$

- superiority of method / neighbor one : $n \leq 28$ all X's in $\simeq$ 1 week CPU only!
- tricks "relevant $\{\alpha,\beta\}$" $\longrightarrow$ divide / 4

The BV invariant (following Allombert–Ch., variant of Bacher–Venkov)

$$L \rightsquigarrow \text{graph} \begin{cases} \text{vertices}: & \{\pm s\}, \quad 0 < s \cdot s \leq 3 \\ \text{edges}: & \{\pm s\} \text{ —— } \{\pm s'\} \text{ if } s \cdot s' \equiv 1 \ (2) \end{cases}$$

$\rightsquigarrow$ adj. matrix $A \rightsquigarrow A^2$ (possibly heavy comp.)

if C column of $A^2 \rightsquigarrow m(C) =$ multiset of entries of C

Definition $BV(L) = \{\{ m(C) \mid C \text{ col. of } A^2 \}\}$

clearly an invariant of $R_{\leq 3}(L)$, fast to compute if $\simeq 10^3$ vertices!

($\simeq 150$ ms)

Miracle: BV sharp on X_n for all $n \leq 29$

Application to classification of other lattices

- rank 32 even unimodular lattices

$\rightsquigarrow$ list of all such L's s.t. $\exists\, A_3 \hookrightarrow L$

go further! (in progress)

- determination of rep. of isom. classes in each genus of <u>even</u> lattices of rank ≤ 28 with det p or $2p$, $p \in \{3,5,7\}$

Thm: the number iso classes are as follows:

$p \backslash n$	2	4	6	8	10	12	14	16	18	20	22	24	26	28
3	1		1		1		2		6		31		678	
5		1		1		2		5		27		352		2 738 211
7	1		1		2		4		20		153		44 955	

Table 1.2: Number of isometry classes of even lattices of even rank $2 \leq n \leq 28$ and odd prime determinant $p \leq 7$ (zero if blank).

$p \backslash n$	1	3	5	7	9	11	13	15	17	19	21	23	25	27
3	1	1	1	1	2	2	3	5	10	19	64	290	2 827	285 825
5	1	1	1	1	3	3	5	10	21	55	210	1 396	38 749	24 545 511
7	1	1	1	2	3	5	8	14	37	119	513	5 535	341 798	659 641 434

Table 1.3: Number of isometry classes of even lattices of odd rank $1 \leq n \leq 27$ and determinant $2p$ with p a prime ≤ 7.

Also give Gram matrices, invariants, statistics etc... on our website (soon on LMFDB?)

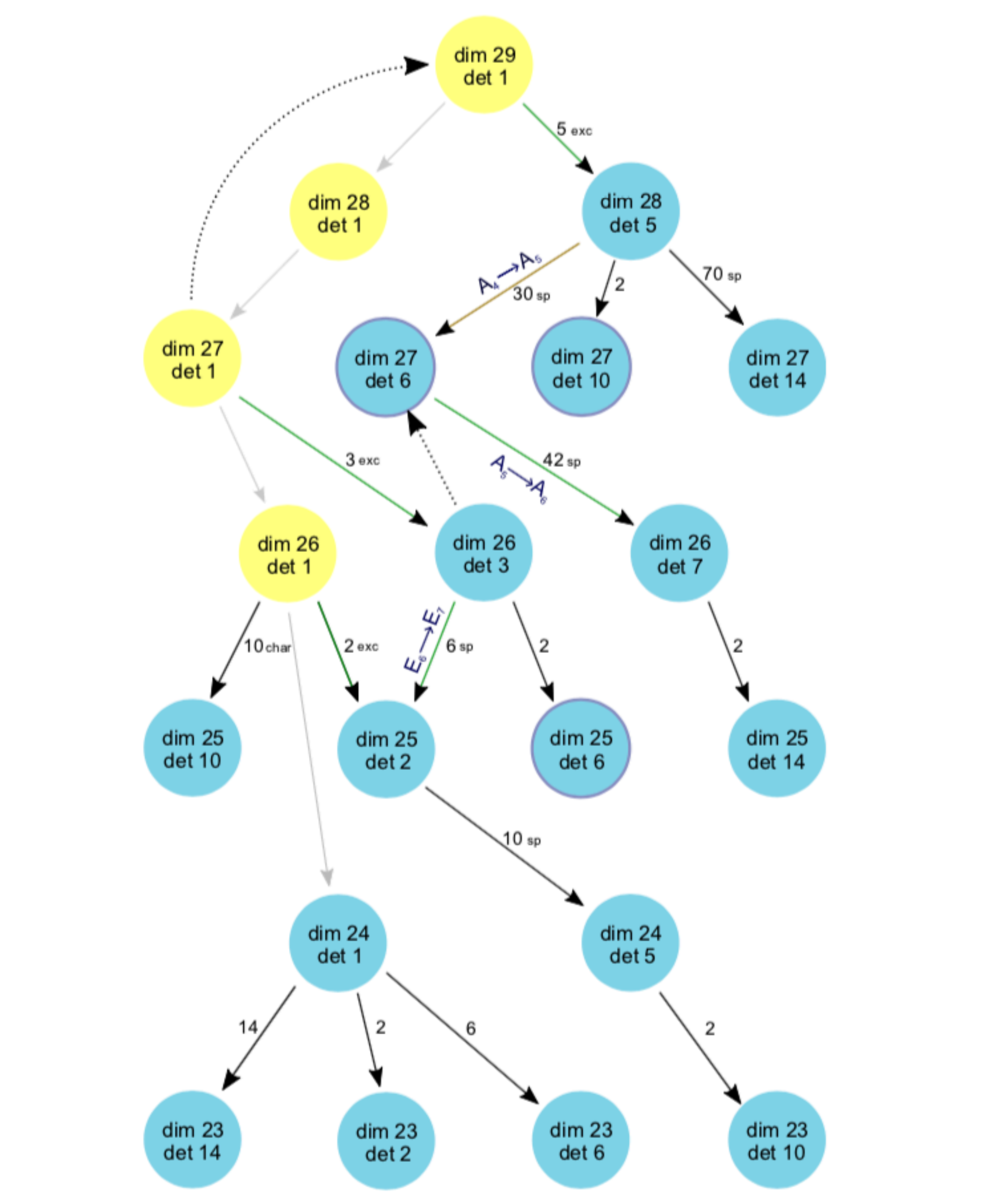

Figure 1: Unimodular lattices rule !
(Leitfaden for the proof of Theorem 7.2)

Idea (Gauss, Kneser, C&S, ...)

Orbit method : $\mathcal{G} \xrightarrow{t} \mathcal{G}'$ if

- orbits of type t vectors of L in $\mathcal{G}$

$\updownarrow$ (L, v)

- iso classes of L' in $\mathcal{G}'$ $\quad L' = v^\perp \cap L$

→ everything deduced from unim. cases !

→ unique orbit phenomena (conceptual pfs)

II. Applications to aut. forms for $GL_n(\mathbb{Z})$

Part of a long term programm (Ch. Renard, Ch. Lannes, Taïbi, Ch. Taïbi, ...)...

<u>Q:</u> $n>1$, $\exists\ \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ irred ?

unram. outside ℓ, cryst. at ℓ, HT wts $0, 1, 2, \ldots, n-1$

<u>aut. side</u> $H^*_{cusp}(GL_n(\mathbb{Z}), \mathbb{C}) \neq 0$? (Borel)

<u>known</u> Mestre, Fermigier, Miller: $n \geq 27$

Boxer-Calegari-Gee: a few ex., smallest $n = 79$

several reasons, we believe $n \simeq 27$ may work, main mot. for I!

16/18

$N_n^{\perp}(k_1, \ldots, k_n) = \#$ cusp aut. rep π of $GL_n/\mathbb{Q}$ with
π_p unram. $\forall p$, π_∞ alg weights $k_1 < \ldots < k_n$
$\pi^\vee \simeq \pi |\cdot|^*$

Thm 1 (Ch. T) explicit formula for $N_n^{\perp}(k_1, \ldots, k_n)$ for $n \leq 26$ & $n \neq 25$

Set $\delta_n^{\perp} = \underset{\text{jump}}{\text{smallest}}(k_n - k_1 - n + 1)$ for π as above ($\delta_n^{\perp} \geq 0$, 0 iff $\mathbb{Q}$ solved)

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$\delta_n^{\perp}$	10	20	16	22	18	17	18	18	14	16	14	14	12
n	15	16	17	18	19	20	21	22	23	24	25	26	27
$\delta_n^{\perp}$	12	10	10	10	8	8	6	6	4	4	≥ 2	2	?

More precisely : $c^{\perp}_{m,i} = N^{\perp}_{m}(0,1,\dots,\hat{i},\dots,\widehat{m+1-i},\dots,m,m+1)$

"allow 2 holes in string of weights" $\quad m$ even, $1 \leq i < \frac{m+1}{2}$

Thm 2 (Ch.T.) • $c_{n,*} = 0$ for all $n \leq 24$

• for $n = 26$, $c^{\perp}_{n,i} = 0$ unless

i	9	11	13
$c^{\perp}_{n,i}$	10	45	25

Thm 3 (Ch.T.) set $N =$ # even lat rank 31 det 2 $/\simeq$

Then we have $N \geq 6{,}689 \cdot 10^9$ and

$$N - 4\,989\,289\,914 = c^{\perp}_{30} + c^{\perp}_{28,14} + c^{\perp}_{28,9} + c^{\perp}_{28,7} + c^{\perp}_{28,5} + 2\,c^{\perp}_{28,3} + 2\,c^{\perp}_{28,1}$$

"# Π with weights $0,1,2,\dots,29$" $\Rightarrow$ at least 1 unknown very big!

General idea

$\mathcal{G}$ genus of even lat as before, compute $\forall\, \underline{P}$

$$\sum_{[L]\in\mathcal{G}} \frac{1}{|O(L)|}\ \#\left\{\gamma \in O(L)\ \middle|\ \begin{array}{l}\text{char poly } \gamma = P \\ \text{spin } \gamma = 1\end{array}\right\}$$

⇝ (local-global method + Taïbi's work) compute certain local orbital int. finite order cl. in $G(\mathbb{Q}_p)$ G split $SO / \mathbb{Z}_p$.

⇝ (Taïbi's thesis) compute geom. term of ATF ⇝ spectral inf. Arthur... ☒

(forthcoming paper!)

Thank you!