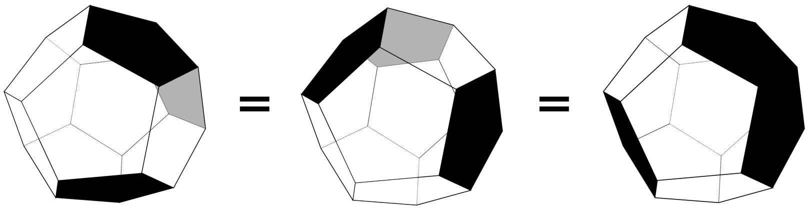
Algebra exam, ENS Ulm, January 16, 2024

No document allowed. Exam duration 3h. It is not necessary to answer all the questions to obtain the maximal score. All answers must be carefully justified.

Problème 1. (Black and white dodecahedra) *Let* D *be a regular dodecahedron. Our aim is to determine the number of ways to color the faces of* D *in black or white, assuming that two such colorings are identified if they can be deduced from each other by a rotation of* D.



(i) Determine the number of 3*-cycles, double-transpositions, and* 5*-cycles, in* A_5.

Let G *be a group acting on a finite set* X*, and let* E *be a finite set of cardinality* m*. For* $g \in G$ *and* $\phi : X \to E$*, we define* $g.\phi : X \to E$ *by the formula* $(g.\phi)(x) = \phi(g^{-1}x)$.

(ii) Check that $(g, \phi) \mapsto g.\phi$ *is an action of* G *on the set* E^X.

(iii) Let $g \in G$*. Show that the number of fixed points of* g *in* E^X *is of the form* $m^{\mathrm{r}_X(g)}$*, where* $\mathrm{r}_X(g)$ *is an integer to be expressed in terms of the type of the cycle decomposition of* g *on* X.

From now on, we assume that G *is the group of proper isometries of a regular dodecahedron, and that* X *is the set of* 12 *faces of this dodecahedron, equipped with its natural action of* G.

(iv) Assume that $g \in G \smallsetminus \{1\}$ *has a fixed point in* X*. Briefly explain why* g *has order* 5 *and exactly two fixed points in* X.

(v) Deduce that the action of G *on* E^X *has exactly* $\frac{1}{60}(m^{12} + 15\,m^6 + 44\,m^4)$ *orbits.*

(vi) Conclude.

Problème 2. (Self-transitive groups and Rotman's simplicity criterion) *We start with a preliminary question. Let* G *be a group acting* k*-transitively*[1] *on a set* X*, with* $k \geq 2$*, and let* $x \in X$.

(o) Check that G_x *acts* $(k-1)$*-transitively on* $X \smallsetminus \{x\}$.

Part I : Self-transitive groups

We are interested in the finite groups G *such that the natural action of the group* $\mathrm{Aut}\, G$ *on the set* $G \smallsetminus \{1\}$*,* $(\alpha, g) \mapsto \alpha(g)$*, is transitive. In questions (i) to (iii) we fix such a group* $G \neq \{1\}$*, and choose a prime* p *dividing* $|G|$.

1. Remember that our convention is that if a group acts k-transitively on a set X, then we have $|X| \geq k$.

(i) Show that any non-trivial element of G has order p.

(ii) Show $\mathrm{Z}(G) = G$.

(iii) Deduce that we have $G \simeq (\mathbb{Z}/p\mathbb{Z})^n$ *for some* $n \geq 1$.

(iv) Conversely, prove that for p prime and $n \geq 1$, the group $(\mathbb{Z}/p\mathbb{Z})^n$ has the required property.

We finally fix an integer $k \geq 2$ and a finite group G such that the natural action of $\operatorname{Aut} G$ *on* $G \setminus \{1\}$ *is k-transitive.*

(v) Assume $k = 2$. Show that we either have $G \simeq \mathbb{Z}/3\mathbb{Z}$, or $G \simeq (\mathbb{Z}/2\mathbb{Z})^n$ and $n \geq 2$.

(vi) Assume $k \geq 3$. Show $k = 3$ and $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Part II : Rotman's simplicity criterion

Let G be a finite group acting faithfully and k-transitively on a finite set X, with $k \geq 2$. Suppose there exists $x \in X$ such that the group G_x is simple. Following J. Rotman, our aim is to show that one of the following assertions is satisfied :

(a) *G is simple,*

(b) *we have $k = 3$ and, either $|X| = 3$ and $G = \mathrm{S}_X$, or $|X|$ is a power of 2,*

(c) *we have $k = 2$ and $|X|$ is a prime power.*

We fix $x_0 \in X$ such that G_{x_0} is simple. Let N be a normal subgroup of G with $N \neq \{1\}$ and $N \neq G$.

(i) Show that N acts transitively X. Hint : consider $x \in X$ with $|Nx| > 1$ and show $X = Nx$.

(ii) Prove $N \cap G_{x_0} = \{1\}$.

(iii) Check that the map $N \setminus \{1\} \longrightarrow X \setminus \{x_0\}, n \mapsto nx_0$, is well-defined and bijective.

(iv) Deduce that the conjugacy action of G_{x_0} on $N \setminus \{1\}$ is $(k-1)$-transitive.

(v) Conclude the proof of Rotman's criterion.

Part III : two applications of Rotman's criterion

(i) Prove that the simplicity of A_5 *implies that of* A_n *for* $n \geq 6$.

É. Mathieu has constructed a subgroup G of S_{24} *acting 5-transitively on* $\{1, 2, \ldots, 24\}$ *and verifying* $G_1 \cap G_2 \cap G_3 \simeq \mathrm{PSL}_3(\mathbb{F}_4)$, *where G_i denotes the stabilizer in G of the element $i \in \{1, 2, \ldots, 24\}$ and $\mathbb{F}_4$ is a field with 4 elements. We denote respectively by* M_{24}, M_{23} *and* M_{22} *the groups G, G_1 and $G_1 \cap G_2$.*

(ii) Show that M_{22}, M_{23} *and* M_{24} *are simple.*[2]

2. These are the three largest sporadic simple groups discovered by Mathieu.

Let K be a field, V a finite-dimensional K-vector space and u an endomorphism of V. Recall that V_u denotes the $K[X]$-module with underlying K-vector space V and $Xv = u(v)$ for all $v \in V$. We denote by ${}^t u$ the endomorphism of the dual V^* of V defined by ${}^t u(\varphi) = \varphi \circ u$ for all $\varphi \in V^*$ (*transpose* of u).

Problème 3. (Duality, and non-isomorphic actions with isomorphic permutation representations) *Let K be a field, V a K-vector space of finite dimension $n \geq 1$, and u an endomorphism of V. Our first aim is to show that the $K[X]$-modules V_u and $(V^*)_{{}^t u}$ are isomorphic.*

(i) Assume that the $K[X]$-module V_u is isomorphic to $K[X]/(P)$ with $P \in K[X]$ monic. Show that there exists $v \in V$ such that $v, Xv, X^2v, \ldots, X^{n-1}v$ is a basis of V, and with $Pv = 0$.

(ii) (continued) Show that the $K[X]$-module $(V^*)_{{}^t u}$ is monogenic, as well as $P\psi = 0$ for all $\psi \in V^*$. Hint : consider a linear form ϕ on V satisfying $\phi(X^i v) = 0$ for $0 \leq i < n-1$, and $\phi(X^{n-1}v) = 1$.

(iii) (continued) Deduce that there is an isomorphism of $K[X]$-modules $(V^*)_{{}^t u} \simeq K[X]/(P)$.

(iv) Assume $V = \oplus_{i=1}^r V_i$ with $V_i \subset V$ a sub-vector space stable by u for $i = 1, \ldots, r$, and set $u_i = u_{|V_i}$. Check that there is a $K[X]$-module isomorphism $\oplus_{i=1}^r (V_i^*)_{{}^t u_i} \simeq (V^*)_{{}^t u}$.

(v) Conclude.

(vi) (Application) Show that any matrix in $\mathrm{M}_n(K)$ is conjugate to its transpose.

We now assume that K is a finite field. We denote by X the set of lines in the vector space K^n and by Y that of hyperplanes in K^n. Those two sets are equipped with natural actions of the group $G = \mathrm{GL}_n(K)$.

(vii) Show that each $g \in G$ has the same number of fixed points in X and in Y.

(viii) Deduce that the $\mathbb{C}[G]$-modules $\mathbb{C}X$ et $\mathbb{C}Y$ are isomorphic.

(ix) Nevertheless, show that the actions of G on X and on Y are not isomorphic for $n \geq 3$.

Problème 4. (Groups of the year) *Our aim is to show that, up to isomorphism, there are exactly 6 groups of order $2024 = 2^3 \cdot 11 \cdot 23$ and with an element of order 8. Let*[3] *G be a group of order 2024.*

(i) Show that G has a unique normal subgroup V of order 23.

(ii) Show that G has a subgroup Q of order 88, as well as $G = V \rtimes Q$.

(iii) Show that Q has a unique normal subgroup O of order 11.

(iv) Deduce that we have $Q = O \rtimes H$ for any 2-Sylow subgroup H of Q.

(v) Show that $\mathrm{Aut}\, O$ is cyclic of order 10, and has a unique order 2 element, namely $x \mapsto x^{-1}$. Show that the group $\mathrm{Aut}\, V$ has exactly 4 subgroups, which are cyclic and respectively generated by the automorphisms $v \mapsto v^i$ with $i \in \{1, -1, 2, -2\}$. We give the congruence $2^{11} \equiv 1 \bmod 23$.

We finally assume that G has an element of order 8, and we fix a 2-Sylow subgroup H of Q.

(vi) Show $H \simeq \mathbb{Z}/8\mathbb{Z}$.

3. We apologize for using the letter V for the french Vingt-trois, Q for Quatre-vingt-huit, O for Onze and H for Huit.

In the sequel, we fix a generator h of H.

- *(vii) Considering a suitable group morphism $H \to \operatorname{Aut} O$, show that either Q is abelian, or we have $hxh^{-1} = x^{-1}$ for all $x \in O$.*
- *(viii) Show that if Q is abelian, then Q is cyclic.*
- *(ix) Assume Q is cyclic. Show that there is a generator g of Q, as well as $i \in \{1, -1, 2, -2\}$, such that for all $v \in V$ we have $gvg^{-1} = v^i$.*
- *(x) Assume Q is not cyclic. Show that there is $i \in \{1, -1\}$ such that for all $v \in V$, and all $x \in O$, we have $hvh^{-1} = v^i$ and $xvx^{-1} = v$.*
- *(xi) Conclude.*