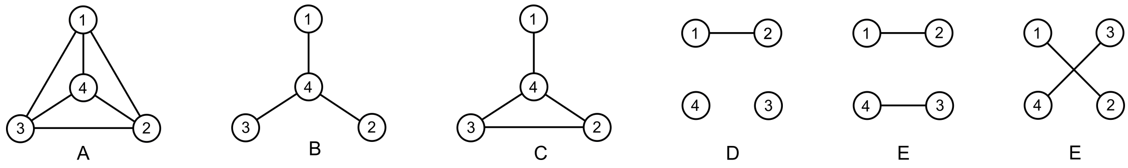
No document allowed. Exam duration 2h. It is not necessary at all to answer all the questions to obtain the maximal score. All answers must be carefully justified.

Problem 1. *Let $n \geq 1$ be an integer. By an n-*graph *we shall mean a subset $\Gamma \subset \mathrm{P}(\{1, \dots, n\})$ consisting of 2-element subsets of $\{1, \dots, n\}$. Those subsets will also be called the* edges *of the graph Γ. For instance, the six figures below define*[1] *five 4-graphs (the last two being the same), in which the edges are represented by lines :*



Let Γ be an n-graph. For $i \in \{1, \dots, n\}$ we set $\mathrm{V}_\Gamma(i) = \{j \in \{1, \dots, n\} \mid \{i, j\} \in \Gamma\}$ (set of neighbors *of i in Γ) and $\mathrm{v}_\Gamma(i) = |\mathrm{V}_\Gamma(i)|$. We finally define the* symmetry group of Γ *as*

$$\mathrm{G}(\Gamma) := \{g \in \mathrm{S}_n \mid \forall i, j \in \{1, \dots, n\},\ \{i, j\} \in \Gamma \implies \{g(i), g(j)\} \in \Gamma\}.$$

(i) Show that $\mathrm{G}(\Gamma)$ is indeed a subgroup of S_n.

(ii) Let $i \in \{1, \dots, n\}$ and $g \in \mathrm{G}(\Gamma)$. Show $g(\mathrm{V}_\Gamma(i)) = \mathrm{V}_\Gamma(g(i))$ and $\mathrm{v}_\Gamma(i) = \mathrm{v}_\Gamma(g(i))$.

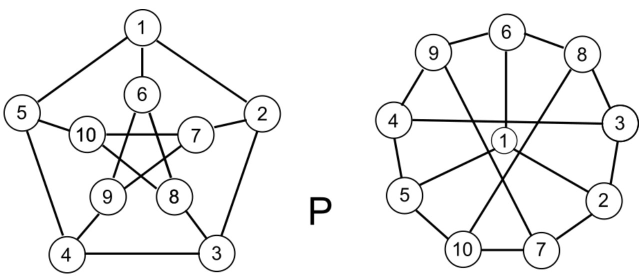
(iii) For each of the five 4-graphs Γ above, give without proof a group isomorphic to $\mathrm{G}(\Gamma)$ in the following list : 1, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, S_3, D_8, H_8, S_4.

(iv) (continuation) For two graphs Γ of your choice among those five, justify your answer. For this, you may first determine the map $\mathrm{v}_\Gamma : \{1, 2, 3, 4\} \to \mathbb{N}$ and list all the elements of $\mathrm{G}(\Gamma)$.

In the sequel, we are interested in the Petersen graph*. This is the 10-graph P defined by*

$\mathrm{P} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}, \{6,8\}, \{8,10\}, \{7,10\}, \{7,9\}, \{6,9\}, \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}\}$.

The two following figures give two representations of P :



Our goal is to show $\mathrm{G}(\mathrm{P}) \simeq \mathrm{S}_5$. As any subgroup of S_{10}, the group $\mathrm{G}(\mathrm{P})$ naturally acts on $\{1, \dots, 10\}$. For $I \subset \{1, \dots, n\}$ we set $\mathrm{G}(\mathrm{P})_I = \{g \in \mathrm{G}(\mathrm{P}) \mid g(i) = i,\ \forall i \in I\}$.

(v) With the help of the figures above, give an order 3 element and an order 5 element in $\mathrm{G}(\mathrm{P})$.

1. Concretely, we have $\mathrm{A} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$, $\mathrm{B} = \{\{1,4\}, \{2,4\}, \{3,4\}\}$, $\mathrm{C} = \{\{1,4\}, \{2,4\}, \{2,3\}, \{3,4\}\}$, $\mathrm{D} = \{\{1,2\}\}$ and $\mathrm{E} = \{\{1,2\}, \{3,4\}\}$.

(vi) Show that $\mathrm{G(P)}$ *acts transitively on* $\{1,\dots,10\}$.

(vii) Show that $\mathrm{G(P)}_{\{1,2,5,6\}}$ *is generated by the element* $(3\,7)(4\,10)(8\,9)$.

(viii) Show that we have a short exact sequence $1 \to \mathrm{G(P)}_{\{1,2,5,6\}} \to \mathrm{G(P)}_{\{1\}} \to \mathrm{S}_{\{2,5,6\}} \to 1$.

(ix) Prove $|\mathrm{G(P)}| = 120$.

A tetrad *of* P *is a 4-element subset* $T \subset \{1,2,\dots,10\}$ *such that* $\forall i,j \in T$, $\{i,\ j\} \notin \mathrm{P}$ *(in other words,* T *does not contain any pair of neighbors in* P*).*

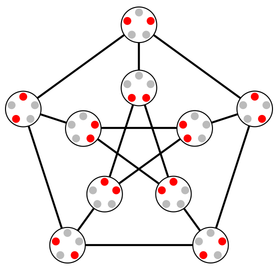
(x) Show that there are exactly two tetrads containing 1*, say* T *and* T' *, as well as* $T \cap T' = \{1\}$.

(xi) Show that P *contains exactly* 5 *tetrads.*

(xii) Show that there is an injective group morphism $\mathrm{G(P)} \longrightarrow \mathrm{S}_5$.

(xiii) Conclude.

(xiv) Prove the existence of an injective morphism $\mathrm{S}_5 \to \mathrm{G(P)}$ *without using the previous questions, but rather by contemplating the equality* $\binom{5}{2} = 10$ *and the following figure :*



Problem 2. (The Miller-Moreno theorem) *A subgroup* G *will be called* subcyclic *if each strict*[2] *subgroup of* G *is cyclic. Our aim is to prove the following theorem, due to Miller and Moreno : if* G *is a finite subcyclic group, then either* G *is isomorphic to one of the groups*

$$\mathbb{Z}/n\mathbb{Z}\ \ (n \geq 1),\ \ \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\ \ (p \text{ prime}),\ \mathrm{H}_8,$$

or we have $G \simeq \mathbb{Z}/p\mathbb{Z} \rtimes_\alpha \mathbb{Z}/q^n\mathbb{Z}$ *with* $n \geq 1$*,* p *and* q *prime with* $q \,|\, p-1$*, and a suitable* α.

Part I : The case of abelian groups

(i) Briefly recall why cyclic groups are subcyclic.

(ii) Show that, for every prime p*, the group* $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ *is subcyclic and not cyclic.*

(iii) Show that if G *is a non cyclic finite abelian group, there is a prime* p *such that* G *contains a subgroup isomorphic to* $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(iv) Conclude the classification, up to isomorphisms, of finite abelian subcyclic groups.

2. A subgroup H of a group G is called strict if we have $H \neq G$.

Part II : Some properties of subabelian groups

A group G will be called subabelian *if each strict subgroup of G is abelian.*

(i) *Let G be a subabelian group and H a normal subgroup of G. Show that both groups H and G/H are subabelian.*

(ii) *Show that every non trivial, finite abelian group, has a subgroup of prime index.*

(iii) *We admit that a finite, non abelian, subabelian group is not simple.* [3] *Show that every finite, non trivial, subabelian group has a normal subgroup of prime index.*

Our aim until the end of Part II *is to show that if G is a finite, non abelian, subabelian group, there are two subgroups P and Q of G such that :* (a) $G = PQ$, (b) *P is normal in G and of order p^m with p prime and $m \geq 0$, and* (c) *Q has order q^n with q prime $\neq p$ and $n \geq 1$.*

(iv) *Let H be a finite abelian group and $n \geq 1$. Show that $H[n] = \{h \in H \mid h^n = 1\}$ is a characteristic subgroup of H, and that each prime dividing $|H[n]|$ divides the integer n.*

(v) *(continuation) Assume $|H| = ab$ with a and b coprime, and set $A = H[a]$ and $B = H[b]$. Show $H = AB$, $A \cap B = \{1\}$, and then $|A| = a$ and $|B| = b$.*

(vi) *Let A and B be two finite subgroups of a group G with $A \triangleleft G$. Show that AB is a subgroup of G whose order divides $|A|\,|B|$.*

(vii) *Let G be a group and H a normal subgroup of G with prime index q. Show that there exists $z \in G$ whose order is a power of q and with $G = H\langle z\rangle$. (Hint : Consider the canonical morphism $G \to G/H$).*

(viii) *(continuation) Assume furthermore that G is subabelian. Show that we have $G = H'Q$ with H' a normal abelian subgroup of G whose order is prime to q, and Q a subgroup of G with order q^n and $n \geq 1$.*

(ix) *Conclude.*

Part III : The Miller-Moreno groups

In this part, we are interested in the finite, non abelian, subcyclic groups G such that $|G|$ is divisible by at least two distinct primes (« Miller-Moreno groups »). Let G be such a group.

(i) *Show that there are elements x and y in G of respective orders p^m and q^n, with p, q distinct primes, $m, n \geq 1$ and $G = \langle x\rangle\langle y\rangle$, as well as $k \in \mathbb{Z}$ with $yxy^{-1} = x^k$ and $k \not\equiv 1 \bmod p^m$.*

(ii) *Show $x\,y^q = y^q\,x$ et $x^p\,y = y\,x^p$. For this, you may consider $\langle x\rangle\,\langle y^q\rangle$ and $\langle x^p\rangle\,\langle y\rangle$.*

(iii) *Deduce $k^q \equiv 1 \bmod p^m$ and $k \equiv 1 \bmod p^{m-1}$.*

(iv) *Show $m = 1$ and $q \mid p - 1$. We recall, for $r \in p\mathbb{Z}$, the congruence $(1 + r)^p \equiv 1 \bmod pr$.*

(v) *Show that there is a group morphism $\alpha : \mathbb{Z}/q^n\mathbb{Z} \to \mathrm{Aut}\,\mathbb{Z}/p\mathbb{Z}$ sending $\overline{1}$ to $z \mapsto \overline{k}z$.*

(vi) *Conclude $G \simeq \mathbb{Z}/p\mathbb{Z} \rtimes_\alpha \mathbb{Z}/q^n\mathbb{Z}$.*

---

3. This result, admitted here and in all the problem, was the subject of Problem 2 of the partial exam 2021-2022.

(vii) Conversely, show that for all primes p, q *with* $q \mid p-1$*, and all* $n \geq 1$*, there is a morphism* $\beta : \mathbb{Z}/q^n\mathbb{Z} \to \operatorname{Aut} \mathbb{Z}/p\mathbb{Z}$ *such that the group* $\mathbb{Z}/p\mathbb{Z} \rtimes_\beta \mathbb{Z}/q^n\mathbb{Z}$ *is subcyclic and non abelian.*

Partie IV : (Bonus) The case of p-groups

We finally assume G *is a subcyclic, non abelian, group of order* p^n*, with* p *prime and* $n \geq 2$*. We fix a normal subgroup* $H = \langle x \rangle$ *of* G *of order* p^{n-1}*, as well as* $y \in G \smallsetminus H$*.*

(i) Show $y^p \in \langle x^p \rangle$ *and* $y\, x^p = x^p\, y$*.*

(ii) Show $yxy^{-1} = x^k$ *with* $k \in \mathbb{Z}$*,* $k \equiv 1 \bmod p^{n-2}$ *and* $n \geq 3$*.*

(iii) Show that, up to replacing y *with* $x^i y$ *for some* $i \in \mathbb{Z}$ *if necessary, we may assume* $y^{2p} = 1$*.*

(iv) By considering the subgroup $\langle t, y \rangle$ *of* G *with* $t = x^{p^{n-2}}$*, show* $p = 2$ *and* $y^2 = t$*.*

(v) Conclude $G \simeq \mathrm{H}_8$*, as well as the proof of the Miller-Moreno theorem.*